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On Atomic Decompositions in Banach Spaces

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Abstract

Atomic decompositions have been studied and some sufficient conditions for their existence have been obtained. It has been proved that an associated Banach space for an atomic decomposition always has a complemented subspace. Further, it has been shown that every Banach space E with $\dim E < \infty$ possesses an (infinite) atomic decomposition. Finally. it has been proved that if two Banach spaces have atomic decompositions, then their product space also has an atomic decomposition.

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1 Introduction

Frames are main tools for use in signal and image processing, compression, sampling theory, optics, filter banks, signal detection etc. In order to have many more uses of frames, several notions generalizing the concept of frames have been introduced and studied, namely; Banach frames [6], pseudo frames [12], oblique frames [3], frames of subspaces (fusion frames) [1,2], G-frames [13] etc.

Coifman and Weiss [4] introduced a concept, similar to that of frames, called atomic decompositions for function spaces. Later, Feichtinger and

Gröchenig [5] extended this notion to Banach spaces. Gröchenig [6] further generalized this concept and introduced Banach frames. But, atomic decompositions for a Banach space have advantage over Banach frames as an atomic decomposition allows every element of the space to be expressed as a linear combination of elements of the atomic decomposition in a stable manner. Thus, the elements of an atomic decomposition for a Banach space serve as building blocks for the space, where the computation of coefficients of an element of the space is convenient and easier.

In the present paper, we shall study atomic decompositions in Banach spaces and obtain conditions for the existence of atomic decompositions for a Banach space. Also, it has been proved that an associated Banach space for an atomic decomposition always has a complemented subspace. Further, it has been shown that every Banach space E with $\dim E < \infty$ possesses an (infinite) atomic decomposition. Finally, it has been proved that if two Banach spaces have atomic decompositions, then their product space also has an atomic decomposition.

2 Preliminaries

Throughout the paper, E will denote an infinite dimensional Banach space over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), E^* the first conjugate space of E, E_d an associated Banach space of scalar-valued sequences indexed by $\mathbb{N}, [f_n]$ the closed linear span of $\{f_n\}$ and $\widetilde{[f_n]}$ the closed linear span of $\{f_n\}$ in the $\sigma(E^*, E)$-topology.

A sequence $\{f_n\} \subset E^*$ is said to be complete if $[f_n] = E^*$ and total over E if $\{x \in E : f_n(x) = 0, n \in \mathbb{N}\} = \{0\}$.

Definition 2.1. *Let E be a Banach and let E_d be an associated Banach space of scalar-valued sequences indexed by $\mathbb{N}$. Let $\{x_n\}$ be a sequence in E and let $\{f_n\}$ be a sequence in E^*. Then, the pair $(\{f_n\}, \{x_n\})$ is called an atomic decomposition for E with respect to E_d, if*

(a) $\{f_n(x)\} \in E_d$, *for all* $x \in E$

(b) *there exist constants* A, B *with* $0 < A \leqslant B < \infty$ *such that*

$$A\|x\|_E \leqslant \|\{f_n(x)\}\|_{E_d} \leqslant B\|x\|_E, \quad x \in E$$

(c) $x = \sum_{n=1}^{\infty} f_n(x)x_n$, *for all* $x \in E$.

The positive constants A, B are called atomic bounds for the atomic decomposition $(\{f_n\}, \{x_n\})$.

