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$rg\alpha$-Interior and $rg\alpha$-Closure in Topological Spaces

A. Vadivel and K. Vairamanickam

Department of Mathematics, Annamalai University
Annamalainagar - 608 002, India
avmaths@gmail.com
kvaufeat@gmail.com

Abstract

In this paper, we introduce $rg\alpha$-interior, $rg\alpha$-closure and some of its basic properties. We define $\tau_{rg\alpha}$ and prove that it forms a topology on X.

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1 Introduction and Preliminaries

N. Levine[8] introduced generalized closed sets in general topology as a generalization of closed sets. This concept was found to be useful and many results in general topology were improved. Many researchers like Balachandran, Sundaram and Maki[3], Bhattacharyya and Lahiri[4], Arockiarani[1], Dunham[6], Gnanambal[7], Malghan[12], Palaniappan and Rao[16], Park[17], Arya and Gupta[2] and Devi[5] have worked on generalized closed sets, their generalizations and related concepts in general topology. In this paper, the notion of $rg\alpha$-interior is defined and some of its basic properties are studied. Also we introduce the concept of $rg\alpha$-closure in topological spaces using the notions of $rg\alpha$-closed sets, and we obtain some related results. We define $\tau_{rg\alpha}$ and prove that it forms a topology on X. For any $A \subset X$, it is proved that the complement of $rg\alpha$-interior of A is the $rg\alpha$-closure of the complement of A.

Throughout the paper, X and Y denote the topological spaces (X, τ) and (Y, σ) respectively and on which no separation axioms are assumed unless otherwise explicitly stated. For any subset A of a space (X, τ), the closure of A, interior of A, w-interior of A, w-closure of A, gpr-interior of A, gpr-closure

of A, α-closure of A, α-interior of A and the complement of A are denoted by $cl\,(A)$ or τ-$cl\,(A)$, $int\,(A)$ or τ-$int\,(A)$, w-$int\,(A)$, w-$cl\,(A)$, gpr-$int\,(A)$, gpr-$cl\,(A)$, α-$int\,(A)$, α-$cl\,(A)$ and A^c or $X - A$ respectively. Sometimes (X, τ) is denoted by simply X if there is no confusion arise.
We begin with the following definitions.

Definition 1.1. *A subset A of a space X is called*
1) a **preopen set** *[13] if $A \subseteq intcl\,(A)$ and a* **preclosed set** *if $clint\,(A) \subseteq A$.*
*2) a α-***open set** *[15] if $A \subseteq intclint\,(A)$ and a α-***closed set** *if $clintcl\,(A) \subseteq A$.*
3) a **regular open set** *[20] if $A = intcl\,(A)$ and a* **regular closed set** *if $A = clint\,(A)$.*

The intersection of all preclosed (resp. α-closed) subsets of X containing A is called pre-closure (resp. α-closure) of A and is denoted by $pcl(A)$(resp. α-$cl\,(A)$).

Definition 1.2. *A subset A of a space X is called*
1) **generalized α-closed set** *(briefly, $g\alpha$-closed) [10] if $\alpha cl\,(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in X.*
2) **α-generalized closed set** *(briefly, αg-closed) [11] if $\alpha cl\,(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.*
3) **regular generalized closed set** *(briefly, rg-closed) [16] if $cl\,(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X.*
4) **generalized preclosed set** *(briefly, gp-closed) [9] if $pcl\,(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.*
5) **weakely generalized closed set** *(briefly, wg-closed) [14] if $clint\,(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.*
6) **weakely closed set** *(briefly, w-closed) [19] if $cl\,(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open in X.*

The complements of the above mentioned closed sets are their respective open sets.

Definition 1.3. *A subset A of a space X is called* **regular α-open set** *(briefly, $r\alpha$-open) [21] if there is a regular open set U such that $U \subset A \subset \alpha cl(U)$.*

Definition 1.4. *A subset A of a space X is called a* **regular generalized α-closed set** *(briefly, $rg\alpha$-closed) [21] if $\alpha cl\,(A) \subset U$ whenever $A \subset U$ and U is regular α-open in X. We denote the set of all $rg\alpha$-closed sets in X by $RG\alpha C\,(X)$.*

We shall use abbreviated from $rg\alpha$-nbhd, for the word $rg\alpha$-neighbourhood.

