Int. Journal of Math. Analysis, Vol. 5, 2011, no. 32, 1583 - 1593

A Construction of a Pairwise Orthogonal Wavelet Frames Using Polyphase Matrix

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Abstract

This paper surveys the progress made on pairwise orthogonal wavelet frames and comments on the construction methods. There are a few known constructions based on the unitary extension principle, a paraunitary matrix and a given modulation matrix. A polyphase matrix based construction method has been presented which satisfies the condition of unitary extension principle yielding pairwise orthogonal tight wavelet frames. This enables us to study the approximation and vanishing moment properties of the resulting frame.

Mathematics Subject Classification: 42

Keywords: Wavelets, Frames, Polyphase Components, Orthogonal frames

1 Introduction

Over the past few years wavelets and frames have made significant development in theory and applications. In some of the applications wavelet bases have been proven to be better than the classical Fourier basis. Being linearly dependent spanning set, frames generalize the orthonormal basis. Like wavelets, frames have some desirable properties like vanishing moments, compact support, approximation order, symmetry etc. This paper focuses on a pair of orthogonal wavelet frames as characterized in [10]. Following this characterization, a pair of orthogonal wavelet frames are constructed in [1, 6] where the construction is based on modulation matrix. The approximation properties for the wavelet system only depend on the scaling filter whereas in case of frames it depends on how the polyphase matrix is completed [9]. The necessary and sufficient conditions for approximation order in multiple spatial dimensions for the wavelet

frames and their duals are provided in [9] using polyphase components of the filters. The modulation matrices have been used in [1, 6] to construct a pair of orthogonal wavelet frames, polyphase matrix of the given wavelet or frame system are purposed here obtaining the polyphase matrices of a pair of orthogonal wavelet frames.
The modulation matrix and the polyphase matrices are given in section 2. The orthogonality conditions are given in section 3 with known results. In section 4 the construction methods of [1, 6] are surveyed and a new construction based on polyphase matrix has been suggested.

2 The Polyphase Matrix

In this paper all the wavelets and frames are multiresolution analysis (MRA) based with dilation factor 2. The Fourier transform of a function $f(x) \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ is defined by

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}dx.$$

A scaling function ϕ is refinable (2-refinable) and satisfies

$$\phi(x) = \frac{1}{\sqrt{2}}\sum_k h_k \phi(2x-k).$$

where the sequence $\{h_k\}$ is square summable. With some additional conditions this function gives rise to an MRA and hence a scaling function [5]. We will assume that only finitely many $h_0, h_1 \cdots h_N$ are non zero which implies that the scaling function has compact support $[0, N]$. Taking Fourier transform on both sides yields

$$\widehat{\phi}(\xi) = m_0(\xi/2)\widehat{\phi}(\xi/2),$$

where

$$m_0(\xi) = \frac{1}{\sqrt{2}}\sum_{k=0}^{N} h_k e^{-2\pi i k\xi}$$

is a trigonometric polynomial satisfying $m(0) = 1$. It is called the *(low-pass) filter*. The wavelet theory (if the dilation factor is 2) consists of a *highpass* filter $m_1(\xi) \in L^\infty([0,1)$ and a wavelet $\hat{\psi}(\xi) = m_1(\xi/2)\hat{\phi}(\xi/2)$ such that the affine system $\psi_{i,j}(x) = \{2^{i/2}\psi(2^i x - j) : i, j \in \mathbb{Z}\}$ forms an orthonormal basis or a Riesz basis for $L_2(\mathbb{R})$ [5]. The wavelet frames are generalization of wavelet orthonormal bases (or Riesz bases). If the affine system $\psi_{k,i,j}(x) = \{2^{i/2}\psi_k(2^i x - j) : k = 1, \cdots r, i, j \in \mathbb{Z}\}$ is a frame (defined below in section 3) for $L_2(\mathbb{R})$ then $\{\psi_{k,i,j}\}$ is called a wavelet frame. This paper deals with frames coming from unitary extension principle (given below) where $\hat{\psi}_k(\xi) = m_k(\xi/2)\hat{\phi}(\xi/2)$. Thus a wavelet frame system has $m_k(\xi)$, $k = 1, \cdots r$ as highpass filters. Such lowpass and highpass filters can be expressed as a sum of 2 (because of the dilation factor 2) unique functions $m_{k,0}(\xi)$ and $m_{k,1}(\xi)$ (trigonometric polynomials in this case),

$$m_k(\xi) = \frac{1}{\sqrt{2}}\left(m_{k,0}(2\xi) + e^{-2\pi i\xi} m_{k,1}(2\xi)\right), \quad k = 0, 1, \cdots r.$$

