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$rg\alpha$-Closed and $rg\alpha$-Open Maps in Topological Spaces

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Abstract

A set A in a topological space (X, τ) is said to be a regular generalized α-closed if $\alpha cl(A) \subset U$ whenever $A \subset U$ and U is regular α-open in X. In this paper, we introduce $rg\alpha$-closed map from a topological space X to a topological space Y as the image of every closed set is $rg\alpha$-closed, and also we prove that the composition of two $rg\alpha$-closed maps need not be $rg\alpha$-closed map. We also obtain some properties of $rg\alpha$-closed maps.

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1 Introduction

Generalized closed mappings were introduced and studied by Malghan[18]. wg-closed maps and rwg-closed maps were introduced and studied by Nagavani[19]. Regular closed maps, gpr-closed maps and rg-closed maps have been introduced and studied by Long[12], Gnanambal[9] and Arockiarani[1] respectively. In this paper, a new class of maps called regular generalized α-closed (briefly, $rg\alpha$-closed) maps, $rg\alpha^*$- closed maps have been introduced and studied their relations with various generalized closed maps. We prove that the composition of two $rg\alpha$-closed maps need not be $rg\alpha$-closed map. We also obtain some properties of $rg\alpha$-closed maps.

Let us recall the following definition which we shall require later.

Definition 1.1. *A subset A of a space (X, τ) is called*
1) a **preopen set***[17] if $A \subseteq intcl\,(A)$ and a* **preclosed set** *if $clint\,(A) \subseteq A$.*
2) a **semiopen set***[10] if $A \subseteq clint\,(A)$ and a* **semiclosed set** *if $intcl\,(A) \subseteq A$.*
3) a **α-open set***[20] if $A \subseteq intclint\,(A)$ and a* **α-closed set** *if $clintcl\,(A) \subseteq A$.*
4) a **semi-preopen set***[1] if $A \subseteq clintcl\,(A)$ and a* **semi-preclosed set** *if $intclint\,(A) \subseteq A$.*
5) a **regular open set***[26] if $A = intcl\,(A)$ and a* **regular closed set** *if $A = clint\,(A)$.*

The intersection of all semiclosed (resp. semiopen) subsets of (X, τ) containing A is called the semi-closure (resp. semi-kernal) of A and is denoted by $scl(A)$(resp. $sker(A)$).

Definition 1.2. *A subset A of a space (X, τ) is called*
1) **generalized closed set** *(briefly, g-closed)[11] if $cl\,(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.*
2) **semi-generalized closed set** *(briefly, sg-closed)[5] if $scl\,(A) \subseteq U$ whenever $A \subseteq U$ and U is semiopen in X.*
3) **generalized semi-closed set** *(briefly, gs-closed)[2] if $scl\,(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.*
4) **generalized α-closed set** *(briefly, gα-closed)[15] if $\alpha cl\,(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in X.*
5) **α-generalized closed set** *(briefly, αg-closed)[14] if $\alpha cl\,(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.*
6) **generalized semi-preclosed set** *(briefly, gsp-closed)[7] if $spcl\,(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.*
7) **regular generalized closed set** *(briefly, rg-closed)[21] if $cl\,(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X.*
8) **generalized preclosed set** *(briefly, gp-closed)[16] if $pcl\,(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.*
9) **generalized preregular closed set** *(briefly, gpr-closed)[9] if $pcl\,(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X.*
10) **weakely generalized closed set** *(briefly, wg-closed)[19] if $clint\,(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.*
11) **strongly generalized semi-closed set***[23] (briefly, g^*-closed) if $cl\,(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open in X.*
12) **π-generalized closed set** *(briefly, πg-closed)[8] if $cl\,(A) \subseteq U$ whenever $A \subseteq U$ and U is π-open in X.*
13) **weakely closed set** *(briefly, w-closed)[25] if $cl\,(A) \subseteq U$ whenever $A \subseteq U$ and U is semiopen in X.*

14) **mildly generalized closed set** *(briefly, mildly g-closed)[22] if* $clint(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is g-open in* X.
15) **semi weakly generalized closed set** *(briefly, swg-closed)[19] if* $clint(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is semi-open in* X.
16) **regular weakly generalized closed set** *(briefly, rwg-closed)[19] if* $clint(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is regular open in* X.
17) **regular semiopen set***[6] if there is a regular open set* U *such that* $U \subset A \subset cl(U)$.
18) **regular α-open set** *(briefly, rα-open)[28] if there is a regular open set* U *such that* $U \subset A \subset \alpha cl(U)$.
19) **regular w-closed set** *(briefly, rw-closed)[4] if* $cl(A) \subset U$ *whenever* $A \subset U$ *and* U *is regular semiopen in* X.
20) **regular generalized α-closed set** *(briefly, rgα-closed)[28] if* $\alpha cl(A) \subset U$ *whenever* $A \subset U$ *and* U *is regular α-open in* X.

The complements of the above mentioned closed sets are their respective open sets.