Definition 1.5. *Let X be a topological space and let $x \in X$. A subset N of X is said to be $rg\alpha$-***nbhd of** *x [21] if there exists a $rg\alpha$-open set G such that $x \in G \subset N$.*

2 $rg\alpha$-interior and $rg\alpha$-closure

We introduce the following definitions

Definition 2.1. *Let A be a subset of X. A point $x \in A$ is said to be* **$rg\alpha$-interior point of** *A if A is a $rg\alpha$-nbhd of x. The set of all $rg\alpha$-interior points of A is called the $rg\alpha$-interior of A and is denoted by $rg\alpha$-int(A).*

Theorem 2.1. *If A be a subset of X. Then $rg\alpha\text{-}int(A) = \cup\{G : G$ is $rg\alpha$-open, $G \subset A\}$.*

Proof. Let A be a subset of X.
$x \in rg\alpha\text{-}int\,(A) \Leftrightarrow x$ is a $rg\alpha$-interior point of A.
$\Leftrightarrow A$ is a $rg\alpha$-nbhd of point x.
$\Leftrightarrow$ there exists $rg\alpha$-open set G such that $x \in G \subset A$.
$\Leftrightarrow x \in \cup\{G : G$is $rg\alpha$-open, $G \subset A\}$.
Hence $rg\alpha\text{-}int\,(A) = \cup\{G : G$ is $rg\alpha$-open, $G \subset A\}$. ∎

Theorem 2.2. *Let A and B be subsets of X. Then*
(i) $rg\alpha\text{-}int(X) = X$ and $rg\alpha\text{-}int(\phi) = \phi$.
(ii) $rg\alpha\text{-}int(A) \subset A$.
(iii) If B is any $rg\alpha$-open set contained in A, then $B \subset rg\alpha\text{-}int(A)$.
(iv) If $A \subset B$, then $rg\alpha\text{-}int(A) \subset rg\alpha\text{-}int(B)$.
(v) $rg\alpha\text{-}int(rg\alpha\text{-}int(A)) = rg\alpha\text{-}int(A)$.

Proof. (i) Since X and ϕ are $rg\alpha$-open sets, by Theorem 2.1. $rg\alpha\text{-}int\,(X) = \cup\{G : G$ is $rg\alpha$-open, $G \subset X\} = X \cup \{$all $rg\alpha$-open sets $\} = X$. That is $rg\alpha\text{-}int\,(X) = X$. Since ϕ is the only $rg\alpha$-open set contained in ϕ, $rg\alpha\text{-}int\,(\phi) = \phi$.
(ii) Let $x \in rg\alpha\text{-}int\,(A) \Rightarrow x$ is a $rg\alpha$-interior point of A.
$\Rightarrow A$ is a $rg\alpha$-nbhd of x.
$\Rightarrow x \in A$.
Thus $x \in rg\alpha\text{-}int\,(A) \Rightarrow x \in A$. Hence $rg\alpha\text{-}int\,(A) \subset A$.
(iii) Let B be any $rg\alpha$-open sets such that $B \subset A$. Let $x \in B$, then since B is a $rg\alpha$-open set contained in A. x is a $rg\alpha$-interior point of A. That is $x \in rg\alpha\text{-}int\,(A)$. Hence $B \subset rg\alpha\text{-}int\,(A)$.
(iv) Let A and B be subsets of X such that $A \subset B$. Let $x \in rg\alpha\text{-}int\,(A)$. Then x is a $rg\alpha$-interior point of A and so A is $rg\alpha$-nbhd of x. Since $B \supset A$, B is also a $rg\alpha$-nbhd of x. This implies that $x \in rg\alpha\text{-}int\,(B)$. Thus we have shown that $x \in rg\alpha\text{-}int\,(A) \Rightarrow x \in rg\alpha\text{-}int\,(B)$. Hence $rg\alpha\text{-}int\,(A) \subset rg\alpha\text{-}int\,(B)$.
(v) Let A be any subset of X. By the definition of $rg\alpha$-interior, $rg\alpha\text{-}int(A) = \cap\{F : A \subset F \in RG\alpha C(X)\}$, if $A \subset F \in RG\alpha C(X)$, then $rg\alpha\text{-}int(A) \subset F$. Since F is $rg\alpha$-closed set containing $rg\alpha\text{-}int(A)$, by (iii) $rg\alpha\text{-}int(rg\alpha\text{-}int(A)) \subset F$. Hence $rg\alpha\text{-}int(rg\alpha\text{-}int(A)) \subset \cap\ \{F : A \subset F \in RG\alpha C(X)\} = rg\alpha\text{-}cl(A)$. That is $rg\alpha\text{-}int(rg\alpha\text{-}int(A)) = rg\alpha\text{-}int(A)$. ∎

