Int. Journal of Math. Analysis, Vol. 4, 2010, no. 12, 589 - 600

Forced Oscillation of a Class of High Order Nonlinear Partial Difference Equations

Figen Özpınar

Bolvadin Vocational School
Afyon Kocatepe University
03300 Afyonkarahisar, Turkey
fozpinar@aku.edu.tr

Zeynep Fidan Koçak

Department of Mathematics
Mugla University, Mugla, Turkey

Abstract

In this paper, some oscillation criteria for the forced nonlinear partial difference equation of the form

$$\Delta_m^r \Delta_n^h x_{m,n} + p_{m,n} f(x_{m-\tau,n-\sigma}) = q_{m,n}$$

are established.

Mathematics Subject Classification: 39A11, 34K11, 34C10

Keywords: Partial difference equation, High order PDEs, Oscillation, Forced Oscillation, Nonlinear Partial Difference Equations

1 Introduction

Partial difference equations arise in applications involving population dynamics with spatial migrations, chemical reactions, mathematical physics and finite difference schemes. The qualitative analysis of partial difference equations has received much attention. Recently, there are many papers that devoted to the development of qualitative theory of partial difference equations, for instance see [2, 4–6].

Y. G. Sun and S. H. Saker had studied the oscillation of the following ordinary difference equation

$$\Delta^m x_n + q_n f(x_{n-\tau}) = e_n$$

(see [3]).

In this paper, we consider the forced nonlinear partial difference equation of the form

$$\Delta_m^r \Delta_n^h x_{m,n} + p_{m,n} f(x_{m-\tau,n-\sigma}) = q_{m,n} \tag{1}$$

where $m, n, \tau, \sigma \in N_0$, $r, h \in N_1$, $N_a = \{a, a+1, a+2, \dots\}$, $p_{m,n}$ and $q_{m,n}$ are double real sequences defined on $N_0^2, xf(x) > 0$ for $x \neq 0$, which includes the special case $f(x) = |x|^\lambda sgnx$ for $\lambda > 0$. The forward partial differences Δ_m and Δ_n are defined as usual, i.e. $\Delta_m x_{m,n} = x_{m+1,n} - x_{m,n}$ and $\Delta_n x_{m,n} = x_{m,n+1} - x_{m,n}$. The high-order partial differences for any positive integers r and h are defined as $\Delta_m^r x_{m,n} = \Delta_m(\Delta_m^{r-1} x_{m,n})$, $\Delta_m^0 x_{m,n} = x_{m,n}$, $\Delta_n^h x_{m,n} = \Delta_n(\Delta_n^{h-1} x_{m,n})$ and $\Delta_n^0 x_{m,n} = x_{m,n}$.

By a solution of (1), we mean a nontrivial double sequence $\{x_{m,n}\}$ which is defined for $m \geq -\tau$ and $n \geq -\sigma$ and satisfies (1) for $m \geq 0$, $n \geq 0$. A solution $\{x_{m,n}\}$ of (1) is said to be eventually positive (or negative) if $x_{m,n} > 0$ (or $x_{m,n} < 0$) for all large m and n. It is said to be oscillatory if it is neither eventually positive nor eventually negative.

2 Preparatory Lemmas

Define two factorial functions $\alpha(m,s)$ and $\beta(n,t)$ as follows:

$$\alpha(m,s) = \alpha_0(m,s) = (m-s)^{(k)} = (m-s)(m-s+1)\dots(m-s+k-1), \; k \geq r \tag{2}$$

and

$$\alpha_i(m,s) = (-1)^i \Delta_s^i \alpha(m,s) \qquad i = 0, 1, \dots, r. \tag{3}$$

Hence, we have

$$\begin{cases} \alpha(m,s) = 0, & m \leq s \leq m+r-1 \\ \alpha_i(m+i+1, m+r) = 0, & i = 0, 1, \dots, r-1 \\ \alpha_r(m,s) \geq 0, & 0 \leq s \leq m-1 \end{cases} \tag{4}$$

and

$$\lim_{m\to\infty} \frac{\alpha_i(m+i+1, m_0)}{\alpha(m+1, m_0)} = 0 \tag{5}$$

for some $m_0 \geq 0$ and $i = 1, 2, \dots, r$.

By the same way;

$$\beta(n,t) = \beta_0(n,t) = (n-t)^{(l)} = (n-t)(n-t+1)\dots(n-t+l-1), \quad l \geq h \tag{6}$$

and

$$\beta_j(n,t) = (-1)^j \Delta_t^j \beta(n,t), \quad j = 0, 1, \dots, h. \tag{7}$$

Then we have

$$\begin{cases} \beta\,(n,t) = 0, & n \le t \le n+h-1 \\ \beta_j\,(n+j+1, n+h) = 0, & j = 0,1,\ldots,h-1 \\ \beta_h\,(n,t) \ge 0, & 0 \le t \le n-1 \end{cases} \tag{8}$$

and

$$\lim_{n\to\infty} \frac{\beta_j\,(n+j+1, n_0)}{\beta\,(n+1, n_0)} = 0 \tag{9}$$

for some $n_0 \ge 0$ and $j = 1, 2, \ldots, h$.