The following result which is referred in this paper is listed in the form of a lemma

Lemma 2.2. *If E is a Banach space and $\{f_n\} \subset E^*$ is total over E, then E is linearly isometric to the associated Banach space $E_d = \{\{f_n(x)\} : x \in E\}$, where the norm is given by $\|\{f_n(x)\}\|_{E_d} = \|x\|_E, x \in E$.*

3 Main Results

Towards the existence of atomic decomposition in a Banach space, we have the following result

Theorem 3.1. *A Banach space E has an atomic decomposition if there exist sequences $\{x_n\} \subset E$ and $\{f_n\} \subset E^*$ such that every $x \in E$ can be expressed as $x = \sum_{n=1}^{\infty} f_n(x)x_n$.*

Proof. Define

$$E_d = \left\{ \{\alpha_n\} \subset \mathbb{K} : \sum_{n=1}^{\infty} \alpha_n x_n \text{ is convergent} \right\}.$$

Then E_d is a Banach space with norm given by

$$\|\{\alpha_n\}\|_{E_d} = \sup_{1 \le n < \infty} \left\| \sum_{i=1}^{n} \alpha_i x_i \right\|.$$

Let $T : E \to E_d$ be defined by $T(x) = \{f_n(x)\}, x \in E$. Then, by Principle of Uniform boundedness,

$$\begin{aligned} \|x\|_E &\le \sup_{1 \le n < \infty} \left\| \sum_{i=1}^{n} f_i(x)x_i \right\| \\ &= \|\{f_n(x)\}\|_{E_d} \\ &\le B\|x\|_E, \quad x \in E, \end{aligned} \tag{3.1}$$

where $B = \sup_{1 \le n < \infty} \|S_n\| < \infty$ and $S_n(x) = \sum_{i=1}^{n} f_i(x)x_i$.

Thus $(\{f_n\}, \{x_n\})$ is an atomic decomposition for E with respect to E_d.

In the next result, we show that an associated Banach space E_d always has a complemented subspace.

Theorem 3.2. *If $(\{f_n\}, \{x_n\})$ is an atomic decomposition for E with respect to E_d, then there exist a complemented subspace G of E_d and an isomorphism T of E into E_d such that $E_d = T(E) \oplus G$.*

Proof. Let $T : E \to E_d$ be defined by $T(x) = \{f_n(x)\}$, $x \in E$.

Then in view of (3.1), T is an isomorphism of E into E_d.

Now, define $S : E_d \to E$ by $S(\{\alpha_n\}) = \sum_{i=1}^{\infty} \alpha_i x_i$. Then S is a bounded linear operator from E_d onto E. Put $G = kerS$. Then G is a closed subspace of E_d. Also, if $\{f_n(x)\} \in G$, then

$$0 = S(\{f_n(x)\}) = \sum_{n=1}^{\infty} f_n(x)x_n = x.$$

So $T(E) \cap G = \{0\}$. Let $\{\alpha_n\} \in E_d$ be any element such that $x = \sum_{i=1}^{\infty} \alpha_i x_i$. Then $\{f_n(x)\} \in T(E)$ such that

$$\sum_{i=1}^{\infty}(\alpha_i - f_i(x))x_i = \sum_{i=1}^{\infty} \alpha_i x_i - \sum_{i=1}^{\infty} f_i(x)x_i = 0.$$

Therefore $\{\alpha_n - f_n(x)\} \in G$ is such that

$$\{\alpha_n\} = \{f_n(x)\} + \{\alpha_n - f_n(x)\}.$$

Hence $E_d = T(E) \oplus G$.

In the following result, we show that an atomic decomposition for a Banach space produces another atomic decomposition for the space.

Theorem 3.3. *If* $(\{f_n\}, \{x_n\})$ *is an atomic decomposition for* E *with respect to* E_d*, then there exists a projection* ν *of* E_d *onto* $T(E)$ *along* G *such that* $(\{f_n\}, \{T^{-1}(\nu(e_n))\})$ *is an atomic decomposition for* E *with respect to* E_d*, where* $\{e_n\}$ *is the sequence of unit vectors in* E_d *and* T, G *are as in Theorem 3.2.*

Proof. Let ν be the projection of E_d onto $T(E)$ along G.