Theorem 2.3. *If a subset A of space X is $rg\alpha$-open, then $rg\alpha\text{-}int\,(A) = A$.*

Proof. Let A be $rg\alpha$-open subset of X. We know that $rg\alpha\text{-}int\,(A) \subset A$. Also, A is $rg\alpha$-open set contained in A. From Theorem 2.2. (iii) $A \subset rg\alpha\text{-}int\,(A)$. Hence $rg\alpha\text{-}int\,(A) = A$. ■

The converse of the above Theorem need not be true, as seen from the following example.

Example 2.1 *Let $X = \{a, b, c, d, e\}$ with topology $\tau = \{X, \phi, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\}$.*

Then $RG\alpha O\,(X) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{a, e\}, \{a, d\}, \{d, e\}, \{b, c\}, \{a, d, e\}, \{a, b, d, e\}, \{a, c, d, e\}\}$. Note that $rg\alpha\text{-}int\,(\{a, b\}) = \{a\} \cup \{b\} \cup \phi = \{a, b\}$, but $\{a, b\}$ is not a $rg\alpha$-open set in X.

Theorem 2.4. *If A and B are subsets of X, then $rg\alpha\text{-}int\,(A) \cup rg\alpha\text{-}int\,(B) \subset rg\alpha\text{-}int\,(A \cup B)$.*

Proof. We know that $A \subset A \cup B$ and $B \subset A \cup B$. We have, by Theorem 2.2. (iv), $rg\alpha\text{-}int\,(A) \subset rg\alpha\text{-}int\,(A \cup B)$ and $rg\alpha\text{-}int\,(B) \subset rg\alpha\text{-}int\,(A \cup B)$. This implies that $rg\alpha\text{-}int\,(A) \cup rg\alpha\text{-}int\,(B) \subset rg\alpha\text{-}int\,(A \cup B)$. ■

Theorem 2.5. *If A and B are subsets of space X, then $rg\alpha\text{-}int\,(A \cap B) = rg\alpha\text{-}int\,(A) \cap rg\alpha\text{-}int\,(B)$.*

Proof. We know that $A \cap B \subset A$ and $A \cap B \subset B$. We have, by Theorem 2.2. (iv), $rg\alpha\text{-}int\,(A \cap B) \subset rg\alpha\text{-}int\,(A)$ and $rg\alpha\text{-}int\,(A \cap B) \subset rg\alpha\text{-}int\,(B)$. This implies that $rg\alpha\text{-}int\,(A \cap B) \subset rg\alpha\text{-}int\,(A) \cap rg\alpha\text{-}int\,(B) \rightarrow (1)$. Again, let $x \in rg\alpha\text{-}int\,(A) \cap rg\alpha\text{-}int\,(B)$. Then $x \in rg\alpha\text{-}int\,(A)$ and $x \in rg\alpha\text{-}int\,(B)$. Hence x is a $rg\alpha$-interior point of each of sets A and B. It follows that A and B are $rg\alpha$-nbhds of x, so that their intersection $A \cap B$ is also a $rg\alpha$-nbhds of x. Hence $x \in rg\alpha\text{-}int\,(A \cap B)$. Thus $x \in rg\alpha\text{-}int\,(A) \cap rg\alpha\text{-}int\,(B)$ implies that $x \in rg\alpha\text{-}int\,(A \cap B)$. Therefore $rg\alpha\text{-}int\,(A) \cap rg\alpha\text{-}int\,(B) \subset rg\alpha\text{-}int\,(A \cap B) \rightarrow (2)$. From (1) and (2), we get $rg\alpha\text{-}int\,(A \cap B) = rg\alpha\text{-}int\,(A) \cap rg\alpha\text{-}int\,(B)$. ■