Definition 1.3. *A map* $f : (X, \tau) \to (Y, \sigma)$ *is called*
(i) **g-continuous***[3] if* $f^{-1}(V)$ *is g-closed set of* (X, τ) *for every closed set* V *of* (Y, σ),
*(ii)***$rg\alpha$-continuous***[30] if the inverse image of every closed set in* (Y, σ) *is rgα-closed set in* (X, τ).
*(iii)***$rg\alpha$-irresolute map***[30] if the inverse image of every rgα-closed set in* (Y, σ) *is rgα-closed in* (X, τ).
*(iv)***strongly $rg\alpha$-continuous***[30] if the inverse image of every rgα-open set in* (Y, σ) *is open in* (X, τ).

Definition 1.4. *A map* $f : (X, \tau) \to (Y, \sigma)$ *is said to be*
(i) **g-closed***[18] if* $f(F)$ *is g-closed in* (Y, σ) *for every closed set* F *of* (X, τ),
(ii) **w-closed***[24] if* $f(F)$ *is w-closed in* (Y, σ) *for every closed set* F *of* (X, τ),
(iii) **wg-closed***[19] if* $f(F)$ *is wg-closed in* (Y, σ) *for every closed set* F *of* (X, τ),
(iv) **rwg-closed***[19] if* $f(F)$ *is rwg-closed in* (Y, σ) *for every closed set* F *of* (X, τ),
(v) **rg-closed***[1] if* $f(F)$ *is rg-closed in* (Y, σ) *for every closed set* F *of* (X, τ),
(vi) **gpr-closed***[9] if* $f(F)$ *is gpr-closed in* (Y, σ) *for every closed set* F *of* (X, τ),
(vii) **regular closed***[13] if* $f(F)$ *is closed in* (Y, σ) *for every regular closed set* F *of* (X, τ).

Definition 1.5. *A map $f : (X, \tau) \to (Y, \sigma)$ is said to be*
*(i) g-**open**[27] if $f(U)$ is g-open in (Y, σ) for every open set U of (X, τ),*
*(ii) w-**open**[24] if $f(U)$ is w-open in (Y, σ) for every open set U of (X, τ),*
*(iii) wg-**open**[19] if $f(U)$ is wg-open in (Y, σ) for every open set U of (X, τ),*
*(iv) rwg-**open**[19] if $f(U)$ is rwg-open in (Y, σ) for every open set U of (X, τ),*
*(v) rg-**open**[1] if $f(U)$ is rg-open in (Y, σ) for every open set U of (X, τ),*
*(vi) gpr-**open**[9] if $f(U)$ is gpr-open in (Y, σ) for every open set U of (X, τ),*
*(vii) **regular open**[13] if $f(U)$ is open in (Y, σ) for every regular open set U of (X, τ).*

2 $rg\alpha$-closed Maps and $rg\alpha$-open Maps

We introduce the following definition

Definition 2.1. *A map $f : (X, \tau) \to (Y, \sigma)$ is said to be* **regular generalized α-closed** *(briefly, $rg\alpha$-closed) if the image of every closed set in (X, τ) is $rg\alpha$-closed in (Y, σ).*

Theorem 2.1. *Every closed map is $rg\alpha$-closed map, but not conversely.*

Proof. The proof follows from the definitions and fact that every closed set is $rg\alpha$-closed. ■

The converse of the above Theorem need not be true, as seen from the following example.

Example 2.1 *Consider $X = Y = \{a, b, c\}$ with topologies $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity map. Then this function is $rg\alpha$-closed but not closed, as the image of closed set $\{c\}$ in X is $\{c\}$ which is not closed set in Y.*

Theorem 2.2. *Every w-closed map is $rg\alpha$-closed map but not conversely.*

Proof. The proof follows from the definitions and fact that every w-closed set is $rg\alpha$-closed. ■

The converse of the above Theorem need not be true, as seen from the following example.

Example 2.2 *Consider $X = Y = \{a, b, c\}$ with topologies $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. Let the map $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = f(b) = b$ and $f(c) = c$. Then this function is $rg\alpha$-closed but not w-closed, as the image of closed set $\{c\}$ in X is $\{c\}$ which is not w-closed set in Y.*

Theorem 2.3. *Every rw-closed map is $rg\alpha$-closed map but not conversely.*

Proof. The proof follows from the definitions and fact that every rw-closed set is $rg\alpha$-closed. ■

The converse of the above Theorem need not be true, as seen from the following example.

Example 2.3 *Consider $X = Y = \{a, b, c, d\}$ with topologies $\tau = \sigma = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Let the map $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = f(b) = b, f(c) = d$ and $f(d) = c$. Then this function is $rg\alpha$-closed but not rw-closed, as the image of closed set $\{d\}$ in X is $\{c\}$ which is not rw-closed set in Y.*

Theorem 2.4. *Every $rg\alpha$-closed map is rg-closed map but not conversely.*

Proof. The proof follows from the definitions and fact that every $rg\alpha$-closed set is rg-closed. ■

The converse of the above Theorem need not be true, as seen from the following example.

Example 2.4 *Consider $X = \{a, b, c\}$, $Y = \{a, b, c, d\}$ with topologies $\tau = \{X, \phi, \{a\}, \{c\}, \{a, c\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\}$. Let the map $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = a$, $f(b) = c$ and $f(c) = d$. Then this function is rg-closed but not $rg\alpha$-closed, as the image of the closed set $\{a, b\}$ in X is $\{a, c\}$ which is not $rg\alpha$-closed set in Y.*

Theorem 2.5. *Every $rg\alpha$-closed map is rwg-closed map but not conversely.*

Proof. The proof follows from the definitions and fact that every $rg\alpha$-closed set is rwg-closed. ■

The converse of the above Theorem need not be true, as seen from the following example.

