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A Slacks-Based Measure of Efficiency in Two-Stage Data Envelopment Analysis

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Abstract

As a non-parametric technique, Data Envelopment Analysis (DEA) evaluates the relative efficiency of peer decision making units (DMUs) that have multiple inputs and outputs. In many situations, DMUs have a two-stage structure, where

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the first stage uses inputs to produce outputs that then become the inputs to the second stage. The first stage outputs are called "intermediate products." In recent years, a great number of DEA studies have focused on developing the two-stage processes, so that researchers, by using radial DEA models, have proposed new models for evaluating the efficiency of two-stage processes. However, because of the existence of intermediate measures, these models do not provide comprehensive information about the efficiency frontier and the projections of inefficient DMUs. This paper introduces a non-radial model in the slacks-based measure (SBM) framework for two-stage production processes that considers the series relationship between two stages. Unlike the radial two-stage DEA models, the new approach enables us to determine efficient projections for inefficient DMUs within the framework of two-stage DEA. An example from the literature is applied to clarify the model.

Keywords: Data envelopment analysis, Decision making units, Efficiency, Two-stage, Slacks-based measure

1. Introduction

Data envelopment analysis (DEA) is a mathematical programming approach for measuring the relative efficiency of a set of production systems, or decision making units (DMUs), with multiple inputs and outputs. DEA was first introduced by Charnes et al. [4] in 1978. In many situations, DMUs have a two-stage structure, where the first stage uses inputs to produce outputs that then become the inputs to the second stage. Thus, the second stage utilizes these first stage outputs to produce its own outputs. We refer to the first stage outputs as intermediate products and assume that they are the only inputs to the second stage. A usual way to deal with this type of production system is to apply the conventional DEA model to measure the efficiency of the whole production process and the two sub-processes independently (see, e.g., Seiford and Zhu [11], Zhu [14] and Sexton and Lewis [12]). One of the drawbacks of these models is that they ignore the impact of the intermediate products on the overall efficiency of the system as a whole, so that may conclude that two inefficient stages lead to an overall efficient DMU with the inputs of the first stage and outputs of the second stage. Also, due to the existence of intermediate products, the usual procedure of adjusting the inputs or outputs by the efficiency scores for an inefficient DMU does not necessarily yield an efficient DMU. For example, suppose the first stage is efficient but the second stage is not. When the second stage, by decreasing the intermediate products via an input-oriented DEA model, improves its performance, the decreased intermediate products may transform the first stage inefficient. In recent years, a number of researchers have developed two-stage DEA models for overcoming these drawbacks. For example, Chen and Zhu [7] developed a linear DEA type model where each stage's efficiency is defined on its own production possibility set. Kao and Hwang [9] developed a different approach where the overall efficiency of the system can be decomposed into the product of the efficiencies of the two-stages. Chen et al. [5] presented a model similar to the Kao and Hwang's model, but in an additive form. Chen et al. [6], by modifying Kao and Hwang's approach, developed an approach for determining the projections of inefficient DMUs within the framework of two-stage processes. These studies apply the radial DEA models for measuring the efficiencies of each stage and whole system.

The aim of this paper is to introduce a non-radial two-stage DEA model in slacks based measure (SBM) formulation for measuring the overall efficiency of two-stage production processes by considering the series relationship between two stages. We will demonstrate that unlike the radial two-stage DEA models, in this approach, the projected DMUs for inefficient DMUs are efficient. The data set consists of 27 firms in the banking industry in US studied by Chen and Zhu [7], is used to illustrate the new approach.

The rest of this paper is organized as follows. Section 2 contains some preliminaries. In Section 3, we propose a SBM model on two-stage processes. Section 4 applies the new approach to the 27 firms in the banking industry in US. Finally, conclusions are provided in the last section.

2. Preliminaries

There are two types of measure in DEA, radial and non-radial, which can evaluate the efficiency of DMUs. Radial models assume proportional change of inputs or outputs and usually disregard the existence of slacks in the efficiency scores. For the first time the radial DEA model was proposed by Charnes, Cooper and Rhodes (CCR) (Charnes et al. [4]) and later extended by Banker, Charnes, and Cooper (BCC) (Banker et al. [2]). In other side, there are non-radial models which regard the slacks of each input or output and the variations of inputs and outputs are not proportional; in other words in non-radial models the inputs/outputs are allowed to decrease/increase at different rates. The non-radial DEA model was first proposed by Färe and Lovell [8], which called the Russell measure and later adapted by Pastor et al. [10]. A new non-radial model by the name of slacks-based measure (SBM) was developed by Tone [13] in 2001. SBM model directly works with input excess and output shortfall slacks, and integrates them into an efficiency measure. In the recent years, it has been commonly used to evaluate efficiencies of production systems.