Theorem 2.6. *If A is a subset of X, then $int\,(A) \subset rg\alpha\text{-}int\,(A)$.*

Proof. Let A be a subset of a space X.
Let $x \in int(A) \Rightarrow x \in \cup\{G : G$ is open, $G \subset A\}$.
$\Rightarrow$ there exists an open set G such that $x \in G \subset A$.
$\Rightarrow$ there exist a $rg\alpha$-open set G such that $x \in G \subset A$, as every open set is a $rg\alpha$-open set in X.
$\Rightarrow x \in \cup\{G : G$ is $rg\alpha$-open, $G \subset A\}$.
$\Rightarrow x \in rg\alpha\text{-}int\,(A)$.
Thus $x \in int\,(A) \Rightarrow x \in rg\alpha\text{-}int\,(A)$. Hence $int\,(A) \subset rg\alpha\text{-}int\,(A)$. ■

Remark 2.1. *Containment relation in the above Theorem 2.6. may be proper as seen from the following example.*

Example 2.2 *Let $X = \{a, b, c\}$ with topology $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Then $RG\alpha O(X) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}\}$. Let $A = \{a, c\}$. Now $rg\alpha$-int $(A) = \{a, c\}$ and int $(A) = \{a\}$. It follows that int $(A) \subset rg\alpha$-int (A) and int $(A) \neq rg\alpha$-int (A).*

Theorem 2.7. *If A is a subset of X, then w-int$(A) \subset rg\alpha$-int(A), where w-int(A) is given by w-int$(A) = \cup\{G : G$ is a w-open, $G \subset A\}$.[18]*

Proof. Let A be a subset of a space X.
Let $x \in w\text{-}int(A) \Rightarrow x \in \cup\{G \subset X : G$ is a w-open, $G \subset A\}$.
$\Rightarrow$ there exists a w-open set G such that $x \in G \subset A$.
$\Rightarrow$ there exists a $rg\alpha$- open set G such that, $x \in G \subset A$, as every w-open set is a $rg\alpha$-open set in X.
$\Rightarrow x \in \cup\{G \subset X : G$ is a $rg\alpha$-open, $G \subset A\}$.
$\Rightarrow x \in rg\alpha\text{-}int\,(A)$.
Thus $x \in w\text{-}int\,(A) \Rightarrow x \in rg\alpha\text{-}int\,(A)$. Hence $w\text{-}int\,(A) \subset rg\alpha\text{-}int\,(A)$. ■

Remark 2.2. *Containment relation in the above Theorem 2.7. may be proper as seen from the following example.*

Example 2.3 *Let $X = \{a, b, c\}$ with topology $\tau = \{X, \phi, \{a\}\}$, Then $RG\alpha O(X) = P(X)$ and $WO(X) = \{X, \phi, \{a\}\}$. Let $A = \{a, b\}$. Then $rg\alpha$-int$(A) = \{a, b\}$ and w-int $(A) = \{a\}$. It follows that w-int $(A) \subset rg\alpha$-int (A) and w-int$(A) \neq rg\alpha$-int (A).*

Theorem 2.8. *If A is a subset of X, then $rg\alpha$-int $(A) \subset gpr$-int (A), where gpr-int (A) is given by gpr-int $(A) = \cup\{G \subset X : G$ is a gpr-open, $G \subset A\}$.*