Example 2.5 *Consider $X = \{a, b, c\}$, $Y = \{a, b, c, d\}$ with topologies $\tau = \{X, \phi, \{b, c\}\}$ and $\sigma = \{Y, \phi, \{d\}, \{a, c\}, \{a, c, d\}\}$. Let the map $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = a$, $f(b) = b$ and $f(c) = d$. Then this function is rwg-closed but not $rg\alpha$-closed, as the image of the closed set $\{a\}$ in X is $\{a\}$ which is not $rg\alpha$-closed set in Y.*

Theorem 2.6. *Every $rg\alpha$-closed map is gpr-closed map but not conversely.*

Proof. The proof follows from the definitions and fact that every $rg\alpha$-closed set is gpr-closed. ■

The converse of the above Theorem need not be true, as seen from the following example.

Example 2.6 *Consider* $X = Y = \{a, b, c, d\}$ *with topologies* $\tau = \sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. *Let the map* $f : (X, \tau) \to (Y, \sigma)$ *be defined by* $f(a) = d$, $f(b) = b$, $f(c) = a$ *and* $f(d) = c$. *Then this function is gpr-closed but not* $rg\alpha$*-closed, as the image of the closed set* $\{c, d\}$ *in* X *is* $\{a, c\}$ *which is not* $rg\alpha$*-closed set in* Y.

Remark 2.1. *The following examples show that the regular closed maps and* $rg\alpha$*-closed maps are independent.*

Example 2.7 *Let* $X = Y = \{a, b, c\}$, *and a map* $f : (X, \tau) \to (Y, \sigma)$ *be the identity map with* $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ *and* $\sigma = \{Y, \phi, \{a\}, \{a, b\}\}$. *Then* f *is* $rg\alpha$*-closed but not regular closed, as the image of the regular closed set* $\{a, c\}$ *in* X *is* $\{a, c\}$ *which is not closed set in* Y.

Example 2.8 *Let* $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a, b\}\}$ *and* $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\}$. *Let a map* $f : (X, \tau) \to (Y, \sigma)$ *be defined by* $f(a) = c$, $f(b) = b$ *and* $f(c) = a$. *Then* f *is regular closed but not* $rg\alpha$*-closed, as the image of the closed set* $\{c\}$ *in* X *is* $\{a\}$ *which is not* $rg\alpha$*-closed in* Y.

Remark 2.2. *The following examples show that the g-closed maps and* $rg\alpha$*-closed maps are independent.*

Example 2.9 *Let* $X = Y = \{a, b, c\}$, *and a map* $f : (X, \tau) \to (Y, \sigma)$ *be the identity map with* $\tau = \{X, \phi, \{a\}, \{b, c\}\}$ *and* $\sigma = \{Y, \phi, \{a\}\}$. *Then* f *is* $rg\alpha$*-closed but not g-closed, as the image of the closed set* $\{a\}$ *in* X *is* $\{a\}$ *which is not g-closed set in* Y.

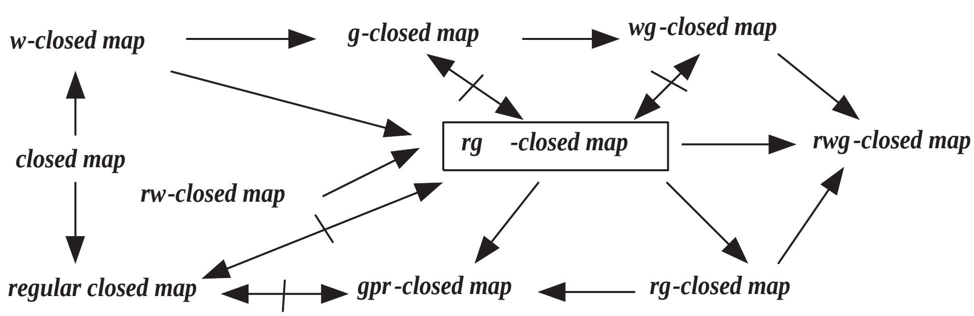
Example 2.10 *Consider* $X = \{a, b, c\}$, $Y = \{a, b, c, d\}$, $\tau = \{X, \phi, \{c\}\}$ *and* $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. *Let a map* $f : (X, \tau) \to (Y, \sigma)$ *be defined by* $f(a) = a$, $f(b) = d$ *and* $f(c) = c$. *Then this function is g-closed but not* $rg\alpha$*-closed, as the image of the closed set* $\{a, b\}$ *in* X *is* $\{a, d\}$ *which is not* $rg\alpha$*-closed in* Y.

Remark 2.3. *The following examples show that the wg-closed maps and* $rg\alpha$*-closed maps are independent.*

Example 2.11 *Let* $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{b, c\}\}$ *and* $\sigma = \{Y, \phi, \{a\}\}$. *Let a map* $f : (X, \tau) \to (Y, \sigma)$ *be the identity map. Then this function is* $rg\alpha$*-closed but not wg-closed, as the image of the closed set* $\{a\}$ *in* X *is* $\{a\}$ *which is not wg-closed set in* Y.