Suppose there are n DMUs and each DMU$_j$ (j=1,…,n), uses m inputs $x_{ij}\ (i=1,\dots,m)$ to produce s outputs $y_{rj}\ (r=1,\dots,s)$. It is assumed that all inputs and outputs are positive. The SBM model, as a non-oriented and non-radial DEA model, to evaluate the efficiency of DMU$_o$ $(o \in \{1,...,n\})$ under the assumption of constant returns to scale (CRS) is defined as follows:

$$\rho^* = \min \rho = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} \frac{s_i^-}{x_{io}}}{1 + \frac{1}{s}\sum_{r=1}^{s} \frac{s_r^+}{y_{ro}}}$$

$$\begin{aligned} \text{s.t. } & x_{io} = \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^-, i = 1,...,m, \\ & y_{ro} = \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+, r = 1,...,s, \\ & \lambda, s^-, s^+ \geq 0, \end{aligned} \tag{1}$$

where $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \Re^n$ is the intensity vector. Also non-negative vectors $s^- = (s_1^-, ..., s_m^-) \in \Re^m$ and $s^+ = (s_1^+, ..., s_s^+) \in \Re^s$ indicate the input excess and output shortfall slacks, respectively. The optimal solution of ρ^* is the SBM efficiency score. It can be obviously identified that $0 < \rho^* \leq 1$ and supports the properties of units invariance and monotone.
Suppose an optimal solution for model (1) be $(\rho^*, \lambda^*, s^{-*}, s^{+*})$.

Definition 1 (*SBM-efficiency*). *A DMU$_o$* is *SBM-efficient* if and only if $\rho^* = 1$.

This is equivalent to $s^{-*} = s^{+*} = 0$. It means that there are no input excesses and output shortfalls in any optimal solution.
Note that the SBM model under the assumption of variable returns to scale (VRS) can be expressed by adding the convexity constraint, namely $\sum_{j=1}^{n} \lambda_j = 1$, into the model (1).

3. A SBM model for two-stage processes

Consider a manufacturing process composed of a two-stage process as shown in Fig. 1.

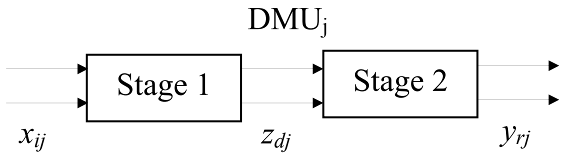

Fig. 1. Two stage production process

Suppose we have n DMUs, that each DMU_j (j=1,…,n) has m inputs x_{ij} (i=1,...,m) to the first stage and p outputs z_{dj} (d=1,…,p) from that stage. These p outputs then become the inputs to the second stage, and are referred to as intermediate products. The outputs from the second stage are y_{rj} (r=1,…,s).

In an effort to estimate the overall efficiency of DMU$_o$ $(o \in \{1,...,n\})$ under the assumption of CRS by taking into account the operation of two sub-processes, first we apply the SBM model for the two individual stages. The efficiency scores of the first and second stages can be calculated by the following two models, respectively:

$$\begin{aligned} \rho_1^* = \min\ & \frac{1-\frac{1}{m}\sum_{i=1}^{m}\frac{s_i^-}{x_{io}}}{1+\frac{1}{p}\sum_{d=1}^{p}\frac{s_d^+}{z_{do}}} \\ \text{s.t.}\ & x_{io} = \sum_{j=1}^{n}\lambda_j x_{ij} + s_i^-,\ i=1,...,m, \\ & z_{do} = \sum_{j=1}^{n}\lambda_j z_{dj} - s_d^+,\ d=1,...,p, \\ & \lambda, s^-, s^+ \geq 0. \end{aligned} \tag{2}$$

and

$$\begin{aligned} \rho_2^* = \min\ & \frac{1-\frac{1}{p}\sum_{d=1}^{p}\frac{s_d^-}{z_{do}}}{1+\frac{1}{S}\sum_{r=1}^{s}\frac{s_r^+}{y_{ro}}} \\ \text{s.t.}\ & z_{do} = \sum_{j=1}^{n}\mu_j z_{dj} + s_d^-,\ d=1,...,p, \\ & y_{ro} = \sum_{j=1}^{n}\mu_j y_{rj} - s_r^+,\ r=1,...,s, \\ & \mu, s^-, s^+ \geq 0. \end{aligned} \tag{3}$$