Proof. Let A be a subset of a space X.
Let $x \in rg\alpha\text{-}int\,(A) \Rightarrow x \in \cup\{G \subset X : G$ is a $rg\alpha$-open, $G \subset A\}$.
$\Rightarrow$ there exists a $rg\alpha$-open set G such that $x \in G \subset A$.
$\Rightarrow$ there exists a gpr-open set G such that $x \in G \subset A$, as every $rg\alpha$-open set is gpr-open set in X.
$\Rightarrow x \in \cup\{G \subset X : G$ is a gpr-open, $G \subset A\}$.
$\Rightarrow x \in gpr\text{-}int\,(A)$.
Thus $x \in rg\alpha\text{-}int\,(A) \Rightarrow x \in gpr\text{-}int\,(A)$.
Hence $rg\alpha\text{-}int\,(A) \subset gpr\text{-}int\,(A)$. ■

Anlogous to closure in a space X, we define $rg\alpha$-closure in a space X as follows.

Definition 2.2. *Let A be a subset of a space X. We define the $rg\alpha$-closure of A to be the intersection of all $rg\alpha$-closed sets containing A. In symbols, $rg\alpha\text{-}cl\,(A) = \cap\{F : A \subset F \in RG\alpha C\,(X)\}$.*

Theorem 2.9. *If A and B are subsets of a space X . Then*
(i) $rg\alpha\text{-}cl\,(X) = X$ and $rg\alpha\text{-}cl\,(\phi) = \phi$. (ii)$A \subset rg\alpha\text{-}cl\,(A)$.
(iii) If B is any $rg\alpha$-closed set containing A, then $rg\alpha\text{-}cl\,(A) \subset B$.
(iv) If $A \subset B$ then $rg\alpha\text{-}cl\,(A) \subset rg\alpha\text{-}cl\,(B)$.
(v) $rg\alpha\text{-}cl(A) = rg\alpha\text{-}cl(rg\alpha\text{-}cl(A))$.

Proof. (i) By the definition of $rg\alpha$-closure, X is the only $rg\alpha$-closed set containing X. Therefore $rg\alpha\text{-}cl\,(X) =$ Intersection of all the $rg\alpha$-closed sets containing X. $= \cap\,\{X\} = X$. That is $rg\alpha\text{-}cl\,(X) = X$. By the definition of $rg\alpha$-closure, $rg\alpha\text{-}cl\,(\phi) =$ Intersection of all the $rg\alpha$-closed sets containing ϕ $= \phi\cap$ any $rg\alpha$-closed sets containing $\phi = \phi$. That is $rg\alpha\text{-}cl\,(\phi) = \phi$.
(ii) By the definition of $rg\alpha$-closure of A , it is obvious that $A \subset rg\alpha\text{-}cl\,(A)$.
(iii) Let B be any $rg\alpha$-closed set containing A. Since $rg\alpha\text{-}cl\,(A)$ is the intersection of all $rg\alpha$-closed sets containing A, $rg\alpha\text{-}cl\,(A)$ is contained in every $rg\alpha$-closed set containing A. Hence in particular $rg\alpha\text{-}cl\,(A) \subset B$.
(iv) Let A and B be subsets of X such that $A \subset B$. By the definition of $rg\alpha$-closure, $rg\alpha\text{-}cl\,(B) = \cap\,\{F : B \subset F \in RG\alpha C\,(X)\}$. If $B \subset F \in RG\alpha C\,(X)$, then $rg\alpha\text{-}cl\,(B) \subset F$. Since $A \subset B$, $A \subset B \subset F \in RG\alpha C\,(X)$, we have $rg\alpha\text{-}cl\,(A) \subset F$. There fore $rg\alpha\text{-}cl\,(A) \subset \cap\,\{F : B \subset F \in RG\alpha C\,(X)\} = rg\alpha\text{-}cl\,(B)$. That is $rg\alpha\text{-}cl\,(A) \subset rg\alpha\text{-}cl\,(B)$.
(v) Let A be any subset of X. By the definition of $rg\alpha$-closure, $rg\alpha\text{-}cl\,(A) = \cap\{F : A \subset F \in RG\alpha C\,(X)\}$, If $A \subset F \in RG\alpha C\,(X)$, then $rg\alpha\text{-}cl\,(A) \subset F$. Since F is $rg\alpha$-closed set containing $rg\alpha\text{-}cl\,(A)$, by (iii) $rg\alpha\text{-}cl(rg\alpha\text{-}cl(A)) \subset F$. Hence $rg\alpha\text{-}cl(rg\alpha\text{-}cl(A)) \subset \cap\{F : A \subset F \in RG\alpha C\,(X)\} = rg\alpha\text{-}cl\,(A)$. That is $rg\alpha\text{-}cl(rg\alpha\text{-}cl(A)) = rg\alpha\text{-}cl\,(A)$. ■