Example 2.12 *Consider* $X = \{a, b, c\}$, $Y = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}\}$ *and* $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. *Let a map* $f : (X, \tau) \to (Y, \sigma)$ *be defined by* $f(a) = c$, $f(b) = b$ *and* $f(c) = d$. *Then this function is wg-closed but not* $rg\alpha$*-closed, as the image of the closed set* $\{b, c\}$ *in* X *is* $\{b, d\}$ *which is not* $rg\alpha$*-closed in* Y.

Remark 2.4. *From the above discussions and known results we have the following implications*
In the following diagram, by
$A \to B$ *we mean* A *implies* B *but not conversely and*
$A \not\leftrightarrow B$ *means* A *and* B *are independent of each other.*



Figure

Theorem 2.7. *If a mapping* $f : (X, \tau) \to (Y, \sigma)$ *is* $rg\alpha$*-closed, then* $rg\alpha$*-*$cl(f(A)) \subset f(cl(A))$ *for every subset* A *of* (X, τ).

Proof. Suppose that f is $rg\alpha$-closed and $A \subset X$. Then $cl(A)$ is closed in X and so $f(cl(A))$ is $rg\alpha$-closed in (Y, σ). We have $f(A) \subset f(cl(A))$, by Theorem 2.9 (iv) in [29], $rg\alpha\text{-}cl(f(A)) \subset rg\alpha\text{-}cl(f(cl(A))) \to$ (i). Since $f(cl(A))$ is $rg\alpha$-closed in (Y, σ), $rg\alpha\text{-}cl(f(cl(A))) = f(cl(A)) \to$ (ii), by the Theorem 2.10 in [29]. From (i) and (ii), we have $rg\alpha\text{-}cl(f(A)) \subset f(cl(A))$ for every subset A of (X, τ). ■

Remark 2.5. *The converse of the above Theorem 2.7. is not true in general as seen from the following example*

Example 2.13 *Let* $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{c\}, \{a, c\}\}$, $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\}$ *Define* $f : (X, \tau) \to (Y, \sigma)$ *by* $f(x) = x$ *for every* $x \in X$. *Then* $rg\alpha\text{-}cl(f(A)) \subset f(cl(A))$ *for every subset* A *of* (X, τ). *But* f *is not* $rg\alpha$*-closed, since* $f(\{b\}) = \{b\}$ *is not* $rg\alpha$*-closed in* (Y, σ).

Corollary 2.1. *If a mapping* $f : (X, \tau) \to (Y, \sigma)$ *is a* $rg\alpha$*-closed, then the image* $f(A)$ *of closed set* A *in* (X, τ) *is* $\tau_{rg\alpha}$*-closed in* (Y, σ).

Proof. Let A be a closed set in (X,τ). Since f is $rg\alpha$-closed, by above Theorem 2.7., $rg\alpha$-$cl(f(A)) \subset f(cl(A)) \to$ (i). Also $cl(A) = A$, as A is a closed set and so $f(cl(A)) = f(A) \to$ (ii). From (i) and (ii), we have $rg\alpha$-$cl(f(A)) \subset f(A)$. We know that $f(A) \subset rg\alpha$-$cl(f(A))$ and so $rg\alpha$-$cl(f(A)) = f(A)$. Therefore $f(A)$ is $\tau_{rg\alpha}$-closed in (Y,σ). ■

Theorem 2.8. *Let (X,τ) be any topological spaces and (Y,σ) be a topological space where "$rg\alpha$-$cl(A) = w$-$cl(A)$ for every subset A of Y" and $f : (X,\tau) \to (Y,\sigma)$ be a map, then the following are equivalent.*
(i) f is $rg\alpha$-closed map.
(ii) $rg\alpha$-$cl(f(A)) \subset f(cl(A))$ for every subset A of (X,τ).

Proof. $(i) \Rightarrow (ii)$ Follows from the Theorem 2.7.
$(ii) \Rightarrow (i)$ Let A be any closed set of (X,τ). Then $A = cl(A)$ and so $f(A) = f(cl(A)) \supset rg\alpha$-$cl(f(A))$ by hypothesis. We have $f(A) \subset rg\alpha$-$cl(f(A))$, by Theorem 2.9(ii) in [29]. Therefore $f(A) = rg\alpha$-$cl(f(A))$. Also $f(A) = rg\alpha$-$cl(f(A)) = w$-$cl(f(A))$, by hypothesis. That is $f(A) = w$-$cl(f(A))$ and so $f(A)$ is w-closed in (Y,σ). Thus $f(A)$ is $rg\alpha$-closed set in (Y,σ) and hence f is $rg\alpha$-closed map. ■

Theorem 2.9. *A map $f : (X,\tau) \to (Y,\sigma)$ is $rg\alpha$-closed if and only if for each subset S of (Y,σ) and each open set U containing $f^{-1}(S) \subset U$, there is a $rg\alpha$-open set V of (Y,σ) such that $S \subset V$ and $f^{-1}(V) \subset U$.*

Proof. Suppose f is $rg\alpha$-closed. Let $S \subset Y$ and U be an open set of (X,τ) such that $f^{-1}(S) \subset U$. Now $X - U$ is closed set in (X,τ). Since f is $rg\alpha$-closed, $f(X-U)$ is $rg\alpha$-closed set in (Y,σ). Then $V = Y - f(X-U)$ is a $rg\alpha$-open set in (Y,σ). Note that $f^{-1}(S) \subset U$ implies $S \subset V$ and $f^{-1}(V) = X - f^{-1}(f(X-U)) \subset X - (X-U) = U$. That is $f^{-1}(V) \subset U$.