As in [9], the polyphase components can also be expressed in terms of the filter via:

$$\begin{aligned} m_{k,0}(\xi) &= \frac{1}{\sqrt{2}}\left(m_k(\frac{\xi}{2}) + m_k(\frac{\xi+1}{2})\right) \\ m_{k,1}(\xi) &= \frac{1}{\sqrt{2}}\left(m_k(\frac{\xi}{2})e^{\pi i\xi} + m_k(\frac{\xi+1}{2})e^{\pi i(\xi+1)}\right) \end{aligned}$$

So the differentiability of the filter is related to its polyphase components as well. Given a lowpass filter with square summable coefficients, we define the *polyphase* and the *modulation* matrices as follows.

$$\begin{aligned} P(\xi) &= \begin{pmatrix} m_{0,0}(\xi) & m_{0,1}(\xi) \\ m_{1,0}(\xi) & m_{1,1}(\xi) \\ \vdots & \vdots \\ m_{r,0}(\xi) & m_{r,1}(\xi) \end{pmatrix} \\ M(\xi) &= \begin{pmatrix} m_0(\xi) & m_0(\xi+1/2) \\ m_1(\xi) & m_1(\xi+1/2) \\ \vdots & \vdots \\ m_r(\xi) & m_r(\xi+1/2) \end{pmatrix}. \end{aligned}$$

In this setting the entries of above matrices are trigonometric polynomials. The constructions provided in [1, 6] are based on the modulation matrix. The following theorem provides the fact that a polyphase matrix can be used in the statement of unitary extension principle.

Theorem 2.1 *$M^*(\xi)M(\xi) = I_2$ if and only if $P^*(\xi)P(\xi) = I_2$.*

Proof: In our setting we have the following relations between the polyphase and modulation matrices.

$$M(\xi) = P(2\xi)E(\xi)$$

where $E(\xi)$ is given by

$$E(\xi) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ e^{-2\pi i\xi} & -e^{-2\pi i\xi} \end{pmatrix}.$$

We notice that $P(2\xi)$ is paraunitary if and only if $P(\xi)$ is paraunitary.

3 Frames and Orthogonality

Let $\mathbb{H}$ be a separable Hilbert space. A sequence $\mathbb{X} = \{x_j\}_j$ in $\mathbb{H}$ called a Bessel sequence if there exists a constant $B > 0$ such that for all $f \in \mathbb{H}$

$$\sum_{j\in\mathbb{J}} | < f, x_j > |^2 \leq B||f||^2.$$

It is said to be a frame if there exists constants $0 < A \leq B$ such that for all $f \in \mathbb{H}$

$$A||f||^2 \leq \sum_{j\in\mathbb{J}} | < f, x_j > |^2 \leq B||f||^2.$$

Let $\mathbb{X} = \{x_j\}_j$ be a Bessel sequence on $\mathbb{H}$. The analysis operator $\Theta_{\mathbb{X}}$: $\mathbb{H} \to l^2(\mathbb{J})$ is defined by

$$\Theta_{\mathbb{X}} : f \to (< f, x_j >)_j;$$

and the synthesis operator is

$$\Theta^*_{\mathbb{X}} : (c_j) \to \sum_{j\in\mathbb{J}} c_j x_j;$$

Note that the analysis and synthesis operators are well defined and bounded because $\mathbb{X}$ is a Bessel sequence [3]. The synthesis operator is the Hilbert space adjoint of the analysis operator. The frame operator $S_{\mathbb{X}}f : \mathbb{H} \to \mathbb{H}$ is defined as

$$S_{\mathbb{X}}f = \Theta^*_{\mathbb{X}}\Theta_{\mathbb{X}}f :\to \sum_j < f, x_j > x_j$$