Theorem 2.10. *If $A \subset X$ is $rg\alpha$-closed, then $rg\alpha\text{-}cl\,(A) = A$.*

Proof. Let A be $rg\alpha$-closed subset of X. We know that $A \subset rg\alpha\text{-}cl\,(A)$. Also $A \subset A$ and A is $rg\alpha$-closed. By Theorem 2.9. (iii) $rg\alpha\text{-}cl\,(A) \subset A$. Hence $rg\alpha\text{-}cl\,(A) = A$.

The converse of the above Theorem need not be true as seen from the following exmaple.

Example 2.4 *Let $X = \{a, b, c, d, e\}$ with topology $\tau = \{X, \phi, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\}$.*

Then $RG\alpha C\,(X) = \{X, \phi, \{b\}, \{c\}, \{b, c\}, \{a, b, c\}, \{a, d, e\}, \{b, c, d\}, \{b, c, e\}, \{a, b, c, d\}, \{a, c, d, e\}, \{b, c, d, e\}, \{a, b, d, e\}, \{a, b, c, e\}\}$. Now $rg\alpha\text{-}cl\,(\{a\}) = \{a\}$, but $\{a\}$ is not $rg\alpha$-closed subset in X.

Theorem 2.11. *If A and B are subsets of a space X, then $rg\alpha\text{-}cl\,(A \cap B) \subset rg\alpha\text{-}cl\,(A) \cap rg\alpha\text{-}cl\,(B)$.*

Proof. Let A and B be subsets of X. Clearly $A \cap B \subset A$ and $A \cap B \subset B$. By Theorem 2.9.(iv), $rg\alpha\text{-}cl\,(A \cap B) \subset rg\alpha\text{-}cl\,(A)$ and $rg\alpha\text{-}cl\,(A \cap B) \subset rg\alpha\text{-}cl\,(B)$. Hence $rg\alpha\text{-}cl\,(A \cap B) \subset rg\alpha\text{-}cl\,(A) \cap rg\alpha\text{-}cl\,(B)$. ■

Theorem 2.12. *If A and B are subsets of a space X, then $rg\alpha\text{-}cl\,(A \cup B) = rg\alpha\text{-}cl\,(A) \cup rg\alpha\text{-}cl\,(B)$.*

Proof. Let A and B be subsets of X. Clearly $A \subset A \cup B$ and $B \subset A \cup B$. Hence $rg\alpha\text{-}cl\,(A) \cup rg\alpha\text{-}cl\,(B) \subset rg\alpha\text{-}cl\,(A \cup B)$—-(1). Now to prove $rg\alpha\text{-}cl\,(A \cup B) \subset rg\alpha\text{-}cl\,(A) \cup rg\alpha\text{-}cl\,(B)$. Let $x \in rg\alpha\text{-}cl\,(A \cup B)$ and suppose $x \notin rg\alpha\text{-}cl\,(A) \cup rg\alpha\text{-}cl\,(B)$. Then there exists $rg\alpha$-closed sets A_1 and B_1 with $A \subset A_1$, $B \subset B_1$ and $x \notin A_1 \cup B_1$. We have $A \cup B \subset A_1 \cup B_1$ and $A_1 \cup B_1$ is $rg\alpha$-closed set by the Theorem 1.6 in [21] such that $x \notin A_1 \cup B_1$. Thus $x \notin rg\alpha\text{-}cl(A \cup B)$ which is a contradiction to $x \in rg\alpha\text{-}cl(A \cup B)$. Hence $rg\alpha\text{-}cl(A \cup B) \subset rg\alpha\text{-}cl(A) \cup rg\alpha\text{-}cl(B)$—(2). From (1) and (2), we have $rg\alpha\text{-}cl(A \cup B) = rg\alpha\text{-}cl(A) \cup rg\alpha\text{-}cl(B)$. ■