For the converse, let F be a closed set of (X,τ). Then $f^{-1}(f(F)^c) \subset F^c$ and F^c is an open in (X,τ). By hypothesis, there exists a $rg\alpha$-open set V in (Y,σ) such that $f(F)^c \subset V$ and $f^{-1}(V) \subset F^c$ and so $F \subset (f^{-1}(V))^c$. Hence $V^c \subset f(F) \subset f(((f^{-1}(V))^c) \subset V^c$ which implies $f(V) \subset V^c$. Since V^c is $rg\alpha$-closed, $f(F)$ is $rg\alpha$-closed. That is $f(F)$ is $rg\alpha$-closed in (Y,σ) and therefore f is $rg\alpha$-closed. ■

Remark 2.6. *The composition of two $rg\alpha$-closed maps need not be $rg\alpha$-closed map in general and this is shown by the following example.*

Example 2.14 *Let $X = Y = Z = \{a,b,c\}$, $\tau = P(X)$, $\sigma = \{Y, \phi, \{c\}, \{a,b\}\}$ and $\eta = \{Z, \phi, \{a\}, \{b\}, \{a,b\}\}$. Define $f : (X,\tau) \to (Y,\sigma)$ by $f(a) = a$, $f(b) = b$ and $f(c) = c$ and $g : (Y,\sigma) \to (Z,\eta)$ be the identity map. Then*

f and g are $rg\alpha$-closed maps, but their composition $g \circ f : (X, \tau) \to (Z, \eta)$ is not $rg\alpha$-closed map, because $F = \{a\}$ is closed in (X, τ), but $g \circ f(F) = g \circ f(\{a\}) = g(f(\{a\})) = g(\{a\}) = \{a\}$ which is not $rg\alpha$-closed in (Z, η).

Theorem 2.10. *If $f : (X, \tau) \to (Y, \sigma)$ is closed map and $g : (Y, \sigma) \to (Z, \eta)$ is $rg\alpha$-closed map, then the composition $g \circ f : (X, \tau) \to (Z, \eta)$ is $rg\alpha$-closed map.*

Proof. Let F be any closed set in (X, τ). Since f is closed map, $f(F)$ is closed set in (Y, σ). Since g is $rg\alpha$-closed map, $g(f(F))$ is $rg\alpha$-closed set in (Z, η). That is $g \circ f(F) = g(f(F))$ is $rg\alpha$-closed and hence $g \circ f$ is $rg\alpha$-closed map. ■

Remark 2.7. *If $f : (X, \tau) \to (Y, \sigma)$ is $rg\alpha$-closed map and $g : (Y, \sigma) \to (Z, \eta)$ is closed map, then the composition need not be $rg\alpha$-closed map as seen from the following example.*

Example 2.15 *Consider $X = Y = Z = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$, $\sigma = \{Y, \phi, \{a\}, \{b, c\}\}$ and $\eta = \{Z, \phi, \{b\}, \{c\}, \{b, c\}\}$ Let $f : (X, \tau) \to (Y, \sigma)$ be the identity map and $g : (Y, \sigma) \to (Z, \eta)$ is defined by $g(a) = g(b) = a$ and $g(c) = b$. Then f is $rg\alpha$-closed map and g is a closed map. But their composition $g \circ f : (X, \tau) \to (Z, \eta)$ is not $rg\alpha$-closed map, since for the closed set $\{c\}$ in (X, τ), but $g \circ f(\{c\}) = g(f(\{c\})) = g(\{c\}) = \{b\}$ which is not $rg\alpha$-closed in (Z, η).*

Theorem 2.11. *Let (X, τ), (Z, η) be topological spaces, and (Y, σ) be topological spaces where "every $rg\alpha$-closed subset is closed". Then the composition $g \circ f : (X, \tau) \to (Z, \eta)$ of the $rg\alpha$-closed maps $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ is $rg\alpha$-closed.*

Proof. Let A be a closed set of (X, τ). Since f is $rg\alpha$-closed, $f(A)$ is $rg\alpha$-closed in (Y, σ). Then by hypothesis, $f(A)$ is closed. Since g is $rg\alpha$-closed, $g(f(A))$ is $rg\alpha$-closed in (Z, η) and $g(f(A)) = g \circ f(A)$. Therefore $g \circ f$ is $rg\alpha$-closed. ■

Theorem 2.12. *If $f : (X, \tau) \to (Y, \sigma)$ is g-closed, $g : (Y, \sigma) \to (Z, \eta)$ be $rg\alpha$-closed and (Y, σ) is $T_{1/2}$-space then their composition $g \circ f : (X, \tau) \to (Z, \eta)$ is $rg\alpha$-closed map.*

