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The Convergence of Family of Integral Operators with Positive Kernel

Sevilay KIRCI SERENBAY

Baskent University
Department of Mathematics Education
06530 Ankara, Turkey
kirci@baskent.edu.tr

Abstract

The aim of this study is to investigate the convergence of family of integral operators with positive kernel in the space $L_1(-\infty,\infty)$ generalized as ;

$$L_\lambda(f,x)=\int_{-\infty}^{\infty}\sum_{k=1}^{\infty}P_{k,\lambda}f\left(x+\alpha_{k,\lambda}t\right)K_\lambda(t)\,dt$$

within this integral operator, when $P_{k,\lambda}$ and $\alpha_{k,\lambda}$ are real numbers for $\forall\lambda\in\Lambda\subset R$, $\lambda\geq 0$ parameter

$$\sum_{k=1}^{\infty}|p_{k,\lambda}|\leq M<\infty$$

M is independent of λ and for $\forall\lambda\in\Lambda$,

$$\sum_{k=1}^{\infty}p_{k,\lambda}=1 \quad \text{and} \quad \sup_{k,\lambda}\{\alpha_{k,\lambda}\}=\alpha^*<\infty.$$

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1 Introduction

The family of the integral operators with positive kernel,

$$L_\lambda(f,x)=\int_{-\infty}^{\infty}\sum_{k=1}^{\infty}P_{k,\lambda}f\left(x+\alpha_{k,\lambda}t\right)K_\lambda(t)\,dt \tag{1}$$

had been handled and asymptotic behaviour in the class of derivative functions had been investigated in [3].

In this study, using a new modulus of continuity, the convergence of the family of the integral operators (1) to the function f in the norm of $L_1(-\infty,\infty)$ has been investigated.

In the family of the integral operators (1) which we have investigated, if λ parameter and $P_{k,\lambda}$ and $\alpha_{k,\lambda}$numbers are chosen properly, a great number of well-known operators can be obtained (see, [1], [2] and [7]). For example, if

$$P_{k,\lambda}=\begin{cases}1, k=1\\ 0, k=2,3,...,\end{cases} \quad \text{and} \quad \alpha_{k,\lambda}=\begin{cases}1, k=1\\ 0, k=2,3,...,\end{cases}$$

is taken particularly and if we select $K_\lambda(t)=\frac{\lambda}{\sqrt{\pi}}e^{-\lambda^2t^2}$ satisfying the condition $\int\limits_{-\infty}^{\infty} K_\lambda(t)\,dt=1$ then

$$L_\lambda(f,x)=\int\limits_{-\infty}^{\infty} f(x+t)\,\frac{\lambda}{\sqrt{\pi}}e^{-\lambda^2t^2}dt$$

Gauss-Weierstrass family of the integral operators is obtained and some different examples can be found [9]. This family of the integral operators that transform from L_1-space into L_1-space converge to the function f in the norm of $L_1(-\infty,\infty)$.

2 Convergence in $L_1(-\infty,\infty)$ Space

Let us first show the existence of the function which is defined following series