The following lemma is taken from [1].

Lemma 1. *Let $F(x) = ax - bx^{\lambda}$ for $x > 0$. If $a \ge 0$, $b > 0$ and $\lambda > 1$, then $F\,(x)$ reaches its maximal*

$$F_{\max} = (\lambda - 1)\,\lambda^{\lambda/(1-\lambda)} a^{\lambda/(\lambda-1)} b^{1/(1-\lambda)}.$$

If $a > 0$, $b \ge 0$ and $0 < \lambda < 1$, then $F\,(x)$ reaches its minimal

$$F_{\min} = (\lambda - 1)\,\lambda^{\lambda/(1-\lambda)} a^{\lambda/(\lambda-1)} b^{1/(1-\lambda)}.$$

3 Main Results

Theorem 1. *Let $p_{m,n} \ge 0$ for $m, n \ge 0$. If*

$$\limsup_{m,n\to\infty} \frac{1}{\alpha\,(m+1, m_0)\,\beta\,(n+1, n_0)} \sum_{s=m_0}^{m+r-1} \sum_{t=n_0}^{n+h-1} \alpha\,(m,s)\,\beta(n,t) q_{s,t} = \infty \tag{10}$$

and

$$\liminf_{m,n\to\infty} \frac{1}{\alpha\,(m+1, m_0)\,\beta\,(n+1, n_0)} \sum_{s=m_0}^{m+r-1} \sum_{t=n_0}^{n+h-1} \alpha\,(m,s)\,\beta(n,t) q_{s,t} = -\infty \tag{11}$$

for some $m_0, n_0 \ge 0$, then every solution of (1) that satisfies

$$\limsup_{m,n\to\infty} |x_{m,n}| < \infty, \tag{12}$$

is oscillatory.

Proof. Let $x_{m,n}$ be a nonoscillatory solution of (1) that satisfies (12). Without loss of generality, we assume that $x_{m,n} > 0$, $x_{m-\tau,n-\sigma} > 0$ for $m \ge m_0 \ge 0$, $n \ge n_0 \ge 0$. First multiplying (1) by $\alpha(m,s)$ and $\beta(n,t)$ later double summing it from m_0 to $m+r-1$ and n_0 to $n+h-1$, we obtain

$$\sum_{s=m_0}^{m+r-1}\sum_{t=n_0}^{n+h-1}\alpha(m,s)\,\beta(n,t)\Delta_s^r\Delta_t^h x_{s,t}+\sum_{s=m_0}^{m+r-1}\sum_{t=n_0}^{n+h-1}\alpha(m,s)\,\beta(n,t)p_{s,t}f\left(x_{s-\tau,t-\sigma}\right)$$
$$=\sum_{s=m_0}^{m+r-1}\sum_{t=n_0}^{n+h-1}\alpha(m,s)\,\beta(n,t)q_{s,t} \tag{13}$$

Since

$$\alpha(m,s)\,\Delta_s x_{s,t}=\Delta_s\left[\alpha(m+1,s)\,x_{s,t}\right]+\alpha_1(m+1,s)\,x_{s,t}$$

and

$$\beta(n,t)\,\Delta_t x_{s,t}=\Delta_t\left[\beta(n+1,t)\,x_{s,t}\right]+\beta_1(n+1,t)\,x_{s,t},$$