Also note that $\mathbb{X}$ is a frame if and only if $S_{\mathbb{X}}$ is bounded and has bounded inverse [3]. In that case the reconstruction formula can be written as [3]

$$f(x) = \sum_j < f, S^{-1}_{\mathbb{X}} x_j > x_j = \sum_j < f, x_j > S^{-1}_{\mathbb{X}} x_j.$$

$S_{\mathbb{X}}$ is preferred to be an identity. In that case $\mathbb{X}$ is called a normalized tight frame or Parseval frame. The frame $\mathbb{Y} = \{y_j\} : j \in \mathbb{J}$ is called the dual frame to $\mathbb{X}$ if

$$f(x) = \sum_j < f, x_j > y_j.$$

Here $S^{-1}_{\mathbb{X}}\{x_j\}$ is a dual called canonical dual to the given $\mathbb{X}$. Let $\mathbb{X} = \{x_j\}, j \in \mathbb{J}$ and $\mathbb{Y} = \{y_j\}, j \in \mathbb{J}$ be two Bessel sequences in $\mathbb{H}$. Using $\mathbb{X}$ to discritize the signal and $\mathbb{Y}$ to reconstruct, one arrives at

$$\Theta^*_{\mathbb{Y}}\Theta_{\mathbb{X}} : \mathbb{H} \to \mathbb{H} : f \to \sum_{j\in\mathbb{J}} < f, x_j > y_j.$$

This is the reconstruction formula if $\mathbb{X}$ and $\mathbb{Y}$ are dual frames. In case if $\mathbb{Y}$ is dual to $\mathbb{X}$ the operator $\Theta_{\mathbb{Y}}^*\Theta_{\mathbb{X}}$ is an identity. Two Bessel sequences $\mathbb{X}$ and $\mathbb{Y}$ are said to be orthogonal [10] if

$$\Theta_{\mathbb{Y}}^*\Theta_{\mathbb{X}} := \sum_{j\in\mathbb{J}} < \cdot, x_j > y_j = 0,$$

Let $f, g \in \mathbb{H}$ be two signals to be transmitted. Let $\mathbb{X}$ and $\mathbb{Y}$ both be Parseval frames and orthogonal to each other. Then

$$f = \sum_j (< f, x_j > + < g, y_j >)x_j$$
$$g = \sum_j (< f, x_j > + < g, y_j >)y_j.$$

So both signals can be reconstructed from the discretized samples of $< f, x_j > + < g, y_j >$. This idea can be used in multiple access communication systems. Note that the operator $S_{\mathbb{X}} = \Theta_{\mathbb{X}}^*\Theta_{\mathbb{X}}$ is called dual Gramian and $G_{\mathbb{X}} := \Theta_{\mathbb{X}}\Theta_{\mathbb{X}}^*$ is called the Gramian. Authors in [6] have used mixed dual Gramian, dual Gramian and Gramians [8] to characterize the affine and quasi-affine (wavelet) systems using the following proposition taken from [6].

Proposition 3.1 *Let $\mathbb{X}$ and $\mathbb{Y}$ be frames for $\mathbb{H}$. They are orthogonal if and only if $G_{\mathbb{Y}}G_{\mathbb{X}} = 0$.*

Proof: Suppose that they are orthogonal, then

$$\Theta_{\mathbb{Y}}^*\Theta_{\mathbb{X}} = 0.$$

So

$$G_{\mathbb{Y}}G_{\mathbb{X}} = \Theta_{\mathbb{Y}}\Theta_{\mathbb{Y}}^*\Theta_{\mathbb{X}}\Theta_{\mathbb{X}}^* = \Theta_{\mathbb{Y}}^* 0 \Theta_{\mathbb{X}} = 0.$$