Then

$$\nu(\{\alpha_n\}) = \left\{ f_n \left(\sum_{i=1}^{\infty} \alpha_i x_i \right) \right\}, \{\alpha_n\} \in E_d.$$

Therefore, for each $k \in \mathbb{N}$,

$$\nu(e_k) = \left\{ f_n \left(\sum_{i=1}^{\infty} \delta_{i,k} x_i \right) \right\} = T(x_k),$$

where

$$\delta_{i,k} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k. \end{cases}$$

Since T is invertible, $x_k = T^{-1}(\nu(e_k))$.

Hence $(\{f_n\}, \{T^{-1}(\nu(e_n))\})$ is an atomic decomposition for E.

Further, we give a sufficient condition for the existence of atomic decomposition for a Banach space together with an associated Banach space.

Theorem 3.4. *Let E be a Banach space and E_d be an associated Banach space indexed by $\mathbb{N}$. Let $\{x_n\}, x_n \neq 0$ be a sequence in E such that for each $x \in E$ there exists a sequence $\{\alpha_n\}$ in E_d such that $x = \sum_{i=1}^{\infty} \alpha_i x_i$, then there exists a sequence $\{f_n\} \subset E^*$ such that $(\{f_n\}, \{x_n\})$ is an atomic decomposition for E with respect to E_d.*

Proof. Let $S : E_d \to E$ be defined by

$$S(\{\alpha_n\}) = \sum_{i=1}^{\infty} \alpha_i x_i, \{\alpha_n\} \in E_d.$$

Then, by Theorem 3.2, $E_d = T(E) \oplus G$. So $S|_{T(E)}$ is an isomorphism of $T(E)$ onto E. Let $x \in E$ be an arbitrary element and let

$$\{f_n(x)\} = (S|_{T(E)})^{-1}(x) \in T(E).$$

Then

$$\{\alpha_n\} = \{f_n(x)\} + \{\beta_n\}, \{\beta_n\} \in G.$$

So, we have $S(\{\alpha_n\}) = S(\{f_n(x)\})$.

Therefore $x = S(\{f_n(x)\}) = \sum_{i=1}^{\infty} f_i(x) x_i$ and each f_n is linear. Also

$$\begin{aligned}
|f_n(x)| &= \frac{1}{\|x_n\|} \|f_n(x) x_n\| \\
&\leq \frac{2}{\|x_n\|} \sup_{1 \leq k < \infty} \left\| \sum_{i=1}^{k} f_i(x) x_i \right\| \\
&\leq \frac{2}{\|x_n\|} \|(S|_{T(E)})^{-1} x\|, \quad x \in E, n \in \mathbb{N} \\
&\leq \frac{2}{\|x_n\|} \|(S|_{T(E)})^{-1} x\|, \quad x \in E, n \in \mathbb{N}.
\end{aligned}$$

Thus $\{f_n\} \subset E^*$. Also, by the Principle of Uniform boundedness,

$$\begin{aligned}
\|x\|_E &= \left\| \sum_{i=1}^{\infty} f_i(x) x_i \right\| \\
&\leq \sup_{1 \leq n < \infty} \left\| \sum_{i=1}^{n} f_i(x) x_i \right\| \\
&= \|\{f_n(x)\}\|_{E_d} \\
&\leq C \|x\|_E,
\end{aligned}$$

where $C = \sup_{1\le n<\infty} \|S_n\| < \infty$ and $S_n(x) = \sum_{i=1}^{n} f_i(x)x_i$.

Next, regarding existence of an atomic decomposition in a finite dimensional Banach space, we prove the following result

Theorem 3.5. *Let E be a Banach space such that $\dim E < \infty$. Then E has an infinite atomic decomposition.*

Proof. Let $\{y_i\}_{i=1}^{n} \subset E$ and $\{f_i\}_{i=1}^{n} \subset E^*$ be sequences such that for every $x \in E$,

$$x = \sum_{i=1}^{n} g_i(x)x_i, \quad i = 1, 2, \dots, n.$$

Define $\{x_n\} \subset E$ and $\{g_n\} \subset E^*$ by

$$\begin{aligned} x_{pn^2+qn+j} &= \frac{1}{2^{p+1}n} y_j \left(\begin{matrix} p = 0, 1, 2, \dots ; \\ q = 0, 1, \dots, n-1; j = 1, 2, \dots, n \end{matrix} \right) \\ g_{pn^2+qn+j} &= f_j \left(\begin{matrix} p = 0, 1, 2, \dots ; \\ q = 0, 1, 2, \dots, n-1; j = 1, 2, \dots, n \end{matrix} \right) \end{aligned}$$