$$\sum_{k=1}^{\infty} p_{k,\lambda} f(x+\alpha_{k,\lambda}t)$$

for $f\in L_1(-\infty,\infty)$. Let φ be the function satisfying the inequality

$$|f(x)|<\varphi(x) \quad , \quad \varphi\in L_1.$$

It is obvious that,

$$\begin{aligned}\left|\sum_{k=1}^{\infty}p_{k,\lambda}f\left(x+\alpha_{k,\lambda}t\right)\right| &\leq \sum_{k=1}^{\infty}\left|p_{k,\lambda}f\left(x+\alpha_{k,\lambda}t\right)\right| \\ &\leq \sum_{k=1}^{\infty}\frac{\left|p_{k,\lambda}f\left(x+\alpha_{k,\lambda}t\right)\right|}{\left|\varphi\left(x+\alpha_{k,\lambda}t\right)\right|}\frac{\left|\varphi\left(x+\alpha_{k,\lambda}t\right)\right|}{\varphi\left(x\right)}\varphi\left(x\right) \\ &\leq \varphi\left(x\right)\sum_{k=1}^{\infty}\left|p_{k,\lambda}\right|\left|\frac{f\left(x+\alpha_{k,\lambda}t\right)}{\varphi\left(x+\alpha_{k,\lambda}t\right)}\right|\frac{\left|\varphi\left(x+\alpha_{k,\lambda}t\right)\right|}{\varphi\left(x\right)} \\ &\leq M\varphi\left(x\right)\mu\left(\alpha^{*}t\right)\end{aligned}$$

where $\alpha^{*}=\sup_{k,\lambda}\left\{\alpha_{k,\lambda}\right\}<\infty$ and $\sum\limits_{k=1}^{\infty}\left|p_{k,\lambda}\right|\leq M<\infty$.

Therefore the series $\sum\limits_{k=1}^{\infty}p_{k,\lambda}f\left(x+\alpha_{k,\lambda}t\right)$ is convergent for each $t\in R$. Here ,

$$\mu\left(t\right)=\sup_{\substack{-\infty<x<\infty\\ |y|\leq t}}\frac{\varphi\left(x+y\right)}{\varphi\left(x\right)}<\infty$$

additionally, due to the fact that for $f\in L_{1}\left(-\infty,\infty\right)$

$$\begin{aligned}\int_{-\infty}^{\infty}\left|\sum_{k=1}^{\infty}p_{k,\lambda}f\left(x+\alpha_{k,\lambda}t\right)\right|dx &\leq \int_{-\infty}^{\infty}M\varphi\left(x\right)\mu\left(\alpha^{*}t\right)dx \\ &\leq M\mu\left(\alpha^{*}t\right)\int_{-\infty}^{\infty}\varphi\left(x\right)dx \\ &= M\mu\left(\alpha^{*}t\right)\left\|\varphi\right\|_{L_{1}}.\end{aligned}$$

Hence the function $\sum\limits_{k=1}^{\infty}p_{k,\lambda}f\left(x+\alpha_{k,\lambda}t\right)$ belongs to $L_{1}\left(-\infty,\infty\right)$ −space.

Luzin's theorem from which we'll benefit to prove our theorems is as follows;

Theorem 2.1 (*Luzin Theorem*) *If $f\in L_{1}\left[a,b\right]$, then a continuous function φ is obtained according to each $\varepsilon>0$ number by getting the following;*

$$\left\|f-\varphi\right\|_{L_{1}\left[a,b\right]}<\varepsilon.$$

(See [8]).

Modulus of continuity used in previous studies (for example [4], [5] and [6]), such as

$$\omega_{1}\left(f,\delta\right)=\sup_{|t|\leq\delta}\int_{-\infty}^{\infty}\left|f\left(x+t\right)-f\left(x\right)\right|dx.$$

This modulus of continuity is not so benefical to be used to estimate convergence with the help of the family of the integral operator with positive kernel;

$$L_{\lambda}(f,x)=\int_{-\infty}^{\infty}\sum_{k=1}^{\infty}P_{k,\lambda}f(x+\alpha_{k,\lambda}t)\,K_{\lambda}(t)\,dt.$$

Because it is a series which is under integral sign in this operator, it is too difficult to displace the integral sign with summation sign. For this reason a more useful modulus of continuity has been defined in this paper as the following;

$$\omega^{*}_{L_1}(f,\delta)=\sup_{\left|\alpha_{k,\lambda}t\right|\leq\delta}\int_{-\infty}^{\infty}\left|\sum_{k=1}^{\infty}p_{k,\lambda}\left[f(x+\alpha_{k,\lambda}t)-f(x)\right]\right|dx.$$

The function $\omega^{*}_{L_1}(f,\delta)$ defines a modulus of continuity in $L_1(-\infty,\infty)$.