To connect two sub-processes as a whole process, a model must describe this series relationship between two stages. Since, z_{dj} $(d = 1,...,p)$, the outputs of stage 1 are the inputs of stage 2, the following constraints guarantee the continuity of two stages:

$$\sum_{j=1}^{n} \lambda_j z_{dj} = \sum_{j=1}^{n} \mu_j z_{dj}, \; d = 1,2,..., p. \tag{4}$$

Using these constraints, we can propose the following programming problem for measuring the overall efficiency of DMU_o.

$$\begin{aligned} \rho^*_{\text{overall}} = \min \; & \frac{1 - \frac{1}{m}\sum_{i=1}^{m} \frac{s_i^-}{x_{io}}}{1 + \frac{1}{s}\sum_{r=1}^{s} \frac{s_r^+}{y_{ro}}} \\ \text{s.t.} \;\; & x_{io} = \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^-, \; i = 1,...,m, \\ & y_{ro} = \sum_{j=1}^{n} \mu_j y_{rj} - s_r^+, \; r = 1,...,s, \\ & \sum_{j=1}^{n} \lambda_j z_{dj} = \sum_{j=1}^{n} \mu_j z_{dj}, \; d = 1,..., p, \\ & \lambda, \mu, s^-, s^+ \geq 0. \end{aligned} \tag{5}$$

By using the model (5), we are able to measure the overall efficiency of DMU_o by considering the operation of two sub-processes.
Note that model (5) is a nonlinear programming problem that can be converted into a linear programming problem by using the Cooper et al. transformation (see [3] for more details).

Theorem 1: A DMU_o is overall efficient if and only if it is efficient for each two stages.

Proof. The proof is clear and, hence, omitted. □

Let an optimal solution of model (5) be $(\lambda^*, \mu^*, s^{-*}, s^{+*})$. Then we have the projection onto frontier as follows:

$$\begin{aligned} & x^*_{io} \leftarrow x_{io} - s_i^{-*}, \; i = 1,...,m, \\ & y^*_{ro} \leftarrow y_{ro} + s_r^{+*}, \; r = 1,...,s, \\ & z^*_{do} \leftarrow (\sum_{j=1}^{n} \lambda^*_j z_{dj} = \sum_{j=1}^{n} \mu^*_j z_{dj}), \; d = 1,..., p. \end{aligned} \tag{6}$$

As indicated by Chen et al. [6], in the two-stage DEA models, due to the existence of intermediate products, the projected DMU for an inefficient DMU is normally not sufficient to yield a frontier projection. We will show that under the new model the projection of an inefficient DMU is efficient.

Theorem 2: *The projected DMU$_o$ defined by (6) is overall efficient*
Proof. Suppose that an optimal solution of the projected DMU$_o$ calculated by model (5) is $(\lambda', \mu', s'^-, s'^+)$. Then, we have

$$
\begin{aligned}
x_{io}^* &= \sum_{j=1}^{n} \lambda'_j x_{ij} + s'^-_i, \ i = 1,\ldots,m, \\
y_{ro}^* &= \sum_{j=1}^{n} \mu'_j y_{rj} - s'^+_r, \ r = 1,\ldots,s.
\end{aligned}
\tag{7}
$$

From (6), by inserting amounts of x_{io}^* and y_{ro}^*, we have

$$
\begin{aligned}
x_{io} &= \sum_{j=1}^{n} \lambda'_j x_{ij} + s'^-_i + s_i^{-*}, \ i = 1,\ldots,m, \\
y_{ro} &= \sum_{j=1}^{n} \mu'_j y_{rj} - s'^+_r - s_r^{+*}, \ r = 1,\ldots,s.
\end{aligned}
\tag{8}
$$

Corresponding to these restrictions, the overall efficiency score of DMU$_o$ can be expressed as:

$$
\rho'_{\text{overall}} = \min \frac{1 - \dfrac{1}{m}\displaystyle\sum_{i=1}^{m} \dfrac{s_i^{-*} + s'^-_i}{x_{io}}}{1 + \dfrac{1}{s}\displaystyle\sum_{r=1}^{s} \dfrac{s_r^{+*} + s'^+_r}{y_{ro}}}. \tag{9}
$$

