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Quasi rw-Open and Quasi rw-Closed Functions

A. Vadivel and K. Vairamanickam

Department of Mathematics, Annamalai University
Annamalainagar - 608 002, India
avmaths@gmail.com
kvaufeat@gmail.com

Abstract

A set A in a topological space (X, τ) is said to be a regular w-closed (briefly, rw-closed) if $cl(A) \subset U$ whenever $A \subset U$ and U is regular semiopen in X. In this paper, we introduce quasi rw-open map from a space X to a space Y as the image of every rw-open set is open. Also, we obtain its characterizations and its basic properties.

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1 Introduction and Preliminaries

Functions and of course open functions stand among the most important notions in the whole of mathematical science. Many different forms of open functions have been introduced over the years. Various interesting problems arise when one consider openness. Its important is significant in various areas of mathematics and related sciences.

N. Levine[10] introduced generalized closed sets in general topology as a generalization of closed sets. This concept was found to be useful and many results in general topology were improved. Many researchers like Balachandran, Sundaram and Maki[3], Bhattacharyya and Lahiri[5], Arockiarani[1], Dunham[8], Gnanambal[9], Malghan[11], Palaniappan and Rao[12], Park[13]

Arya and Gupta[2] and Devi[7], Benchalli and Wali[4] have worked on generalized closed sets, their generalizations and related concepts in general topology.

Recently, as a generalization of closed sets, the notion of rw-closed sets were introduced and studied in [4]. In this paper, we will continue the study of related functions by involving rw-open sets. We introduce and characterize the concept of quasi rw-open fuctions.

Throughtout this paper, by spaces we mean topological spaces on which no separation axioms are assumed unless otherwise mentioned and $f:(X,\tau)\rightarrow(Y,\sigma)$ (or simply $f:X\rightarrow Y$) denotes a function f of a space (X,τ) into a space (Y,σ). For a subset A of X, $cl(A)$ and $int(A)$ denote the closure and interior of A, respectively.

2 Preliminaries

Definition 2.1. *A subset A of a space X is called a* **regular open set***[14] if $A = intcl\,(A)$ and a regular closed set if $A = clint\,(A)$.*

Definition 2.2. *A subset A of a space X is called* **regular semiopen***[6] if there is a regular open U such $U \subset A \subset cl(U)$.*

Definition 2.3. *A subset A of a space X is called a* **regular w-closed set** *(briefly, rw-closed)[4] if $cl\,(A) \subset U$ whenever $A \subset U$ and U is regular semiopen in X. We denote the set of all rw-closed sets in X by $RWC\,(X)$.*

The complements of the rw-closed sets are their respective rw-open sets.

Definition 2.4. *A map $f:(X,\tau)\rightarrow(Y,\sigma)$ is called a* **rw-open map***[4] if the image $f\,(A)$ is rw-open in (Y,σ) for each open set A in (X,τ).*

Definition 2.5. *A map $f:(X,\tau)\rightarrow(Y,\sigma)$ is called a* **rw-irresolute***[4] if the inverse image of every rw-closed in (Y,σ) is rw-closed in (X,τ).*

3 Quasi rw-open Functions

We have introduced the following definition:

Definition 3.1. *A function $f:X\rightarrow Y$ is said to be* **quasi rw-open** *if the image of every rw-open set in X is open in Y.*

It is evident that, the concepts of quasi rw-openness and rw-continuity coincide if the function is a bijection.

Theorem 3.1. *A function $f : X \to Y$ is said to be quasi rw-open iff for every subset U of X, $f(rw\text{-}int(U)) \subset int(f(U))$.*

Proof. Let f be a quasi rw-open function. Now, we have $int(U) \subset U$ and $rw\text{-}int(U)$ is a rw-open set. Hence we obtain that $f(rw\text{-}int(U)) \subset f(U)$. As $f(rw\text{-}int(U))$ is open, $f(rw\text{-}int(U)) \subset int(f(U))$.

Conversely, assume that U is a rw-open set in X. Then $f(U) = f(rw\text{-}int(U)) \subset int(f(U))$ but $int(f(U)) \subset f(U)$. Consequently $f(U) = int(f(U))$ and hence f is quasi rw-open. ■

Lemma 3.1. *If a function $f : X \to Y$ is quasi rw-open, then $rw\text{-}int(f^{-1}(G)) \subset f^{-1}(int(G))$ for every subset G of Y.*

Proof. Let G be any arbitrary subset of Y. Then, $rw\text{-}int(f^{-1}(G))$ is a rw-open set in X and f is quasi rw-open, then $f(rw\text{-}int(f^{-1}(G))) \subset int(f(f^{-1}(G))) \subset int(G)$. Thus $rw\text{-}int(f^{-1}(G)) \subset f^{-1}(int(G)))$. ■

Recall that a subset S is called a **rw-neighbourhood**[4] of a point x of X, if there exists a rw-open set U such that $x \in U \subset S$.