It is obvious that $\omega^{*}_{L_1}(f,\delta)$function is positive and monotone increasing function compared to δ.

Certain basic properties of this modulus of continuity has been given with below mentioned theorem.

Theorem 2.2 ***a)***

$$\lim_{\delta\to 0}\omega^{*}_{L_1}(f,\delta)=0.$$

b) *Let* $m\in N$,

$$\omega^{*}_{L_1}(f,m\delta)\leq m\,\omega^{*}_{L_1}(f,\delta).$$

c) *Where* $\lambda>0$ *is an arbitrary real number,*

$$\omega^{*}_{L_1}(f,\lambda\delta)\leq(1+\lambda)\,\omega^{*}_{L_1}(f,\delta).$$

Proof. a) Because of the fact that f and $\sum_{k=1}^{\infty}p_{k,\lambda}f(x+\alpha_{k,\lambda}t)$ functions belong to $L_1(-\infty,\infty)$ space, for $\forall\varepsilon>0$, there exists a real number a that will give the inequalities shown below as follows;

$$a)\int_{-\infty}^{-a}\left|\sum_{k=1}^{\infty}p_{k,\lambda}f(x+\alpha_{k,\lambda}t)\right|dx<\frac{\varepsilon}{4},\quad b)\int_{a}^{\infty}\left|\sum_{k=1}^{\infty}p_{k,\lambda}f(x+\alpha_{k,\lambda}t)\right|dx<\frac{\varepsilon}{4}$$

$$c)\int_{-\infty}^{-a}\left|\sum_{k=1}^{\infty}p_{k,\lambda}f(x)\right|dx<\frac{\varepsilon}{4},\quad d)\int_{a}^{\infty}\left|\sum_{k=1}^{\infty}p_{k,\lambda}f(x)\right|dx<\frac{\varepsilon}{4}.\qquad(2)$$

Furthermore for any $\delta > 0$, from (2)

$$\int_{-\infty}^{-a-\delta} \left| \sum_{k=1}^{\infty} p_{k,\lambda} f\left(x + \alpha_{k,\lambda} t\right) \right| dx < \frac{\varepsilon}{4}, \int_{a+\delta}^{\infty} \left| \sum_{k=1}^{\infty} p_{k,\lambda} f\left(x + \alpha_{k,\lambda} t\right) \right| dx < \frac{\varepsilon}{4}$$

$$\int_{-\infty}^{-a-\delta} \left| \sum_{k=1}^{\infty} p_{k,\lambda} f\left(x\right) \right| dx < \frac{\varepsilon}{4}, \int_{a+\delta}^{\infty} \left| \sum_{k=1}^{\infty} p_{k,\lambda} f\left(x\right) \right| dx < \frac{\varepsilon}{4}.$$

We can write these inequalies as follows;

$$\begin{aligned} &\int_{-\infty}^{\infty} \left| \sum_{k=1}^{\infty} p_{k,\lambda} \left(f\left(x + \alpha_{k,\lambda} t\right) - f\left(x\right) \right) \right| dx \\ \leq &\int_{-a-\delta}^{a+\delta} \left| \sum_{k=1}^{\infty} p_{k,\lambda} \left(f\left(x + \alpha_{k,\lambda} t\right) - f\left(x\right) \right) \right| dx \\ &+ \int_{-\infty}^{-a-\delta} \left| \sum_{k=1}^{\infty} p_{k,\lambda} \left(f\left(x + \alpha_{k,\lambda} t\right) - f\left(x\right) \right) \right| dx \\ &+ \int_{a+\delta}^{\infty} \left| \sum_{k=1}^{\infty} p_{k,\lambda} \left(f\left(x + \alpha_{k,\lambda} t\right) - f\left(x\right) \right) \right| dx \\ \leq &\int_{-a-\delta}^{a+\delta} \left| \sum_{k=1}^{\infty} p_{k,\lambda} \left(f\left(x + \alpha_{k,\lambda} t\right) - f\left(x\right) \right) \right| dx + \varepsilon \end{aligned}$$