Then

$$\begin{aligned} \sum_{i=1}^{\infty} g_i(x)x_i &= \sum_{p=0}^{\infty} \sum_{q=0}^{n-1} \sum_{j=1}^{n} g_{pn^2+qn+j}(x) x_{pn^2+qn+j} \\ &= \sum_{p=0}^{\infty} \sum_{j=1}^{n} \frac{1}{2^{p+1}n} f_j(x) y_j \\ &= \sum_{j=1}^{n} f_j(x) y_j \\ &= x, x \in E \end{aligned}$$

Define $E_d = \{\{g_n(x)\}; x \in E\}$ with the norm defined by

$$\|\{g_n(x)\}\|_{E_d} = \sup_{1\le n<\infty} \left\| \sum_{i=1}^{n} g_i(x)x_i \right\|.$$

Then E_d is a Banach space. Therefore, by the Principle of Uniform boundedness, we have

$$\|x\|_E \le \sup_{1\le n<\infty} \left\| \sum_{i=1}^{n} g_i(x)x_i \right\| = \|\{g_n(x)\}\|_{E_d} \le C\|x\|_E,$$

where $C = \sup_{1\le n<\infty} \|S_n\| < \infty$ and $S_n(x) = \sum_{i=1}^{n} f_i(x)x_i$.

In [7], it has been proved that if two Banach spaces have Banach frames, then their product space also has a Banach frame. The following is a similar result regarding atomic decompositions.

Theorem 3.6. *Let E and F be Banach spaces such that $(\{f_n\}, \{x_n\})$ is an atomic decomposition for E with respect to E_d and $(\{g_n\}, \{y_n\})$ is an atomic decomposition for F with respect to F_d, then there exist $\{z_n\} \subset E \times F$ and $\{h_n\} \subset (E \times F)^*$ such that $(\{h_n\}, \{z_n\})$ is an atomic decomposition of $E \times F$ with respect to some associated Banach space $(E \times F)_d$, where $E \times F$ is the product space with some suitable norm.*

Proof. Define $\{h_n\} \subset (E \times F)^*$ by

$$\begin{aligned} h_{2n-1}(x,y) &= f_n(x) \\ h_{2n}(x,y) &= g_n(y), (x,y) \in E \times F, n \in \mathbb{N} \end{aligned}$$

and $\{z_n\} \subset E \times F$ by

$$\begin{cases} z_{2n} = (0, y_n) \\ z_{2n-1} = (x_n, 0), n \in \mathbb{N}\,. \end{cases}$$

Then, proceeding as in the proof of Theorem 5.1 in [7], there exists an associated Banach space $(E \times F)_d = \{\{h_n(x,y)\} : x \in E, y \in F\}$ with norm given by

$$\|\{h_n(x,y)\}\|_{(E\times F)_d} = \|(x,y)\|_{E\times F}, \quad (x,y) \in E \times F.$$

Also, we have

$$\begin{aligned} \sum_{n=1}^{\infty} h_n(x,y) z_n &= \sum_{n=1}^{\infty} h_{2n}(x,y) z_{2n} + \sum_{n=1}^{\infty} h_{2n-1}(x,y) z_{2n-1} \\ &= \left(\sum_{n=1}^{\infty} f_n(x) x_n, \sum_{n=1}^{\infty} g_n(y) y_n \right). \end{aligned}$$

Hence $(\{z_n\}, \{h_n\})$ is an atomic decomposition for $E \times F$ with respect to $(E \times F)_d$.

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