If any s'^-_i or s'^+_r be positive, then we have $\rho'_{\text{overall}} < \rho^*_{\text{overall}}$. This is a contradiction, since ρ^*_{overall} is optimal. Therefore, we have $s'^-_i = s'^+_r = 0; \forall i, r$. Hence, the projected DMU is overall efficient. □

4. Numerical example

In this section, the new approach is applied to the 27 firms in the banking industry in US as studied by Chen and Zhu [7]. They divided the production process of the banking industry into two stages: fund collection and profit generation. The inputs to the first stage are IT investment (x_1), fixed assets (x_2) and the number of employees (x_3); and the outputs from the second stage are profit (y_1) and fraction of loans recovered (y_2). There is also an intermediate product between the two stages, namely the deposits generated (z).

The efficiency of the first stage evaluates the performance in collecting fund of the firm while the efficiency of the second stage measures the performance in generating profit. Table 1 reports the inputs, intermediate products and outputs of the 27 firms in the banking industry in US.

By applying the models (2), (3) and (5), the efficiencies of the first stage, second stage and the whole process of the 27 firms are calculated. The results are shown in Table 2. It can be seen that except DMU18, all of the DMUs are inefficient in at least one of stages. Hence, according to Theorem 1, under the two-stage SBM model only DMU18 performs efficiently for the whole process.

Table 1
Data set

DMU	x_1 ($ billions)	x_2 ($ billions)	x_3 (thousands)	z ($ billions)	y_1 ($ billions)	y_2
1	0.150	0.713	13.3	14.478	0.232	0.986
2	0.170	1.071	16.9	19.502	0.340	0.986
3	0.235	1.224	24.0	20.952	0.363	0.986
4	0.211	0.363	15.6	13.902	0.211	0.982
5	0.133	0.409	18.485	15.206	0.237	0.984
6	0.497	5.846	56.42	81.186	1.103	0.955
7	0.060	0.918	56.42	81.186	1.103	0.986
8	0.071	1.235	12.0	11.441	0.199	0.985
9	1.500	18.120	89.51	124.072	1.858	0.972
10	0.120	1.821	19.8	17.425	0.274	0.983
11	0.120	1.915	19.8	17.425	0.274	0.983
12	0.050	0.874	13.1	14.342	0.177	0.985
13	0.370	6.918	12.5	32.491	0.648	0.945
14	0.440	4.432	41.9	47.653	0.639	0.979
15	0.431	4.504	41.1	52.630	0.741	0.981
16	0.110	1.241	14.4	17.493	0.243	0.988
17	0.053	0.450	7.6	9.512	0.067	0.980
18	0.345	5.892	15.5	42.469	1.002	0.948
19	0.128	0.973	12.6	18.987	0.243	0.985
20	0.055	0.444	5.9	7.546	0.153	0.987
21	0.057	0.508	5.7	7.595	0.123	0.987
22	0.098	0.370	14.1	16.906	0.233	0.981
23	0.104	0.395	14.6	17.264	0.263	0.983
24	0.206	2.680	19.6	36.430	0.601	0.982
25	0.067	0.781	10.5	11.581	0.120	0.987
26	0.100	0.872	12.1	22.207	0.248	0.972
27	0.0106	1.757	12.7	20.670	0.253	0.988

If the operations of sub-processes are not taken into account, i.e., the overall efficiency is calculated using the conventional SBM model with fixed assets, number of employees and IT investment as the inputs and profits and fraction of loans recovered as the outputs and ignore the intermediate measure, the efficiency scores of 27 DMUs are displayed in the right hand side of the Table 2 under the heading ρ^*. It also can be seen that by using the conventional SBM model, DMUs 4, 7, 17, 20, 21, 22, 23 and 27 are overall efficient while these DMUs are inefficient in at least one of stages. Especially DMUs 17, 21, 22 and 23 are inefficient in both stages but their overall efficiency scores are equal to one. As a result, the overall efficiency calculated from our model is more meaningful than that calculated from the conventional SBM model.

Finally, if we obtain the projection DMUs for all inefficient DMUs, we can see that the overall efficiency scores, calculated by model (5), for all new DMUs are equal to one. This is means that, under our model, the projections of inefficient DMUs are efficient.