in view of (2), (3), (4) and (6), (7), (8) we obtain

$$\sum_{s=m_0}^{m+r-1}\sum_{t=n_0}^{n+h-1}\alpha(m,s)\,\beta(n,t)\Delta_s^r\Delta_t^h x_{s,t}$$
$$=\sum_{s=m_0}^{m+r-1}\alpha(m,s)\,\Delta_s^r\left\{\sum_{t=n_0}^{n+h-1}\beta(n,t)\Delta_t^h x_{s,t}\right\}$$
$$=\sum_{s=m_0}^{m+r-1}\alpha(m,s)\,\Delta_s^r\left\{\sum_{t=n_0}^{n+h-1}\Delta_t\left[\beta(n+1,t)\Delta_t^{h-1}x_{s,t}\right]+\sum_{t=n_0}^{n+h-1}\beta_1(n+1,t)\Delta_t^{h-1}x_{s,t}\right\}$$
$$=\sum_{s=m_0}^{m+r-1}\alpha(m,s)\,\Delta_s^r\left\{-\beta(n+1,n_0)\Delta_t^{h-1}x_{s,n_0}+\sum_{t=n_0}^{n+h-1}\beta_1(n+1,t)\Delta_t^{h-1}x_{s,t}\right\}$$
$$=\sum_{s=m_0}^{m+r-1}\alpha(m,s)\,\Delta_s^r\Bigg\{-\beta(n+1,n_0)\Delta_t^{h-1}x_{s,n_0}+\sum_{t=n_0}^{n+h-1}\Delta_t\left[\beta_1(n+2,t)\Delta_t^{h-2}x_{s,t}\right]$$
$$+\sum_{t=n_0}^{n+h-1}\beta_2(n+2,t)\Delta_t^{h-2}x_{s,t}\Bigg\}$$
$$=\sum_{s=m_0}^{m+r-1}\alpha(m,s)\,\Delta_s^r\Bigg\{-\beta(n+1,n_0)\Delta_t^{h-1}x_{s,n_0}-\beta_1(n+2,n_0)\Delta_t^{h-2}x_{s,n_0}$$
$$+\sum_{t=n_0}^{n+h-1}\beta_2(n+2,t)\Delta_t^{h-2}x_{s,t}\Bigg\}.$$

And following similar arguments, we get

$$
\begin{aligned}
&\sum_{s=m_0}^{m+r-1}\sum_{t=n_0}^{n+h-1}\alpha\,(m,s)\,\beta(n,t)\Delta_s^r\Delta_t^h x_{s,t}\\
&=\sum_{s=m_0}^{m+r-1}\alpha\,(m,s)\,\Delta_s^r\Bigg\{-\sum_{j=0}^{h-1}\beta_j(n+j+1,n_0)\Delta_t^{h-j-1}x_{s,n_0}+\sum_{t=n_0}^{n+h-1}\beta_h(n+h,t)x_{s,t}\Bigg\}\\
&=-\sum_{j=0}^{h-1}\beta_j(n+j+1,n_0)\Delta_t^{h-j-1}\sum_{s=m_0}^{m+r-1}\alpha\,(m,s)\,\Delta_s^r x_{s,n_0}\\
&\quad+\sum_{t=n_0}^{n+h-1}\beta_h(n+h,t)\sum_{s=m_0}^{m+r-1}\alpha\,(m,s)\,\Delta_s^r x_{s,t}\\
&=-\sum_{j=0}^{h-1}\beta_j(n+j+1,n_0)\Delta_t^{h-j-1}\Bigg\{-\sum_{i=0}^{r-1}\alpha_i\,(m+i+1,m_0)\,\Delta_s^{r-i-1}x_{m_0,n_0}\\
&\quad+\sum_{s=m_0}^{m+r-1}\alpha_r\,(m+r,s)\,x_{s,n_0}\Bigg\}+\sum_{t=n_0}^{n+h-1}\beta_h(n+h,t)\Bigg\{-\sum_{i=0}^{r-1}\alpha_i\,(m+i+1,m_0)\,\Delta_s^{r-i-1}x_{m_0,t}\\
&\quad+\sum_{s=m_0}^{m+r-1}\alpha_r\,(m+r,s)\,x_{s,t}\Bigg\}\\
&=\sum_{i=0}^{r-1}\sum_{j=0}^{h-1}\alpha_i\,(m+i+1,m_0)\,\beta_j(n+j+1,n_0)\Delta_s^{r-i-1}\Delta_t^{h-j-1}x_{m_0,n_0}\\
&\quad-\sum_{s=m_0}^{m+r-1}\sum_{j=0}^{h-1}\alpha_r\,(m+r,s)\,\beta_j(n+j+1,n_0)\Delta_t^{h-j-1}x_{s,n_0}\\
&\quad-\sum_{i=0}^{r-1}\sum_{t=n_0}^{n+h-1}\alpha_i\,(m+i+1,m_0)\,\beta_h(n+h,t)\Delta_s^{r-i-1}x_{m_0,t}\\
&\quad+\sum_{s=m_0}^{m+r-1}\sum_{t=n_0}^{n+h-1}\alpha_r\,(m+r,s)\,\beta_h(n+h,t)x_{s,t}. \qquad (14)
\end{aligned}
$$

Note that $\alpha\,(m,s)\geq 0$ for $m_0\leq s\leq m+r-1$ and $\beta\,(n,t)\geq 0$ for $n_0\leq t\leq n+h-1$.