Conversely let,

$$G_{\mathbb{Y}}G_{\mathbb{X}} = 0,$$

so

$$\Theta_{\mathbb{Y}}\Theta_{\mathbb{Y}}^*\Theta_{\mathbb{X}}\Theta_{\mathbb{X}}^* = 0 = (\Theta_{\mathbb{Y}}^*\Theta_{\mathbb{Y}})(\Theta_{\mathbb{Y}}^*\Theta_{\mathbb{X}})(\Theta_{\mathbb{X}}^*\Theta_{\mathbb{X}}) = S_{\mathbb{Y}}(\Theta_{\mathbb{Y}}^*\Theta_{\mathbb{X}})S_{\mathbb{X}}.$$

Since the operators $S_{\mathbb{X}}$ and $S_{\mathbb{Y}}$ are invertible, $\mathbb{X}$ and $\mathbb{Y}$ being the frames,we have that $\Theta_{\mathbb{Y}}^*\Theta_{\mathbb{X}} = 0$.
The Parseval frames satisfy the following equations [2, 1] .

Theorem 3.1 *Suppose $\{\psi_1, \psi_2, \cdots, \psi_r\}$ and $\{\eta_1, \eta_2, \cdots, \eta_r\}$ generate wavelet frames for $L_2(\mathbb{R})$. The frames are dual if and only if*

$$\sum_{k=1}^{r}\sum_{j\in\mathbb{Z}} \hat{\psi}_k(2^j\xi)\overline{\hat{\eta}_k(2^j\xi)} = 1, \quad a.e.\,\xi;$$

and for every odd integer q

$$\sum_{k=1}^{r}\sum_{j=0}^{\infty} \hat{\psi}_k(2^j\xi)\overline{\hat{\eta}_k(2^j(\xi+q))} = 0, \quad a.e.\,\xi.$$

In particular $\{\psi_1, \psi_2, \cdots, \psi_r\}$ *forms a Parseval frame for* $L_2(\mathbb{R})$ *if and only if the above equations are satisfied with* $\psi_i = \eta_i$.

The following theorem provides the orthogonality of Bessel sequences [10].

Theorem 3.2 *Suppose* $\{\psi_1, \psi_2, \cdots, \psi_r\}$ *and* $\{\eta_1, \eta_2, \cdots, \eta_r\}$ *generate affine Bessel sequences for* $L_2(\mathbb{R})$. *They are orthogonal if and only if*

$$\sum_{k=1}^{r} \sum_{j\in\mathbb{Z}} \hat{\psi}_k(2^j\xi)\overline{\hat{\eta}_k(2^j\xi)} = 0, \quad a.e.\, \xi;$$

and for every odd ineger q

$$\sum_{k=1}^{r} \sum_{j=0}^{\infty} \hat{\psi}_k(2^j\xi)\overline{\hat{\eta}_k(2^j(\xi+q))} = 0, \quad a.e.\, \xi.$$

The orthogonal wavelets frames constructed in [1] are based on the following theorems [4, 1].

Theorem 3.3 *(Unitary Extension Principle). Suppose* $\phi \in L_2(\mathbb{R})$ *is a refinable function with lowpass filter* $m_0(\xi)$, *which satisfies the following two conditions:*

$$lim_{\xi\to 0}\hat{\phi}(\xi) = 1;$$
$$\text{there exists } K > 0 \text{ such that} \sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi+k)|^2 \leq K, a.e.\, \xi.$$

Let $m_k \in L^\infty([0,1))$ $k = 1, 2\cdots, r$ *be such that the modulation matrix satisfies the equation*

$$M^*(\xi)M(\xi) = I_2.$$

Then the affine system generated by $\{\psi_k\}$ *where* $\widehat{\psi}_k(2\xi) = m_k(\xi)\widehat{\phi}(\xi)$, $k = 1\cdots r$ *is a Parseval wavelet frame for* $L_2(\mathbb{R})$.

In the following theorem taken from [1] we assume that all the filters $m_k(\xi), n_k(\xi)$ satisfy the conditions of the unitary extension principle.