Theorem 2.13. *For an $x \in X$, $x \in rg\alpha\text{-}cl(A)$ if and only if $V \cap A \neq \phi$ for every $rg\alpha$-open sets V containing x.*

Proof. Let $x \in X$ and $x \in rg\alpha\text{-}cl(A)$. To prove $V \cap A \neq \phi$ for every $rg\alpha$-open set V containing x. Prove the result by contradiction. Suppose there exists a $rg\alpha$-open set V containing x such that $V \cap A = \phi$. Then $A \subset X - V$ and $X - V$ is $rg\alpha$-closed. We have $rg\alpha\text{-}cl(A) \subset X - V$. This shows that $x \notin rg\alpha\text{-}cl(A)$, which is contradiction. Hence $V \cap A \neq \phi$ for every $rg\alpha$-open set V containing x.

Conversely, let $V \cap A \neq \phi$ for every $rg\alpha$-open set V containing x. To prove $x \in rg\alpha\text{-}cl(A)$. We prove the result by contradiction. Suppose $x \notin rg\alpha\text{-}cl(A)$. Then there exists a $rg\alpha$-closed subset F containing A such that $x \notin F$. Then $x \in X - F$ and $X - F$ is $rg\alpha$-open. Also $(X - F) \cap A = \phi$, which is a contradiction. Hence $x \in rg\alpha\text{-}cl(A)$. ■

Theorem 2.14. *If A is subset of a space X, then $rg\alpha\text{-}cl(A) \subset cl(A)$.*

Proof. Let A be a subset of a space X. By the definition of closure, $cl(A) = \cap\{F \subset X : A \subset F \in C(X)\}$. If $A \subset F \in C(X)$, then $A \subset F \in RG\alpha C(X)$, because every closed set is $rg\alpha$-closed. That is $rg\alpha\text{-}cl(A) \subset F$. Therefore $rg\alpha\text{-}cl(A) \subset \cap\{F \subset X : A \subset F \in C(X)\} = cl(A)$. Hence $rg\alpha\text{-}cl(A) \subset cl(A)$. ■

Remark 2.3. *Containment relation in the above Theorem 2.14., may be proper as seen from following example.*

Example 2.5 *Let $X = \{a, b, c\}$ with topology $\tau = \{X, \phi, \{a\}, \{a, b\}\}$. Then $rg\alpha$-$cl(\{a\}) = \{a\}$ and $cl(\{a\}) = X$. It follows that $rg\alpha$-$cl(\{a\}) \subset cl(\{a\})$ and $rg\alpha$-$cl(\{a\}) \neq cl(\{a\})$.*

Theorem 2.15. *If A is subset of a space X, then $rg\alpha$-$cl(A) \subset w$-$cl(A)$, where w-$cl(A)$ is given by w-$cl(A) = \cap\{F \subset X : A \subset F$ and F is w-closed set in $X\}$.*

Proof. Let A be a subset of X. By definition of w-closure w-$cl(A) = \cap\{F \subset X : A \subset F$ and F is w-closed set in $X\}$. If $A \subset F$ and F is w-closed subset of X, then $A \subset F \in RG\alpha C(X)$, because every w-closed is $rg\alpha$-closed subset in X. That is $rg\alpha$-$cl(A) \subset F$. Therefore $rg\alpha$-$cl(A) \subset \cap\{F \subset X : A \subset F$ and F is w-closed$\} = w$-$cl(A)$. Hence $rg\alpha$-$cl(A) \subset w$-$cl(A)$. ■

Remark 2.4. *Containment relation in the above Theorem 2.15. may be proper as seen from following example.*