Substitute (14) into (13) and divide through by $\alpha(m+1,m_0)\,\beta(n+1,n_0)$, we have

$$\begin{aligned}&\frac{1}{\alpha(m+1,m_0)\,\beta(n+1,n_0)}\sum_{s=m_0}^{m+r-1}\sum_{t=n_0}^{n+h-1}\alpha(m,s)\,\beta(n,t)q_{s,t}\\ \geq&\sum_{i=0}^{r-1}\sum_{j=0}^{h-1}\frac{\alpha_i(m+i+1,m_0)}{\alpha(m+1,m_0)}\frac{\beta_j(n+j+1,n_0)}{\beta(n+1,n_0)}\Delta_s^{r-i-1}\Delta_t^{h-j-1}x_{m_0,n_0}\\ &-\frac{1}{\alpha(m+1,m_0)}\sum_{s=m_0}^{m+r-1}\sum_{j=0}^{h-1}\alpha_r(m+r,s)\frac{\beta_j(n+j+1,n_0)}{\beta(n+1,n_0)}\Delta_t^{h-j-1}x_{s,n_0}\\ &-\frac{1}{\beta(n+1,n_0)}\sum_{i=0}^{r-1}\sum_{t=n_0}^{n+h-1}\frac{\alpha_i(m+i+1,m_0)}{\alpha(m+1,m_0)}\beta_h(n+h,t)\Delta_s^{r-i-1}x_{m_0,t}\end{aligned}\qquad(15)$$

which yields a contradiction with (11) by applying (5) and (9). □

Example 1. *Consider the following forced partial difference equation*

$$\Delta_m^3\Delta_n x_{m,n}+\frac{n}{(m+1)^3}\left|x_{m-1,n-2}\right|^{\lambda}sgn\left(x_{m-1,n-2}\right)=m^3n^2\cos m\sin n,\qquad(16)$$

where $\lambda>0$.

By Theorem1 every solution of (16) *that satisfies* (12) *is oscillatory. We find the solution of the forced partial difference equation*(16) *for* $\lambda=0,2$

$$\Delta_m^3\Delta_n x_{m,n}+\frac{n}{(m+1)^3}\left|x_{m-1,n-2}\right|^{0,2}sgn\left(x_{m-1,n-2}\right)=m^3n^2\cos m\sin n\qquad(17)$$

as in the following Fig. 1 and Fig. 2, which illustrate our theorem.

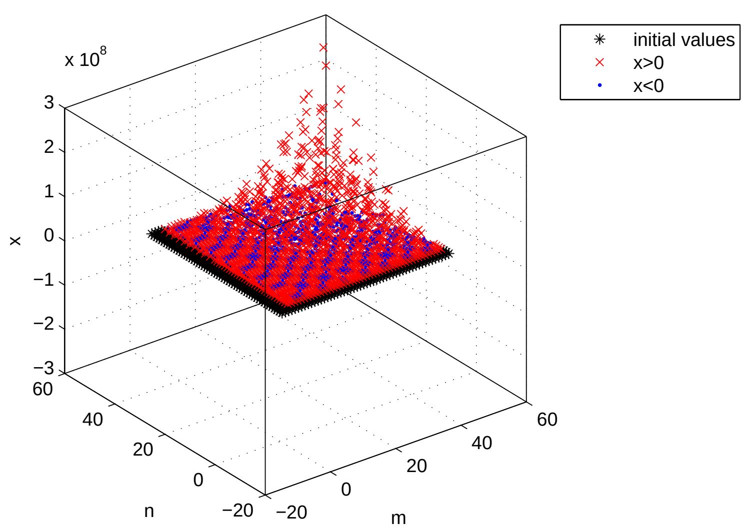

Fig. 1. Solution of (17) *with side view*

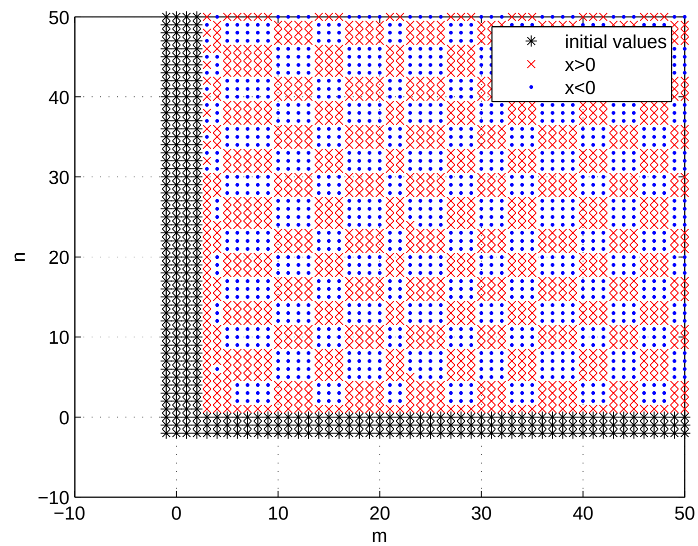

Fig. 2. Solution of (17) *with top view*

We also find the solution of the forced partial difference equation (16) *for* $\lambda = 2$