Theorem 3.4 *Suppose* $\phi \in L_2(\mathbb{R})$ *is a refinable function with low pass filter* $m_0(\xi)$. *Let* $\mathcal{M} = \{m_0(\xi), m_1(\xi), \cdots, m_r(\xi)\}$ *and* $\mathcal{N} = \{n_0(\xi), n_1(\xi), \cdots, n_r(\xi)\}$ *be filters with* $m_0(\xi) = n_0(\xi)$. *Let* $M(\xi)$ *and* $N(\xi)$ *be the corresponding modulation matrices and* $\bar{M}(\xi)$ *and* $\bar{N}(\xi)$ *be same as* $M(\xi)$ *and* $N(\xi)$ *with the first row deleted from* $M(\xi)$ *and* $N(\xi)$ *respectively. Suppose*

1. $M^*(\xi)M(\xi) = I_2$ *for almost every* ξ
2. $N^*(\xi)N(\xi) = I_2$ *for almost every* ξ
3. $\bar{M}^*(\xi)\bar{N}(\xi) = 0,$ *for almost every* ξ.

Let $\hat{\psi}_k(2\xi) = m_k(\xi)\hat{\phi}(\xi)$ *and* $\hat{\eta}_k(2\xi) = n_k(\xi)\hat{\phi}(\xi)$, $1 \leq k \leq r$. *Then the affine system generated by* $\{\psi_1, \psi_2, \cdots, \psi_r\}$ *and* $\{\eta_1, \eta_2, \cdots, \eta_r\}$ *are orthogonal Parseval wavelet frames for* $L_2(\mathbb{R})$.

4 A New Construction Using Polyphase Matrix

A method for the construction of a orthogonal wavelet frames using Vaidyanathan's paraunitery matrices has been suggested in [1]. Their scheme is outlined as follows:

Theorem 4.1 *Let ϕ, ψ be a pair of scaling function and wavelet with filters $m_0(\xi)$ and $m_1(\xi)$. Let $A = \{a_{i,j}\}$ be any $r \times r$ paraunitary matrix with $1/2$ periodic trigonometric polynomial entries. Let A_i and A_j be the i^{th} and j^{th} columns of A;*

$$A_i(\xi) = \begin{pmatrix} a_{1,i}(\xi) \\ a_{2,i}(\xi) \\ \vdots \\ a_{r,i}(\xi) \end{pmatrix}, \quad A_j(\xi) = \begin{pmatrix} a_{1,j}(\xi) \\ a_{2,j}(\xi) \\ \vdots \\ a_{r,j}(\xi) \end{pmatrix}$$

Define

$$\widehat{\psi}^l_k(2\xi) = a_{k,l}(\xi)m_1(\xi)\widehat{\phi}(\xi), \;\; l = i, j, \;\; k = 1, 2 \cdots r.$$

Then the affine systems generated by ψ^i_k and ψ^j_k $k = 1, 2, \cdots r$, $i \neq j$ provide orthogonal Parseval wavelet frames for $L_2(\mathbb{R})$.

Here any two distinct columns provide a pair of orthogonal frames. In fact several pair of orthogonal frames can be obtained from this scheme. The orthogonality for the pair obtained is local in the sense that no cancelation occurs across the scale. The support of wavelets obtained grows as but remains compact. The starting wavelet system has to be orthogonal so that the modulation matrix satisfies the conditions of unitary extension principle. Because of the construction of paraunitary matrices, the systems $\{\psi^i_{k_1}\}$ and $\{\psi^j_{k_2}\}$ can have zero elements for some $k_1 \neq k_2$. The new frames depend on the entries of the paraunitary matrix considered, not on the submatrix.
A similar construction is provided in [6] using dual Gramian to prove the orthogonality where they start from a frame system instead of wavelet system. Their construction is given by the following theorem.

Theorem 4.2 *Let $\Psi = \{\psi_1, \psi_2, \cdots, \psi_r\}$ be a tight wavelet frame for $L_2(\mathbb{R})$ obtained from the unitary extension principle and let $V = \{a_{i,j}\}$ be any $2r \times 2r$ unitary matrix with $1/2$ periodic trigonometric polynomial entries. Let V_1 and V_2 denote the first r and last r columns of V. Let $M(\xi)$, $\widetilde{P}_1$ and $\widetilde{P}_2$ be the following matrices.*

$$M(\xi) = \begin{pmatrix} m_0(\xi) & m_0(\xi + 1/2) \\ \vdots & \vdots \\ m_r(\xi) & m_r(\xi + 1/2) \end{pmatrix}$$

$$\widetilde{P}_1 = \begin{pmatrix} 1 & 0 \\ 0 & V_1 \end{pmatrix}; \quad \widetilde{P}_2 = \begin{pmatrix} 1 & 0 \\ 0 & V_2 \end{pmatrix}$$