Proof. Let A be a closed set of (X, τ). Since f is g-closed, $f(A)$ is g-closed in (Y, σ). Since (Y, σ) is $T_{1/2}$-space, $f(A)$ is closed in (Y, σ). Since g is $rg\alpha$-closed, $g(f(A))$ is $rg\alpha$-closed in (Z, η) and $g(f(A)) = g \circ f(A)$. Therefore $g \circ f$ is $rg\alpha$-closed. ■

Theorem 2.13. *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be two mappings such that their composition $g \circ f : (X, \tau) \to (Z, \eta)$ be $rg\alpha$-closed mapping. Then the following statements are true.*
(i) If f is continuous and surjective, then g is $rg\alpha$-closed.
(ii) If g is $rg\alpha$-irresolute and injective, then f is $rg\alpha$-closed.
(iii) If f is g-continuous, surjective and (X, τ) is a $T_{1/2}$-space, then g is $rg\alpha$-closed.
(iv) If g is strongly $rg\alpha$-continuous and injective, then f is $rg\alpha$-closed.

Proof. (i) Let A be a closed set of (Y, σ). Since f is continuous, $f^{-1}(A)$ is closed in (X, τ) and since $g \circ f$ is $rg\alpha$-closed, $(g \circ f)(f^{-1}(A))$ is $rg\alpha$-closed in (Z, η). That is $g(A)$ is $rg\alpha$-closed in (Z, η), since f is surjective. therefore g is $rg\alpha$-closed.
(ii) Let B be a closed set of (X, τ). Since $g \circ f$ is $rg\alpha$- closed, $g \circ f(B)$ is $rg\alpha$-closed in (Z, η). Since g is $rg\alpha$-irresolute, $g^{-1}(g \circ f(B))$ is $rg\alpha$-closed set in (Y, σ). That is $f(B)$ is $rg\alpha$-closed in (Y, σ), since f is injective. Therefore f is $rg\alpha$-closed.
(iii) Let C be a closed set of (Y, σ). Since f is g-continuous, $f^{-1}(C)$ is g-closed set in (X, τ). Since (X, τ) is a $T_{1/2}$-space, $f^{-1}(C)$ is closed set in (X, τ). Since $g \circ f$ is $rg\alpha$-closed, $(g \circ f)(f^{-1}(C))$ is $rg\alpha$-closed in (Z, η). That is $g(C)$ is $rg\alpha$-closed in (Z, η), since f is surjective. Therefore g is $rg\alpha$-closed.
(iv) Let D be a closed set of (X, τ). Since $g \circ f$ is $rg\alpha$-closed, $(g \circ f)(D)$ is $rg\alpha$-closed in (Z, η). Since g is strongly $rg\alpha$-continous, $g^{-1}((g \circ f)(D)$ is closed set in (Y, σ). That is $f(D)$ is closed set in (Y, σ), since g is injective, Therefore f is closed. ∎

Theorem 2.14. *If $f : (X, \tau) \to (Y, \sigma)$ is an open, continuous, $rg\alpha$-closed surjection and $cl(F) = F$ for every $rg\alpha$-closed set in (Y, σ), where X is regular, then Y is regular.*

Proof. Let U be an open set in Y and $p \in U$. Since f is surjection, there exists a point $x \in X$ such that $f(x) = p$. Since X is regular and f is continuous, there is an open set V in X such that $x \in V \subset cl(V) \subset f^{-1}(U)$. Here $p \in f(V) \subset f(cl(V)) \subset U \to (i)$. Since f is $rg\alpha$-closed, $f(cl(V))$ is $rg\alpha$-closed set contained in the open set U. By hypothesis, $cl(f(cl(V))) = f(cl(V))$ and $cl(f(V)) = cl(f(cl(V))) \to (ii)$. From (i) and (ii), we have $p \in f(V) \subset cl(f(V)) \subset U$ and $f(V)$ is open, since f is open. Hence Y is regular. ∎

Theorem 2.15. *If a map $f : (X, \tau) \to (Y, \sigma)$ is $rg\alpha$-closed and A is closed set of X, then $f_A : (A, \tau_A) \to (Y, \sigma)$ is $rg\alpha$-closed.*

Proof. Let F be a closed set of A. Then $F = A \cap E$ for some closed set E of (X, τ) and so F is closed set of (X, τ). Since f is $rg\alpha$-closed, $f(F)$ is $rg\alpha$-closed set in (Y, σ). But $f(F) = f_A(F)$ and therefore $f_A : (A, \tau_A) \to (Y, \sigma)$ is $rg\alpha$-closed. ■

Analogous to $rg\alpha$-closed maps, we define $rg\alpha$-open map as follows.

Definition 2.2. *A map $f : (X, \tau) \to (Y, \sigma)$ is called a $rg\alpha$-**open map** if the image $f(A)$ is $rg\alpha$-open in (Y, σ) for each open set A in (X, τ).*

From the definitions we have the following results.

Theorem 2.16. *(i)Every open map is $rg\alpha$-open but not conversely.*
(ii) Every w-open map is $rg\alpha$-open but not conversely.
(iii) Every $rg\alpha$-open map is rg-open but not conversely.
(iv) Every $rg\alpha$-open map is rwg-open but not conversely.
(v) Every $rg\alpha$-open map is gpr-open but not conversely.