Example 2.6 *Let $X = \{a, b, c\}$ with topology $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Let $A = \{a\}$. Then $rg\alpha$-$cl(A) = \{a\}$ and w-$cl(A) = \{a, c\}$. That is $rg\alpha$-$cl(A) \subset w$-$cl(A)$ and $rg\alpha$-$cl(A) \neq w$-$cl(A)$.*

Theorem 2.16. *If A is a subset of a space X, then gpr-$cl(A) \subset rg\alpha$-$cl(A)$ where gpr-$cl(A)$ is given by gpr-$cl(A) = \cap\{F \subset X : A \subset F \in GPRC(X)\}$*

Proof. Let A be a subset of X. By the definition of $rg\alpha$-closure, $rg\alpha$-$cl(A) = \cap\{F \subset X : A \subset F \in RG\alpha C(X)\}$. If $A \subset F \in RG\alpha C(X)$, then $A \subset F \in GPRC(X)$, because every $rg\alpha$-closed set is gpr-closed set. That is gpr-$cl(A) \subset F$. Therefore gpr-$cl(A) \subset \cap\{F \subset X : A \subset F \in RG\alpha C(X)\} = rg\alpha$-$cl(A)$. Hence gpr-$cl(A) \subset rg\alpha$-$cl(A)$. ■

Definition 2.3. *Let $\tau_{rg\alpha}$ be the topology on X generated by $rg\alpha$-closure in the usual manner. That is $\tau_{rg\alpha} = \{U \subset X : rg\alpha$-$cl(U^c)\} = U^c\}$.*

Theorem 2.17. *For any topology τ on X, $\tau \subset \tau_w \subset \tau_{rg\alpha}$, where $\tau_w = \{U \subset X : w$-$cl(U^c)\} = U^c\}$ [18]*

Proof. We know that $\tau \subset \tau_w$ from [18]. To prove $\tau_w \subset \tau_{rg\alpha}$. Let $U \in \tau_w$ which implies w-$cl(U^c) = U^c$, it follows that U^c is a w-closed set. Now U^c is $rg\alpha$-closed, as every w-closed set is $rg\alpha$-closed and so $rg\alpha$-$cl(U^c) = U^c$. That is $U \in \tau_{rg\alpha}$ and so $\tau_w \subset \tau_{rg\alpha}$. Hence $\tau \subset \tau_w \subset \tau_{rg\alpha}$. ■

Theorem 2.18. *Let A be any subset of X. Then*
(i) $(rg\alpha$-$int(A))^c = rg\alpha$-$cl(A^c)$ (ii) $rg\alpha$-$int(A) = (rg\alpha$-$cl(A^c))^c$
(iii)$rg\alpha$-$cl(A) = (rg\alpha$-$int(A^c))^c$

Proof. Let $x \in (rg\alpha\text{-}int(A))^c$. Then $x \notin rg\alpha\text{-}int(A)$. That is every $rg\alpha$-open set U containing x is such that $U \not\subset A$. That is every $rg\alpha$-open set U containing x is such that $U \cap A^c \neq \phi$. By Theorem 2.13., $x \in rg\alpha\text{-}cl(A^c)$ and therefore $(rg\alpha\text{-}int(A))^c \subset rg\alpha\text{-}cl(A^c)$. Conversely, let $x \in rg\alpha\text{-}cl(A^c)$. Then by Theorem 2.13., every $rg\alpha$-open set U containing x is such that $U \cap A^c \neq \phi$. That is every $rg\alpha$-open set U containing x is such that $U \not\subset A$. This implies by Definition of $rg\alpha$- interior of A, $x \notin rg\alpha\text{-}int(A)$. That is $x \in (rg\alpha\text{-}int(A))^c$ and $rg\alpha\text{-}cl(A^c) \subset (rg\alpha\text{-}int(A))^c$. Thus $(rg\alpha\text{-}int(A))^c = rg\alpha\text{-}cl(A^c)$.
(ii) Follows by taking complements in (i).
(iii) Follows by replacing A by A^c in (i). ■

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