$$\Delta_m^3 \Delta_n x_{m,n} + \frac{n}{(m+1)^3} \left| x_{m-1,n-2} \right|^2 sgn\left(x_{m-1,n-2}\right) = m^3 n^2 \cos m \sin n \tag{18}$$

as we can see in Fig. 3 and Fig. 4

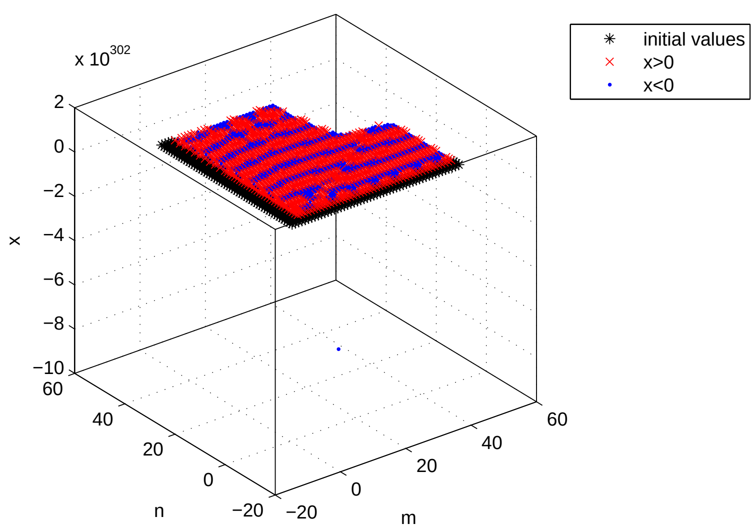

Fig. 3. Solution of (18) *with side view*

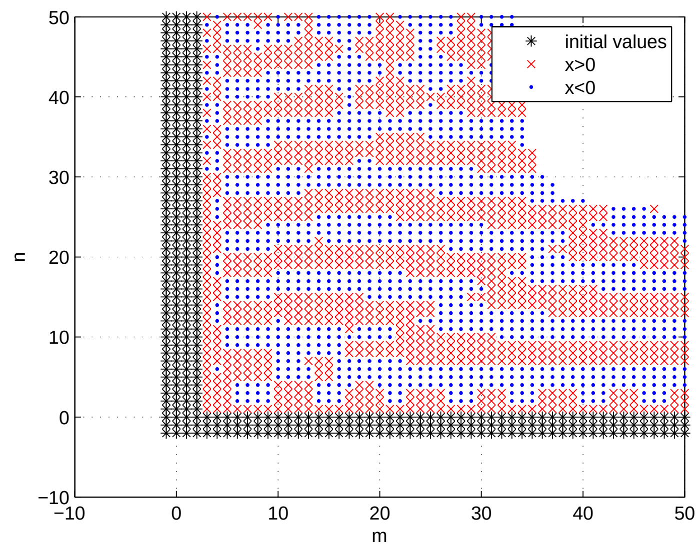

Fig. 4. Solution of (18) *with top view*

Theorem 2. *Assume that* $p_{m,n} < 0$ *for* $m, n \geq 0$ *and there exist* $c > 0$ *and* $\lambda > 1$ *such that* $|f(x)| \geq c|x|^{\lambda}$. *If*

$$\limsup_{m,n\to\infty} \frac{1}{\alpha(m+1,m_0)\beta(n+1,n_0)} \left[\sum_{s=m_0}^{m+r-1} \sum_{t=n_0}^{n+h-1} \alpha(m,s)\beta(n,t)q_{s,t} - \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} \Phi_1(m,n,s,t) \right] = \infty \tag{19}$$

$$\liminf_{m,n\to\infty} \frac{1}{\alpha(m+1,m_0)\beta(n+1,n_0)} \left[\sum_{s=m_0}^{m+r-1} \sum_{t=n_0}^{n+h-1} \alpha(m,s)\beta(n,t)q_{s,t} - \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} \Phi_1(m,n,s,t) \right] = -\infty \tag{20}$$

for some $m_0, n_0 > 0$, *where*

$$\Phi_1(m,n,s,t) = (\lambda-1)\lambda^{\lambda/(1-\lambda)} \left[\alpha_r(m+r,s)\beta_h(n+h,t)\right]^{\lambda/(\lambda-1)} \left[c\alpha(m,s)\beta(n,t)|p_{s,t}|\right]^{1/(1-\lambda)},$$

then each solution of (1) *with* $\tau = \sigma = 0$ *that satisfies*

$$\limsup_{m,n\to\infty} \frac{|x_{m,n}|}{\alpha(m+1,m_0)\beta(n+1,n_0)} < \infty, \tag{21}$$

is oscillatory.