Theorem 3.2. *For a function $f : X \to Y$, the following are equivalent.*
(i) f is quasi rw-open.
(ii) For each subset U of X, $f(rw\text{-}int(U)) \subset int(f(U))$.
(iii) For each $x \in X$ and each rw-neighbourhood U of x in X, there exists a neighbourhood V of $f(x)$ in Y such that $V \subset f(U)$.

Proof. $(i) \Rightarrow (ii)$: It follows from Theorem 3.1.
$(ii) \Rightarrow (iii)$: Let $x \in X$ and U be an arbitrary rw-neighbourhood of x in X. Then there exist a rw-open set V in X such that $x \in V \subset U$. Then by (ii), we have $f(V) = f(rw\text{-}int(V)) \subset int(f(V))$ and hence $f(V) = int(f(V))$. Therefore, it follows that $f(V)$ is open in Y such that $f(x) \in f(V) \subset f(U)$.
$(iii) \Rightarrow (i)$: Let U be an arbitrary rw-open set in X. Then for each $y \in f(U)$, by (iii) there exist a neighbourhood V_y of y in Y such that $V_y \subset f(U)$. As V_y is a neighbourhood of y, there exist an open set W_y in Y such that $y \in W_y \subset V_y$. Thus $f(U) = \bigcup\{W_y : y \in f(U)\}$ which is a open set in Y. This implies that f is quasi rw-open function. ■

Theorem 3.3. *A function $f : X \to Y$ is quasi rw-open iff for any subset B of Y and for any rw-closed set F of X, containing $f^{-1}(B)$, there exist a closed set G of Y containing B such that $f^{-1}(G) \subset F$.*

Proof. Suppose f is quasi rw-open. Let $B \subset Y$ and F be a rw-closed set of X containing $f^{-1}(B)$. Now, put $G = Y - f(X - F)$. It is clear that $f^{-1}(B) \subset F$

implies $B \subset G$. Since f is quasi rw-open, we obtain G as a closed set of Y. Moreover, we have $f^{-1}(G) \subset F$.

Conversely, Let U be a rw-open set of X and put $B = Y \backslash f(U)$. Then $X \backslash U$ is a rw-closed set in X containing $f^{-1}(B)$. By hypothesis, there exists a closed set F of Y such that $B \subset F$ and $f^{-1}(F) \subset X \backslash U$. Hence, we obtain $f(U) \subset Y \backslash F$. On the otherhand, it follows that $B \subset F, Y \backslash F \subset Y \backslash B = f(U)$. Thus, we obtain $f(U) = Y \backslash F$ which is open and hence f is a quasi rw-open function. ■

Theorem 3.4. *A function $f : X \to Y$ is quasi rw-open iff $f^{-1}(cl(B)) \subset rw$-$cl(f^{-1}(B))$ for every subset B of Y.*

Proof. Suppose thst f is quasi rw-open. For any subset B of Y, $f^{-1}(B) \subset rw\text{-}cl(f^{-1}(B))$. Therefore by Theorem 3.3., there exists a closed set F in Y such that $B \subset F$ and $f^{-1}(F) \subset rw\text{-}cl(f^{-1}(B))$. Therefore, we obtain $f^{-1}(cl(B)) \subset f^{-1}(F) \subset rw\text{-}cl(f^{-1}(B))$.

Conversely, let $B \subset Y$ and F be a rw-closed set of X containing $f^{-1}(B)$. Put $W = cl_Y(B)$, then we have $B \subset W$ and W is closed set and $f^{-1}(W) \subset rw\text{-}cl(f^{-1}(B)) \subset F$. Then by Theorem 3.3., f is quasi rw-open. ■

Lemma 3.2. *Let $f : X \to Y$ and $g : Y \to Z$ be two functions and $g \circ f : X \to Z$ is quasi rw-open. If g is continuous injective, then f is quasi rw-open.*

Proof. Let U be a rw-open set in X, then $(g \circ f)(U)$ is open in Z. Since $g \circ f$ is quasi rw-open. Again g is an injective continuous function, $f(U) = g^{-1}(g \circ f(U)))$ is open in Y. This shows that f is quasi rw-open. ■

4 Quasi rw-closed Functions

Definition 4.1. *A function $f : X \to Y$ is said to* **quasi rw-closed** *if the image of each rw-closed set in X is closed in Y.*

Clearly, every quasi rw-closed function is closed as well as rw-closed.