Let $P_1 = \widetilde{P}_1 M(\xi)$ and $P_2 = \widetilde{P}_2 M(\xi)$. Then the first columns of P_1 and P_2 provide the filters for a pair of orthogonal wavelet frames.

The frames obtained are still compactly supported. The starting matrix is a modulation matrix for a given wavelet system. One can start with a frame system as well and still satisfy the conditions of unitary extension principle. The new frames obtained depend on the submatrix of the chosen paraunitary matrix.

A new method similar to above but based on the polyphase component of the filters is presented. Let P denote the polyphase matrix of lowpass highpass filters of an MRA based wavelet system. We assume that the entries of P are trigonometric polynomials. Let $A = \{a_{i,j}\}$ be any $2r \times 2r$ paraunitary matrix. Let V_1 and V_2 denote the first r and last r columns of A. Let P_1 and P_2 be following matrices.

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & V_1 \end{pmatrix}; \quad P_2 = \begin{pmatrix} 1 & 0 \\ 0 & V_2 \end{pmatrix}$$

Let $\widetilde{P}_1 = P_1 P$ and $\widetilde{P}_2 = P_2 P$. Thus

$$\widetilde{P}_1(\xi) = \begin{pmatrix} m_{0,0}(\xi) & m_{0,1}(\xi) \\ \sum_{i=1}^{r} a_{1,i}(\xi) m_{i,0}(\xi) & \sum_{i=1}^{r} a_{1,i}(\xi) m_{i,1}(\xi) \\ \sum_{i=1}^{r} a_{2,i}(\xi) m_{i,0}(\xi) & \sum_{i=1}^{r} a_{2,i}(\xi) m_{i,1}(\xi) \\ \vdots & \vdots \\ \sum_{i=1}^{r} a_{2r,i}(\xi) m_{i,0}(\xi) & \sum_{i=1}^{r} a_{2r,i}(\xi) m_{i,1}(\xi) \end{pmatrix}.$$

We now obtain a pair of filters $\tilde{m}_k(\xi)$ and $\tilde{n}_k(\xi)$, $k = 1, 2, \cdots 2r$ as follows.

$$\begin{aligned} \tilde{m}_k(\xi) &= \frac{1}{\sqrt{2}} \left([\widetilde{P}_1(2\xi)]_{k,1} + e^{-2\pi i \xi} [\widetilde{P}_1(2\xi)]_{k,2} \right) \\ &= \frac{1}{\sqrt{2}} \left(\sum_{i=1}^{r} a_{k,i}(2\xi) m_{i,0}(\xi) + e^{-2\pi i \xi} \sum_{i=1}^{r} a_{k,i}(2\xi) m_{i,1}(2\xi) \right) \\ &= \sum_{i=1}^{r} a_{k,i}(2\xi) \frac{1}{\sqrt{2}} \left(m_{i,0}(2\xi) + e^{-2\pi i \xi} m_{i,1}(2\xi) \right) \\ &= \sum_{i=1}^{r} a_{k,i}(2\xi) m_i(\xi). \end{aligned} \tag{4.1}$$

Similarly let

$$\begin{aligned}\tilde{n}_k(\xi) &= \frac{1}{\sqrt{2}}[\widetilde{P}_2(2\xi)]_{k,1} + e^{-2\pi i\xi}[\widetilde{P}_2(2\xi)]_{k,2} \\ &= \sum_{i=1}^{r} a_{k,r+i}(2\xi)m_i(\xi). \end{aligned} \tag{4.2}$$

So we have the following theorem:

Theorem 4.3 *Let $\tilde{m}_k(\xi)$ and $\tilde{n}_k(\xi)$ be as given by* (4.1) *and* (4.2)

$$\widehat{\tilde{\psi}}_k(2\xi) = \tilde{m}_k(\xi)\widehat{\phi}(\xi),$$
$$\widehat{\tilde{\eta}}_k(2\xi) = \tilde{n}_k(\xi)\widehat{\phi}(\xi)$$

for $k = 1, 2, \cdots, 2r$. Then $\tilde{\psi}_k$ and $\tilde{\eta}_k$ constitute a pair of orthogonal Parseval wavelet frames for $L_2(\mathbb{R})$.