then

$$\begin{aligned} &\sup_{\left|\alpha_{k,\lambda} t\right| \leq \delta} \int_{-\infty}^{\infty} \left| \sum_{k=1}^{\infty} p_{k,\lambda} \left(f\left(x + \alpha_{k,\lambda} t\right) - f\left(x\right) \right) \right| dx \\ \leq &\sup_{\left|\alpha_{k,\lambda} t\right| \leq \delta} \int_{-a-\delta}^{a+\delta} \left| \sum_{k=1}^{\infty} p_{k,\lambda} \left(f\left(x + \alpha_{k,\lambda} t\right) - f\left(x\right) \right) \right| dx + \varepsilon. \end{aligned}$$

Thus, the proof is completed if we can show that

$$\sup_{\left|\alpha_{k,\lambda} t\right| \leq \delta} \int_{-a-\delta}^{a+\delta} \left| \sum_{k=1}^{\infty} p_{k,\lambda} \left(f\left(x + \alpha_{k,\lambda} t\right) - f\left(x\right) \right) \right| dx < \varepsilon. \tag{3}$$

It has been known by Luzin's theorem that for an arbitrary number a, there exists a continuous function Ψ such that

$$\|f-\Psi\|_{L_1}<\varepsilon$$

in the interval $[-a-2\delta,a+2\delta]$. Integral defined in (3) can be seperated into three integrals as follows;

$$\begin{aligned}
&\int_{-a-\delta}^{a+\delta}\left|\sum_{k=1}^{\infty}p_{k,\lambda}\left(f\left(x+\alpha_{k,\lambda}t\right)-f\left(x\right)\right)\right|dx\\
\leq\ &\int_{-a-\delta}^{a+\delta}\sum_{k=1}^{\infty}\left|p_{k,\lambda}\right|\left|f\left(x+\alpha_{k,\lambda}t\right)-\Psi\left(x+\alpha_{k,\lambda}t\right)\right|dx\\
&+\int_{-a-\delta}^{a+\delta}\sum_{k=1}^{\infty}\left|p_{k,\lambda}\right|\left|\Psi\left(x+\alpha_{k,\lambda}t\right)-\Psi\left(x\right)\right|dx\\
&+\int_{-a-\delta}^{a+\delta}\sum_{k=1}^{\infty}\left|p_{k,\lambda}\right|\left|f\left(x\right)-\Psi\left(x\right)\right|dx\\
=\ &I_1\left(t\right)+I_2\left(t\right)+I_3\left(t\right)
\end{aligned}$$

Let us investigate each integral one by one. First, let us take $I_1\left(t\right)$ integral,

$$I_1\left(t\right)=\int_{-a-\delta}^{a+\delta}\sum_{k=1}^{\infty}\left|p_{k,\lambda}\right|\left|f\left(x+\alpha_{k,\lambda}t\right)-\Psi\left(x+\alpha_{k,\lambda}t\right)\right|dx$$

when supremum of both sides are taken according to $\left|\alpha_{k,\lambda}t\right|\leq\alpha^{*}t\leq\delta$, then

$$\begin{aligned}
&\sup_{\left|\alpha_{k,\lambda}t\right|\leq\alpha^{*}t}\int_{-a-\delta}^{a+\delta}\sum_{k=1}^{\infty}\left|p_{k,\lambda}\right|\left|f\left(x+\alpha_{k,\lambda}t\right)-\Psi\left(x+\alpha_{k,\lambda}t\right)\right|dx\\
\leq\ &\int_{-a-2\delta}^{a+2\delta}\sum_{k=1}^{\infty}\left|p_{k,\lambda}\right|\left|f\left(x\right)-\Psi\left(x\right)\right|dx\\
\leq\ &M\int_{-a-2\delta}^{a+2\delta}\left|f\left(x\right)-\Psi\left(x\right)\right|dx\\
\leq\ &\varepsilon\ M.
\end{aligned}$$