Table 2
Result

DMU	First stage	Second stage	System	
	ρ_1^*	ρ_2^*	ρ_{overall}^*	ρ^*
1	0.352	0.628	0.221	0.575
2	0.364	0.533	0.194	0.500
3	0.289	0.506	0.146	0.395
4	0.367	0.627	0.230	1
5	0.359	0.602	0.216	0.780
6	0.426	0.158	0.075	0.166
7	1	0.163	0.163	1
8	0.295	0.745	0.220	0.612
9	0.367	0.111	0.041	0.077
10	0.276	0.554	0.153	0.427
11	0.274	0.554	0.152	0.422
12	0.386	0.564	0.218	0.706
13	0.760	0.363	0.274	0.619
14	0.331	0.254	0.084	0.196
15	0.371	0.236	0.088	0.203
16	0.374	0.530	0.198	0.519
17	0.414	0.482	0.120	1
18	1	1	1	1
19	0.520	0.447	0.224	0.553
20	0.394	1	0.394	1
21	0.398	0.886	0.352	1
22	0.492	0.537	0.264	1
23	0.480	0.551	0.264	1
24	0.802	0.329	0.192	0.484
25	0.354	0.573	0.203	0.556
26	1	0.416	0.240	0.632
27	1	0.455	0.410	1

5. Conclusions

In the real world there are production systems which are composed of two stages. In DEA literature, several studies have focused on the two-stage production systems. These studies apply the radial DEA models for measuring the efficiencies of each stage and whole system. In this paper, we have proposed a non-radial two-stage DEA model in slacks-based measure (SBM) formulation to evaluate the overall efficiency of DMUs, by taking into account the series relationship between two-stages. The proposed model deals with slacks of each input and output and is suitable for measuring the efficiency in the situations that inputs and outputs may change non-proportionally. Also, this approach enables us to determine the frontier points for the inefficient DMUs within the framework of two-stage processes.

In the present paper, the proposed model is based on the assumption of constant returns to scale (CRS). By adding the convexity constraint into the models, the discussion can be expanded to variable returns to scale (VRS) assumption.

There are manufacturing processes which are composed of more than two sub-processes (e.g., Amado [12]). Therefore our approach can be extended easily to systems of multiple stages connected in series.

References

[1] C.F. Amado, Exploring the use of data envelopment analysis for evaluation in primary health care, Ph.D. thesis, The University of Warwick, 2003.

[2] R.D. Banker, A. Charnes, and W.W. Cooper, Some methods for estimating technical and scale efficiencies in DEA, Management Science, 30 (1984), 1078–1092.

[3] A. Charnes and W.W. Cooper, Programming with linear fractional functional, Naval Research Logistics Quarterly, 15 (1962), 333–334.

[4] A. Charnes, W.W. Cooper and E. Rhodes, Measuring the efficiency of decision making units, European Journal of Operational Research, 2 (1978), 429–444.

[5] Y. Chen, W.D. Cook, N. Li, and J. Zhu, Additive efficiency decomposition in two stage DEA, European Journal of Operational Research, 196 (2009), 1170–1176.

[6] Y. Chen, W.D. Cook, and J. Zhu, Deriving the DEA Frontier for Two-Stage Processes, European Journal of Operational Research, 202 (2010), 138–42.

[7] Y. Chen and J. Zhu, Measuring information technology's indirect impact on firm performance, Information Technology and Management journal ,5 (1–2) (2004), 9–22.

[8] R. Färe, and C.A. K. Lovell, Measuring the Technical Efficiency of Product-ion, Journal of Economic Theory, 19 (1978) 150-162.

[9] C. Kao and S-N. Hwang, Efficiency decomposition in two-stage data envelopment analysis: An application to non-life insurance companies in Taiwan, European Journal of Operational Research, 185 (1) (2008), 418–429.

[10] J.T. Pastor, J.L. Ruiz, and I. Sirvent, An Enhanced DEA Russell-Graph Efficiency Measure, European Journal of Operational Research, 115 (1999) 596-607.

[11] L.M. Seiford and J. Zhu, Profitability and marketability of the top 55 US commercial banks, Management Science, 45 (9) (1999), 1270–1288.

[12] T.R. Sexton and H.F. Lewis, Two-stage DEA: An application to major league baseball, Journal of Productivity Analysis, 19 (2003), 227–249.

[13] K. Tone, A slacks-based measure of efficiency in data envelopment analysis, European Journal of Operational Research, 130 (2001), 498–509.

[14] J. Zhu, Multi-factor performance measure model with an application to Fortune 500 companies, European Journal of Operational, 123(1) (2000), 105-124.

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