Proof. Let $x_{m,n}$ satisfying (21) be a nonoscillatory solution of (1) with $\tau = \sigma = 0$. Without loss of generality, we assume that $x_{m,n} > 0$ and $x_{m,n} \leq k\alpha(m+1,m_0)\beta(n+1,n_0)$ for

$m \geq m_0 \geq 0$, $n \geq n_0 \geq 0$, where $k > 0$ is a constant. First multiply (1) by $\alpha(m,s)$ and $\beta(n,t)$ later double summing it from m_0 to $m+r-1$ and n_0 to $n+h-1$, we obtain

$$\begin{aligned}
&\frac{1}{\alpha(m+1,m_0)\,\beta(n+1,n_0)}\sum_{s=m_0}^{m+r-1}\sum_{t=n_0}^{n+h-1}\alpha(m,s)\,\beta(n,t)q_{s,t}\\
&=\sum_{i=0}^{r-1}\sum_{j=0}^{h-1}\frac{\alpha_i(m+i+1,m_0)}{\alpha(m+1,m_0)}\frac{\beta_j(n+j+1,n_0)}{\beta(n+1,n_0)}\Delta_s^{r-i-1}\Delta_t^{h-j-1}x_{m_0,n_0}\\
&\quad-\frac{1}{\alpha(m+1,m_0)}\sum_{s=m_0}^{m+r-1}\sum_{j=0}^{h-1}\alpha_r(m+r,s)\,\frac{\beta_j(n+j+1,n_0)}{\beta(n+1,n_0)}\Delta_t^{h-j-1}x_{s,n_0}\\
&\quad-\frac{1}{\beta(n+1,n_0)}\sum_{i=0}^{r-1}\sum_{t=n_0}^{n+h-1}\frac{\alpha_i(m+i+1,m_0)}{\alpha(m+1,m_0)}\beta_h(n+h,t)\Delta_s^{r-i-1}x_{m_0,t}\\
&\quad+\frac{1}{\alpha(m+1,m_0)\,\beta(n+1,n_0)}\sum_{s=m_0}^{m+r-1}\sum_{t=n_0}^{n+h-1}\big[\alpha_r(m+r,s)\,\beta_h(n+h,t)x_{s,t}\\
&\quad-\alpha(m,s)\,\beta(n,t)\,|p_{s,t}|\,f(x_{s,t})\big].
\end{aligned}$$

Since $\alpha(m,s) = 0$ for $s = m, m+1, \ldots, m+r-1$ and $\beta(n,t) = 0$ for $t = n, n+1, \ldots, n+h-1$, we have

$$\begin{aligned}
&\frac{1}{\alpha(m+1,m_0)\,\beta(n+1,n_0)}\sum_{s=m_0}^{m+r-1}\sum_{t=n_0}^{n+h-1}\alpha(m,s)\,\beta(n,t)q_{s,t}\\
&=\sum_{i=0}^{r-1}\sum_{j=0}^{h-1}\frac{\alpha_i(m+i+1,m_0)}{\alpha(m+1,m_0)}\frac{\beta_j(n+j+1,n_0)}{\beta(n+1,n_0)}\Delta_s^{r-i-1}\Delta_t^{h-j-1}x_{m_0,n_0}\\
&\quad-\frac{1}{\alpha(m+1,m_0)}\sum_{s=m_0}^{m+r-1}\sum_{j=0}^{h-1}\alpha_r(m+r,s)\,\frac{\beta_j(n+j+1,n_0)}{\beta(n+1,n_0)}\Delta_t^{h-j-1}x_{s,n_0}\\
&\quad-\frac{1}{\beta(n+1,n_0)}\sum_{i=0}^{r-1}\sum_{t=n_0}^{n+h-1}\frac{\alpha_i(m+i+1,m_0)}{\alpha(m+1,m_0)}\beta_h(n+h,t)\Delta_s^{r-i-1}x_{m_0,t}\\
&\quad+\frac{1}{\alpha(m+1,m_0)\,\beta(n+1,n_0)}\Bigg\{\sum_{s=0}^{r-1}\sum_{t=0}^{h-1}\alpha_r(r,s)\,\beta_h(h,t)x_{s+m,t+n}\\
&+\sum_{s=0}^{r-1}\sum_{t=n_0}^{n-1}\alpha_r(r,s)\,\beta_h(n+h,t)x_{s+m,t}+\sum_{s=m_0}^{m-1}\sum_{t=0}^{h-1}\alpha_r(m+r,s)\,\beta_h(h,t)x_{s,t+n}\Bigg\}\\
&+\frac{1}{\alpha(m+1,m_0)\,\beta(n+1,n_0)}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\big[\alpha_r(m+r,s)\,\beta_h(n+h,t)x_{s,t}\\
&-\alpha(m,s)\,\beta(n,t)\,|p_{s,t}|\,f(x_{s,t})\big].
\end{aligned}$$