Proof: Since $\widetilde{P}_1^*\widetilde{P}_2 = P^*P_1^*P_2P = I$, the unitary extension principle applied to the matrices $\widetilde{P}_1$ and $\widetilde{P}_2$ together with theorem 2.1 implies that the above frames are Parseval wavelet frames. To prove the orthogonality, let A_1 and A_2 be the matrices obtained from deleting the first rows of $\widetilde{P}_1$ and $\widetilde{P}_2$ respectively. Then it is clear that $A_1^*A_2 = 0$. So the orthogonality follows from theorem 3.4.

The paraunitary matrix doesn't have to be 1/2 periodic as required in [1]. Consider the following polyphase matrix for the frame taken from [9].

$$P(\xi) = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/2 & -1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

Shown below are the frequency responses of a pair of orthogonal wavelet frames that are obtained from the above polyphase filters using a randomly generated paraunitary matrix.

These constructions can be extended to higher dimensions in a very straight forward way. One can start with the polyphase matrix of filters of frames as obtained from the unitary extension principle instead of wavelet system since its the paraunitary matrix that makes them orthogonal. More than one Vaidyanathan's paraunitary matrices can be used, however it is not very clear whether it will improve the approximation order or not. The frequency responses of the filters obtained are all different, unlike in one of the schemes mentioned above there was a possible repetition. A new scheme using paraunitary symmetric matrices has been suggested in [7].

Acknowledgements

The author would like to thank Fritz Keinert for his helpful suggestions and support.

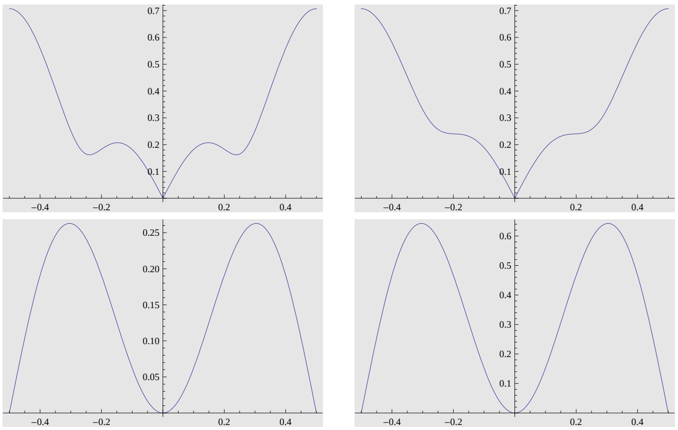

Table 1: Frequency response of high-pass filters

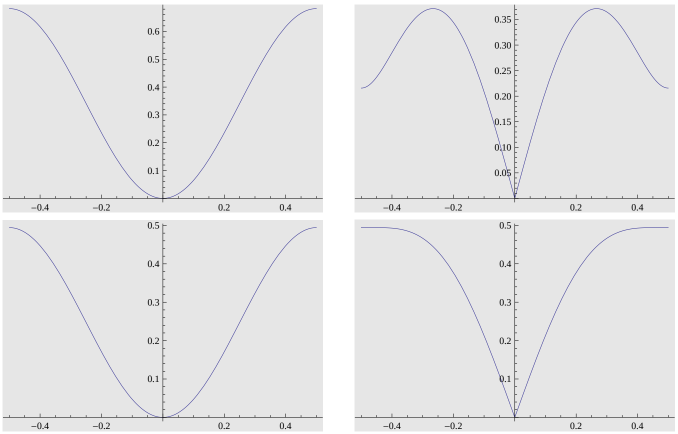

Table 2: Frequency response of the corresponding orthogonal pair

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Received: March, 2011