Theorem 2.17. *For any bijection map $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*
(i) $f^{-1} : (Y, \sigma) \to (X, \tau)$ is $rg\alpha$-continous.
(ii) f is $rg\alpha$-open map and (iii) f is $rg\alpha$-closed map.

Proof. $(i) \Rightarrow (ii)$ Let U be an open set of (X, τ). By assumption, $(f^{-1})^{-1}(U) = f(U)$ is $rg\alpha$-open in (Y, σ) and so f is $rg\alpha$-open.
$(ii) \Rightarrow (iii)$ Let F be a closed set of (X, τ). Then F^c is open set in (X, τ). By assumption, $f(F^c)$ is $rg\alpha$-open in (Y, σ). That is $f(F^c) = f(F)^c$ is $rg\alpha$-open in (Y, σ) and therefore $f(F)$ is $rg\alpha$-closed in (Y, σ). Hence f is $rg\alpha$-closed.
(iii) $\Rightarrow (i)$ Let F be a closed set of (X, τ). By assumption, $f(F)$ is $rg\alpha$-closed in (Y, σ). But $f(F) = (f^{-1})^{-1}(F)$ and therefore f^{-1} is continuous.

Theorem 2.18. *If a map $f : (X, \tau) \to (Y, \sigma)$ is $rg\alpha$-open, then $f(int(A)) \subset rg\alpha\text{-}int(f(A))$ for every subset A of (X, τ).*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a open map and A be any subset of (X, τ). Then $int(A)$ is open in (X, τ) and so $f(int(A))$ is $rg\alpha$-open in (Y, σ). We have $f(int(A)) \subset f(A)$. Therefore by Theorem 2.2 (iii) in [29], $f(int(A)) \subset rg\alpha\text{-}int(f(A))$. ■

Remark 2.8. *The converse of the above Theorem need not be true in general as seen from the following example.*

Example 2.16 *Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, c\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\}$. Let f be the identity map. In (Y, σ), $rg\alpha$-$int(f(A)) = f(A)$ for every subset A of (X, τ). So $f(int(A)) \subset f(A) = rg\alpha$-$int(f(A))$ for every subset A of X. But f is not $rg\alpha$-open map, since for the open set $\{a, c\}$ of (X, τ), $f(\{a, c\}) = \{a, c\}$ which is not $rg\alpha$-open in (Y, σ).*

Theorem 2.19. *If a map $f : (X, \tau) \to (Y, \sigma)$ is $rg\alpha$-open, then for each neighbourhood U of x in (X, τ), there exists a $rg\alpha$-neighbourhood W of $f(x)$ in (Y, σ) such that $W \subset f(U)$.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a $rg\alpha$-open map. Let $x \in X$ and U be an arbitrary neighbourhood of x in (X, τ). Then there exists an open set G in (X, τ) such that $x \in G \subset U$. Now $f(x) \in f(G) \subset f(U)$ and $f(G)$ is $rg\alpha$-open set in (Y, σ), as f is a $rg\alpha$-open map. By Theorem 3.8 in [28], $f(G)$is $rg\alpha$-neighbourhood of each of its points. Taking $f(G) = W$, W is a $rg\alpha$-neighbourhood of $f(x)$ in (Y, σ) such that $W \subset f(U)$. ∎

Theorem 2.20. *A map $f : (X, \tau) \to (Y, \sigma)$ is $rg\alpha$-open if and only if for any subset S of (Y, σ) and any closed set of (X, τ) containing $f^{-1}(S)$, there exists a $rg\alpha$-closed set K of (Y, σ) containing S such that $f^{-1}(K) \subset F$.*

Proof. Suppose f is $rg\alpha$-open map. Let $S \subset Y$ and F be a closed set of (X, τ) such that $f^{-1}(S) \subset F$. Now $X - F$ is an open set in (X, τ). Since f is $rg\alpha$-open map, $f(X - F)$ is $rg\alpha$-open set in (Y, σ). Then $K = Y - f(X - F)$ is a $rg\alpha$-closed set in (Y, σ). Note that $f^{-1}(S) \subset F$ implies $S \subset K$ and $f^{-1}(K) = X - f^{-1}(X - F) \subset X - (X - F) = F$. That is $f^{-1}(K) \subset F$.

For the converse, let U be an open set of (X, τ). Then $f^{-1}((f(U))^c) \subset U^c$ and $U^c$is a closed set in (X, τ). By hypothesis, there exists a $rg\alpha$-closed set K of (Y, σ) such that $(f(U))^c \subset K$ and $f^{-1}(K) \subset U^c$ and so $U \subset (f^{-1}(K))^c$. Hence $K^c \subset f(U) \subset f((f^{-1}(K)))^c \subset K^c$ which implies $f(U) = K^c$. Since K^c is a $rg\alpha$-open, $f(U)$ is $rg\alpha$-open in (Y, σ) and therefore f is $rg\alpha$-open map. ∎