Using the assumption on $f(x)$, we have that there exist a constant $M>0$ such that

$$\begin{aligned}&\frac{1}{\alpha(m+1,m_0)\,\beta(n+1,n_0)}\sum_{s=m_0}^{m+r-1}\sum_{t=n_0}^{n+h-1}\alpha(m,s)\,\beta(n,t)q_{s,t}\\ \le& M+\frac{1}{\alpha(m+1,m_0)\,\beta(n+1,n_0)}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\big[\alpha_r(m+r,s)\,\beta_h(n+h,t)x_{s,t}\\ &-c\alpha(m,s)\,\beta(n,t)\,|p_{s,t}|\,x_{s,t}^{\lambda}\big].\end{aligned}\tag{22}$$

Let $a=\alpha_r(m+r,s)\,\beta_h(n+h,t)$ and $b=c\alpha(m,s)\,\beta(n,t)\,|p_{s,t}|$. According to Lemma 1 we have

$$\begin{aligned}&\alpha_r(m+r,s)\,\beta_h(n+h,t)x_{s,t}-c\alpha(m,s)\,\beta(n,t)\,|p_{s,t}|\,x_{s,t}^{\lambda}\\ \le&(\lambda-1)\,\lambda^{\lambda/(1-\lambda)}\left[\alpha_r(m+r,s)\,\beta_h(n+h,t)\right]^{\lambda/(\lambda-1)}\left[c\alpha(m,s)\,\beta(n,t)\,|p_{s,t}|\right]^{1/(1-\lambda)}=\Phi_1(m,n,s,t)\end{aligned}$$

Thus from (22) we have

$$\frac{1}{\alpha(m+1,m_0)\,\beta(n+1,n_0)}\left[\sum_{s=m_0}^{m+r-1}\sum_{t=n_0}^{n+h-1}\alpha(m,s)\,\beta(n,t)q_{s,t}-\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\Phi_1(m,n,s,t)\right]\le M,$$

which yields a contradiction with (19). □

Theorem 3. *Assume that there exist two positive constants $c>0$ and $0<\lambda<1$ such that $|f(x)|\le c\,|x|^{\lambda}$. If*

$$\limsup_{m,n\to\infty}\frac{1}{\alpha(m+1,m_0)\,\beta(n+1,n_0)}\sum_{s=m_0}^{m+r-1}\sum_{t=n_0}^{n+h-1}\left[\alpha(m,s)\,\beta(n,t)q_{s,t}-\Phi_2(m,n,s,t)\right]=\infty\tag{23}$$

$$\liminf_{m,n\to\infty}\frac{1}{\alpha(m+1,m_0)\,\beta(n+1,n_0)}\sum_{s=m_0}^{m+r-1}\sum_{t=n_0}^{n+h-1}\left[\alpha(m,s)\,\beta(n,t)q_{s,t}-\Phi_2(m,n,s,t)\right]=-\infty\tag{24}$$

for some $m_0,n_0>0$, where

$$\Phi_2(m,n,s,t)=(\lambda-1)\,\lambda^{\lambda/(1-\lambda)}\left[\alpha_r(m+r,s)\,\beta_h(n+h,t)\right]^{\lambda/(\lambda-1)}\left[c\alpha(m,s)\,\beta(n,t)\overline{p}_{s,t}\right]^{1/(1-\lambda)},$$

and $\overline{p}_{s,t}=\max\{-p_{s,t},0\}$, then every solution of (1) with $\tau=\sigma=0$ that satisfies (12), is oscillatory.

Proof. Let $x_{m,n}$ be a nonoscillatory solution of (1) with $\tau = \sigma = 0$ that satisfies (12). Without loss of generality, we assume that $x_{m,n} > 0$ for $m \geq m_0 \geq 0$, $n \geq n_0 \geq 0$. Similar to the proof of Theorem 2, we obtain