Remark 4.1. *Every rw-closed (resp. closed) function need not be quasi rw-closed as shown by the following example.*

Example 4.1 *Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b, c\}\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = b, f(b) = c$, and $f(c) = a$. Then clearly f is rw-closed as well as closed but not quasi rw-closed.*

Lemma 4.1. *If a function $f : X \to Y$ is quasi rw-closed, then $f^{-1}(int(B)) \subset rw\text{-}int(f^{-1}(B))$ for every subset B of Y.*

Proof. This proof is smilar to the proof of Lemma 3.1. ■

Theorem 4.1. *A function $f : X \to Y$ is quasi rw-closed iff for any subset B of Y and for any rw-open set G of X containing $f^{-1}(B)$, there exists an open set U of Y containing B such that $f^{-1}(U) \subset G$.*

Proof. This proof is similar to that Theorem 3.3. ■

Definition 4.2. *A function $f : X \to Y$ is called* rw^***-closed** *if the image of rw-closed subset of X is rw-closed set in Y.*

Theorem 4.2. *If $f : X \to Y$ and $g : Y \to Z$ are two quasi rw-closed function, then $g \circ f$ is quasi rw-closed function.*

Proof. Obvious. ■

Furthermore, we have the following Theorem:

Theorem 4.3. *Let $f : X \to Y$ and $g : Y \to Z$ be any two functions. Then*
(i) *If f is rw-closed and g is quasi rw-closed, then $g \circ f$ is closed.*
(ii) *If f is quasi rw-closed and g is rw-closed, then $g \circ f$ is rw^*-closed.*
(iii) *If f is rw^*-closed and g is quasi rw-closed, then $g \circ f$ is quasi rw-closed.*

Proof. Obvious. ■

Theorem 4.4. *Let $f : X \to Y$ and $g : Y \to Z$ be two functions such that $g \circ f : X \to Z$ is quasi rw-closed.*
(i) *If f is rw-irresolute surjective, then g is closed.*
(ii) *If g is rw-continuous injective, then f is rw^*-closed.*

Proof. (i) Suppose that F is an arbitrary closed set in Y. As f is rw-irresolute, $f^{-1}(F)$ is rw-closed in X. Since $g \circ f$ is quasi rw-closed and f is surjective, $g \circ f(f^{-1}(F)) = g(F)$ which is closed in Z. This implies g is a closed function.

(ii) Suppose F is any rw-closed set in X. Since $g \circ f$ is quasi rw-closed, $(g \circ f)(F)$ is closed in Z. Again g is a rw-continuous injective function, $g^{-1}(g \circ f(F)) = f(F)$, which is rw-closed in Y. This shows that f is rw^*-closed. ■

Theorem 4.5. *Let X and Y be topological spaces. Then the function $g : X \to Y$ is a quasi rw-closed if and only if $g(X)$ is closed in Y and $g(V) \backslash g(X \backslash V)$ is open in $g(X)$ whenever V is rw-open in X.*

Proof. Necessity: Suppose $g : X \to Y$ is a quasi rw-closed function. Since X is a rw-closed, $g(X)$ is closed in Y and $g(V) \backslash g(X \backslash V) = g(V) \cap g(X) \backslash g(X \backslash V)$ is open in $g(X)$ when V is rw-open in X.
Sufficiency: Suppose $g(X)$ is closed in Y, $g(V) \backslash g(X \backslash V)$ is open in $g(X)$ when V is rw-open in X, and let C be closed in X. Then $g(C) = g(X) \backslash g(X \backslash C) \backslash g(C)$ is closed in $g(X)$ and hence, closed in Y. ■

Corollary 4.1. *Let X and Y be topological spaces. Then a surjective function $g : X \to Y$ is quasi rw-closed iff $g(V) \backslash g(X \backslash V)$ is open in Y whenever U is rw-open in X.*

Proof. Obvious. ■

Corollary 4.2. *Let X and Y be topological spaces and let $g : X \to Y$ be a rw-continuous quasi rw-closed surjective function. Then the topology on Y is $\{g(V) \backslash g(X \backslash V) : V$ is rw-open in $X\}$.*

Proof. Let W be open in Y. Then $g^{-1}(W)$ is rw-open in X, and $g(g^{-1}(W)) \backslash g(X \backslash g^{-1}(W)) = W$. Hence, all open sets in Y are of the form $g(V) \backslash g(X \backslash V)$, V is rw-open in X. On the otherhand, all sets of the form $g(V) \backslash g(X \backslash V)$, V is rw-open in X, are open in Y from Corollary 4.1. ■

Definition 4.3. *A topological space (X, τ) is said to be* **rw-normal**, *if for any pair of disjoint rw-closed subsets F_1 and F_2 of X, there exist disjoint open sets U and V such that $F_1 \subset U$ and $F_2 \subset V$.*

Theorem 4.6. *Let X and Y be topological spaces with X is rw-normal. If $g : X \to Y$ is rw-continuous quasi rw-closed surjective function. Then Y is normal.*

Proof. Let K and M be disjoint closed subsets of Y. Then $g^{-1}(K)$, $g^{-1}(M)$ are disjoint rw-closed subsets of X. Since X is rw-normal, there exists disjoint open sets V and W such that $g^{-1}(K) \subset V$ and $g^{-1}(M) \subset W$. Then $K \subset g(V) \backslash g(X \backslash V)$ and $M \subset g(W) \backslash g(X \backslash W)$. Further by Corollary 4.1., $g(V) \backslash g(X \backslash V)$ and $g(W) \backslash g(X \backslash W)$ are open sets in Y and clearly $(g(V) \backslash g(X \backslash V)) \cap g(W) \backslash g(X \backslash W)) = \phi$. This shows that Y is normal. ■

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