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(0;0,1) Interpolation On The Unit Circle

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Abstract

In this paper, we consider explicit forms and convergence of Pál-type $(0;0,1)$ -interpolation on two disjoint set of nodes , which are obtained by projecting vertically the zeros of $(1-x^2)\,P_n\,(x)$ and $P_n'\,(x)$ repectively on the unit circle, where $P_n\,(x)$ stands for n^{th} Legendre polynomial.

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1. Introduction

In a paper, T.N.T.Goodman and A.Sharma [5] considered convergnce and divergence behaviour of Hermite interpolation in the circle of radius $\rho^{\frac{3}{2}}$. Also T.N.T. Goodman, K.G. Ivanov and A. Sharma [6] considered the behaviour of the Hermite interpolant in the roots of unity. In 1998, author [1] proved the convergence of quasi- Hermite interpolation on the nodes obtained by projecting vertically the zeros of $(1-x^2)\,P_n\,(x)$ on the unit circle. Later on convergence of Hermite interpolation is considered by [8] on the same set of nodes.. Recently E. Berriochoaa, , A. Cachafeirob, , and E. Martínez Breyb [3]studied the convergence of the Hermite–Fejér and the Hermite interpolation polynomials, which are constructed by taking equally spaced nodes on the unit circle and concerned with the behaviour outside and inside the unit circle, when they considered analytic functions on a suitable domain. As a consequence, they achieved some improvements on Hermite interpolation problems on the real line. In 2003, H. P. Dikshit [4] studied the regularity of certain Pál-type interpolation problems involving the Möbius transforms of the zeros of $z^n + 1$and $z^n - 1$ along with one additional point ξ or two additional points ξ and 1. Author [2] has also considered $(0, 1; 0)$ -interpolation on the unit circle

In this paper , we consider two disjoint set of the nodes. The Lagrange data is prescribed on the one set of the nodes ,which are obtained by projecting vertically the zeros of $(1 - x^2)\, P_n\,(x)$ on the unit circle, whereas the Hermite data is prescribed on the other set of nodes, which are vertically projected zeros of $P_n'(x)$ on the unit circle, where $P_n\,(x)$ stands for n^{th} Legendre polynomial.

Let $\quad Z_n = \{z_k : k = 0\,(1)\,2n+1\}$ satisfying

(1.1) $\quad z_0 = 1,\ z_{2n+1} = -1,\ z_k = \cos\theta_k + i\sin\theta_k,\ z_{n+k} = -z_k, \quad k = 1\,(1)\,n$

and $\quad T_n = \{t_k : k = 1\,(1)\,2n-2\}$ such that

(1.2) $\quad t_k = \cos\phi_k + i\sin\phi_k,\ t_{n+k} = -t_k, \quad k = 1\,(1)\,n-1.$

In section 2 we give some preliminaries and describe the problem, in section, we give the explicit formulae of the interpolatory polynomials.In sections 4 & 5 estimation of interpolatory polynomials and convergence problem are given.

2. Preliminaries

In this section , we shall give some well-known results, which we shall use. The differential equation satisfied by $P_n\,(x)$ is given by

(2.1) $$(1 - x^2)\, P_n''\,(x) - 2xP_n'\,(x) + n\,(n+1)\, P_n\,(x) = 0.$$

(2.2) $$W\,(z) = \prod_{k=1}^{2n} (z - z_k) = K_n P_n \left(\tfrac{1+z^2}{2z}\right) z^n.$$

(2.3) $$H\,(z) = \prod_{k=1}^{2n-2} (z - t_k) = K_n^* P_n' \left(\tfrac{1+z^2}{2z}\right) z^{n-1}$$

We shall require the fundamental polynomials of Lagrange interpolation on Z_n and T_n , *i.e*

(2.4) $$L_k\,(z) = \tfrac{W(z)}{(z-z_k)W'(z_k)}.$$

(2.5) $$l_k\,(z) = \tfrac{H(z)}{(z-t_k)H'(t_k)}.$$

For $-1 \leq x \leq 1$.

(2.6) $$|P_n(x)| \leq 1,$$

(2.7) $$(1-x^2)^{\frac{1}{4}} |P_n(x)| \leq \left(\frac{2}{\pi n}\right)^{\frac{1}{2}}.$$

For more details, see[9, 10].

The Problem: Let $\{z_k\}_{k=0}^{2n+1}$ and $\{t_k\}_{k=1}^{2n-2}$ be the disjoint set of nodes obtained by projecting vertically the zeros of the $(1-x^2)P_n(x)$ and $P_n'(x)$ on the unit circle respectively. we determine the interpolatory polynomials $R_n(z)$ of degree $\leq 6n-3$ satisfying the conditions:

(2.8) $$R_n(z_k) = \alpha_k,\ k = 0\,(1)\,2n+1;$$

For $k = 1\,(1)\,2n-2$

(2.9) $$\begin{cases} R_n(t_k) = \beta_k, \\ R_n'(t_k) = \gamma_k\ . \end{cases}$$

where α_k, β_k and γ_k are complex constants.