Theorem 2.21. *If a function $f : (X, \tau) \to (Y, \sigma)$ is $rg\alpha$-open, then $f^{-1}(rg\alpha$-$cl(B)) \subset cl(f^{-1}(B))$ for each subset B of (Y, σ).*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a $rg\alpha$-open map and B be any subset of (Y, σ). Then $f^{-1}(B) \subset cl(f^{-1}(B))$ and $cl(f^{-1}(B))$ is closed set in (X, τ). By above Theorem 2.20., there exists a $rg\alpha$-closed set K of (Y, σ) such that $B \subset K$ and $f^{-1}(K) \subset cl(f^{-1}(B))$. Now $rg\alpha$-$cl(B) \subset rg\alpha$-$cl(K) = K$, by Theorems 2.9 and 2.10 in [28], as K is $rg\alpha$-closed set of (Y, σ). Therefore $f^{-1}(rg\alpha$-$cl(B)) \subset f^{-1}(K)$ and so $f^{-1}(rg\alpha$-$cl(B)) \subset f^{-1}(K) \subset cl(f^{-1}(B))$. Thus $f^{-1}(rg\alpha$-$cl(B)) \subset cl(f^{-1}(B))$ for each subset of B of (Y, σ). ∎

Remark 2.9. *The converse of the above Theorem need not be true in general as seen from the following example.*

Example 2.17 *Let* $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{b\}, \{b, c\}\}$ *and* $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\}$. *Let* f *be the identity map. In* (Y, σ), $rg\alpha$-$cl(B) = B$ *for every subset* B *of* (Y, σ). *So* $f^{-1}(rg\alpha\text{-}cl(B)) = f^{-1}(B) \subset cl(f^{-1}(B))$ *for every subset* B *of* (Y, σ). *But* f *is not* $rg\alpha$*-open map, since for the open set* $\{b, c\}$ *of* (X, τ), $f(\{b, c\}) = \{b, c\}$ *which is not* $rg\alpha$*-open in* (Y, σ).

We define another new class of maps called $rg\alpha^*$-closed maps which are stronger than $rg\alpha$-closed maps.

Definition 2.3. *A map* $f : (X, \tau) \to (Y, \sigma)$ *is said to be* $rg\alpha^*$**-closed map** *if the image* $f(A)$ *is* $rg\alpha$*-closed in* (Y, σ) *for every* $rg\alpha$*-closed set* A *in* (X, τ).

Theorem 2.22. *Every* $rg\alpha^*$*-closed map is* $rg\alpha$*-closed map but not conversely.*

Proof. The proof follows from the definitions and fact that every closed set is $rg\alpha$-closed.

The converse of the above Theorem is not true in general as seen from the following example.

Example 2.18 *Let* $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ *and* $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\}$ *and* $f : (X, \tau) \to (Y, \sigma)$ *be the identity map. Then* f *is* $rg\alpha$*-closed map but not* $rg\alpha^*$*-closed map. Since* $\{a\}$ *is* $rg\alpha$*-closed set in* (X, τ)*, but its image under* f *is* $\{a\}$*, which is not* $rg\alpha$*-closed in* (Y, σ).

Theorem 2.23. *If* $f : (X, \tau) \to (Y, \sigma)$ *and* $g : (Y, \sigma) \to (Z, \eta)$ *are* $rg\alpha^*$*-closed maps, then their composition* $g \circ f : (X, \tau) \to (Z, \eta)$ *is also* $rg\alpha^*$*-closed.*

Proof. Let F be a $rg\alpha$-closed set in (X, τ). Since f is $rg\alpha^*$-closed map, $f(F)$ is $rg\alpha$-closed set in (Y, σ). Since g is $rg\alpha^*$-closed map, $g(f(F))$ is $rg\alpha$-closed set in (Z, η). Therefore $g \circ f$ is $rg\alpha^*$-closed map.

Analogous to $rg\alpha^*$-closed map, we define another new class of maps called $rg\alpha^*$-open maps which are stronger than $rg\alpha$-open maps.

Definition 2.4. *A map* $f : (X, \tau) \to (Y, \sigma)$ *is said to be* $rg\alpha^*$**-open map** *if the image* $f(A)$ *is* $rg\alpha$*-open set in* (Y, σ) *for every* $rg\alpha$*-open set* A *in* (X, τ).

Remark 2.10. *Since every open set is a* $rg\alpha$*-open set, we have every* $rg\alpha^*$*-open map is* $rg\alpha$*-open map. The converse is not true in general as seen from the following example.*

Example 2.19 *Let* $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}\}$ *and* $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\}$. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be the identity map. Then* f *is* $rg\alpha$*-open map but not* $rg\alpha^*$*-open map, since for the* $rg\alpha$*-open set* $\{a, c\}$ *in* (X, τ), $f(\{a, c\}) = \{a, c\}$ *which is not* $rg\alpha$*-open set in* (Y, σ).

Theorem 2.24. *If* $f : (X, \tau) \to (Y, \sigma)$ *and* $g : (Y, \sigma) \to (Z, \eta)$ *are* $rg\alpha^*$*-open maps, then their composition* $g \circ f : (X, \tau) \to (Z, \eta)$ *is also* $rg\alpha^*$*-open.*

Proof. Proof is smilar to the Theorem 2.23.

Theorem 2.25. *For any bijection map* $f : (X, \tau) \to (Y, \sigma)$, *the following statements are equivalent:*
(i) $f^{-1} : (Y, \sigma) \to (X, \tau)$ *is* $rg\alpha$ *irresolute*
(ii) f *is* $rg\alpha^*$*-open map*
(iii) f *is* $rg\alpha^*$*-closed map.*

Proof. Proof is similar to that of Theorem 2.17.

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