$$\begin{aligned}
&\frac{1}{\alpha(m+1,m_0)\,\beta(n+1,n_0)} \sum_{s=m_0}^{m+r-1} \sum_{t=n_0}^{n+h-1} \alpha(m,s)\,\beta(n,t) q_{s,t} \\
\geq &\sum_{i=0}^{r-1}\sum_{j=0}^{h-1} \frac{\alpha_i(m+i+1,m_0)}{\alpha(m+1,m_0)} \frac{\beta_j(n+j+1,n_0)}{\beta(n+1,n_0)} \Delta_s^{r-i-1}\Delta_t^{h-j-1} x_{m_0,n_0} \\
&- \frac{1}{\alpha(m+1,m_0)} \sum_{s=m_0}^{m+r-1}\sum_{j=0}^{h-1} \alpha_r(m+r,s) \frac{\beta_j(n+j+1,n_0)}{\beta(n+1,n_0)} \Delta_t^{h-j-1} x_{s,n_0} \\
&- \frac{1}{\beta(n+1,n_0)} \sum_{i=0}^{r-1}\sum_{t=n_0}^{n+h-1} \frac{\alpha_i(m+i+1,m_0)}{\alpha(m+1,m_0)} \beta_h(n+h,t) \Delta_s^{r-i-1} x_{m_0,t} \\
&+ \frac{1}{\alpha(m+1,m_0)\,\beta(n+1,n_0)} \sum_{s=m_0}^{m+r-1}\sum_{t=n_0}^{n+h-1} \big[\alpha_r(m+r,s)\,\beta_h(n+h,t) x_{s,t} \\
&- c\alpha(m,s)\,\beta(n,t)\overline{p}_{s,t} x_{s,t}^{\lambda}\big].
\end{aligned} \tag{25}$$

Let $a = \alpha_r(m+r,s)\,\beta_h(n+h,t)$ and $b = c\alpha(m,s)\,\beta(n,t)\overline{p}_{s,t}$. According to Lemma 1 we get

$$\begin{aligned}
&\alpha_r(m+r,s)\,\beta_h(n+h,t) x_{s,t} - c\alpha(m,s)\,\beta(n,t)\overline{p}_{s,t} x_{s,t}^{\lambda} \\
\geq &(\lambda-1)\,\lambda^{\lambda/(1-\lambda)} \left[\alpha_r(m+r,s)\,\beta_h(n+h,t)\right]^{\lambda/(\lambda-1)} \left[c\alpha(m,s)\,\beta(n,t)\overline{p}_{s,t}\right]^{1/(1-\lambda)} \\
= &\Phi_2(m,n,s,t)
\end{aligned}$$

Then, by using (25) we get

$$\begin{aligned}
&\frac{1}{\alpha(m+1,m_0)\,\beta(n+1,n_0)} \sum_{s=m_0}^{m+r-1} \sum_{t=n_0}^{n+h-1} \left[\alpha(m,s)\,\beta(n,t) q_{s,t} - \Phi_2(m,n,s,t)\right] \\
\geq &\sum_{i=0}^{r-1}\sum_{j=0}^{h-1} \frac{\alpha_i(m+i+1,m_0)}{\alpha(m+1,m_0)} \frac{\beta_j(n+j+1,n_0)}{\beta(n+1,n_0)} \Delta_s^{r-i-1}\Delta_t^{h-j-1} x_{m_0,n_0} \\
&- \frac{1}{\alpha(m+1,m_0)} \sum_{s=m_0}^{m+r-1}\sum_{j=0}^{h-1} \alpha_r(m+r,s) \frac{\beta_j(n+j+1,n_0)}{\beta(n+1,n_0)} \Delta_t^{h-j-1} x_{s,n_0} \\
&- \frac{1}{\beta(n+1,n_0)} \sum_{i=0}^{r-1}\sum_{t=n_0}^{n+h-1} \frac{\alpha_i(m+i+1,m_0)}{\alpha(m+1,m_0)} \beta_h(n+h,t) \Delta_s^{r-i-1} x_{m_0,t},
\end{aligned}$$

which yields a contradiction with (24) by applying (5) and (9). □

References

[1] G. H. Hardy, J. E. Littlewood, G. Polya, Inequalites, second ed., Cambridge University Press, Cambridge, 1952.

[2] C. F. Li and Y. Zhou, Existence of bounded and unbounded non-oscillatory solutions for partial difference equations, Mathematical and Computer Modelling, 45 (2007), 825–833.

[3] Y. G. Sun and S. H. Saker, Forced oscillation of higher nonlinear difference equations, Applied Mathematics and Computation, 187 (2007), 868–872.

[4] B. G. Zhang and R. P. Agarwal, The oscillation and stability of delay partial difference equations, Computers Math. Applic., 45 (2003), 1253–1295.

[5] B. G. Zhang and Y. Zhou, Qualitative Analysis of Delay Partial Difference Equations, Hindawi Publishing Corporation, New York, 2007.

[6] B. G. Zhang, Y.Zhou and Y. Q. Huang, Existence of positive solutions for certain nonlinear partial difference equations, Mathemetical and Computer Modelling, 38 (2003), 331–337.

Received: October, 2009