3. Explicit representation of interpolatory polynomials:

We shall write $R_n(z)$ satisfying (2.8) and (2.9) as

(3.1) $$R_n(z) = \sum_{k=0}^{2n+1} \alpha_k A_k(z) + \sum_{k=1}^{2n-2} \beta_k B_k(z) + \sum_{k=1}^{2n-2} \gamma_k C_k(z)$$

where $A_k(z)$, $B_k(z)$ and $C_k(z)$ are unique polynomials each of degree $\leq 6n-3$ determined by the following conditions:

For $k = 0\,(1)\,2n+1$

(3.2) $$\begin{cases} A_k(z_j) = \delta_{kj}, & j = 0\,(1)\,2n+1, \\ A_k(t_j) = 0, & j = 1\,(1)\,2n-2, \\ A'(t_j) = 0, & j = 1\,(1)\,2n-2. \end{cases}$$

for $k = 1\,(1)\,2n-2$

(3.3) $$\begin{cases} B_k(z_j) = 0, & j = 0\,(1)\,2n+1, \\ B_k(t_j) = \delta_{kj}, & j = 1\,(1)\,2n-2, \\ B_k'(t_j) = 0, & j = 1\,(1)\,2n-2. \end{cases}$$

$$(3.4)\qquad \begin{cases} C_k(z_j) = 0, & j = 0\,(1)\,2n+1; \\ C_k(t_j) = 0, & j = 1\,(1)\,2n-2, \\ C'_k(t_j) = \delta_{kj}, & j = 1\,(1)\,2n-2. \end{cases}$$

Theorem 1: For $k = 1\,(1)\,2n-2$,

$$(3.5)\qquad C_k(z) = \frac{(z^2-1)W(z)H(z)}{(t_k^2-1)W(t_k)H'(t_k)} l_k(z).$$

Theorem 2: For $k = 1\,(1)\,2n-2$, we have

$$(3.6)\qquad B_k(z) = \frac{(z^2-1)W(z)}{(z_k^2-1)W(t_k)} l_k^2(z) - \left\{\frac{R'(t_k)}{R(t_k)} + 2l'_k(t_k)\right\} \frac{(z^2-1)W(z)H(z)}{(t_k^2-1)W(t_k)H'(t_k)}$$

where $\quad R(z) = (z^2-1)\,W(z)$

Theorem For $k = 1\,(1)\,2n$, we have

$$(3.7)\qquad A_k(z) = \frac{(z^2-1)H^2(z)}{(z_k^2-1)H^2(z_k)} L_k(z).$$

Further we have

$$(3.8)\qquad A_0(z) = \frac{(1+z)W(z)H^2(z)}{2W(1)H^2(1)},$$

$$(3.9)\qquad A_{2n+1}(z) = \frac{(1-z)W(z)H^2(z)}{2W(1)H^2(1)},$$

One can easily prove the theorems 1, 2 & 3 owing to conditions (3.2), (3.3) and (3.4) respectively.

4. Estimation of fundamental polynomials

Lemma 1: For $|z| \le 1$, we have

$$(4.1)\qquad \sum_{k=1}^{2n} |\, A_k(z)| \le c \log n.$$

$$(4.2)\qquad |A_0(z)| \le c, \quad |A_{2n+1}(z)| \le c$$

Lemma 2: For $|z| \le 1$, we have

$$(4.3) \qquad \sum_{k=1}^{2n-2} |B_k(z)| \le c \log n.$$

Lemma 3: For $|z| \le 1$, we have

$$(4.4) \qquad \sum_{k=1}^{2n-2} |C_k(z)| \le c \frac{\log n.}{n}$$

For proof see lemma 1 and inequalities (4.5) & (4.6) in [10] and using $(2.1) - (2.7)$. After a little computation , lemmas 1 , 2 & 3 follows.

5. Convergence

Let $f(z)$ be analytic for $|z|<1$ and continuous for $|z| \le 1$ and $\omega(f, \delta)$ be the modulus of continuity of $f(e^{ix})$. The function $f(z)$ is said to belong to the Dini-Lipschitz class *, if*

$$(5.1) \qquad \omega(f, \delta) \log \tfrac{1}{\delta} \longrightarrow 0 \text{ as } \delta \longrightarrow 0.$$

Theorem 4: Let $f(z) \in$ and

$$(5.3) \qquad |\gamma_k| = O\left(\frac{n}{\log n}\right),$$

then the sequence of interpolatory polynomials $R_n(z)$ defined in (3.1) converges uniformly to $f(z)$ on $|z| \le 1$ as $n \longrightarrow \infty$.

Proof: Using Jackson polynomials and ideas in O.Kiš [7] , the theorem follows.

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