Hence, if we take limit for $\delta \to 0$,the following is obtained.

$$\lim_{\delta\to 0} \sup_{\left|\alpha_{k,\lambda}t\right|\leq\delta} I_1(t) = 0.$$

By following similar ways for I_3, we get

$$\lim_{\delta\to 0} \sup_{\left|\alpha_{k,\lambda}t\right|\leq\delta} I_3(t) = 0.$$

If we benefit from the continuity of function Ψ to prove $I_2(t)$, then we have following inequality,

$$\begin{aligned} I_2(t) &= \int_{-a-\delta}^{a+\delta} \sum_{k=1}^{\infty} |p_{k,\lambda}| \left|\Psi(x+\alpha_{k,\lambda}t) - \Psi(x)\right| dx \\ &\leq \varepsilon M(2a+2\delta). \end{aligned}$$

When we take limit for $\delta \to 0$,we obtain

$$\lim_{\delta\to 0} I_2(t) = 0.$$

In consequence,

$$\sup_{\left|\alpha_{k,\lambda}t\right|\leq\delta} \int_{-a-\delta}^{a+\delta} \left|\sum_{k=1}^{\infty} p_{k,\lambda}\left(f(x+\alpha_{k,\lambda}t) - f(x)\right)\right| dx < \varepsilon.$$

Hence the proof is completed for a).

b) and c) can be easily obtained.

Now, we are going to prove the family of the integral operators with positive kernel,

$$L_\lambda(f,x) = \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} P_{k,\lambda} f(x+\alpha_{k,\lambda}t) K_\lambda(t)\, dt$$

converges to the function $f \in L_1(-\infty,\infty)$ in the norm of $L_1(-\infty,\infty)$.

Theorem 2.3 *For each function $f \in L_1(-\infty,\infty)$ satisfying the following inequality*

$$|f(x)| < \varphi(x) \quad , \quad \varphi \in L_1(-\infty,\infty)$$

and for $\forall\delta > 0$,

$$\lim_{\lambda\to\infty} \int_{\delta}^{\infty} \mu(\alpha^* t) K_\lambda(t)\, dt = 0 \quad , \quad \lim_{\lambda\to\infty} \int_{\delta}^{\infty} K_\lambda(t)\, dt = 0$$

and when $K_\lambda(t)$ is non-negative even function satisfying the condition, $\int\limits_{-\infty}^{\infty} K_\lambda(\ t)\,dt = 1$ then for $\lambda \to \infty$,

$$\|L_\lambda f - f\|_{L_1} \to 0.$$

Where

$$L_\lambda(f,x) = \int\limits_{-\infty}^{\infty} \sum_{k=1}^{\infty} P_{k,\lambda} f(x + \alpha_{k,\lambda} t)\, K_\lambda(t)\,dt$$

is the family of the integral operators with positive kernel.

Proof. Owing to the fact that $\int\limits_{-\infty}^{\infty} K_\lambda(t)\,dt = 1$ it comes as follows $f(x)\int\limits_{-\infty}^{\infty} K_\lambda(t)\,dt = f(x)$ then

$$L_\lambda(f,x) - f(x) = \int\limits_{-\infty}^{\infty} \sum_{k=1}^{\infty} p_{k,\lambda}\left[f(x + \alpha_{k,\lambda} t) - f(x)\right] K_\lambda(t)\,dt$$

If we pass to the norm of $L_1(-\infty,\infty)$ on both sides, it can be written as follows;

$$\begin{aligned}
&\int\limits_{-\infty}^{\infty} \left|L_\lambda(f,x) - f(x)\right| dx \\
\leq\ & \int\limits_{-\infty}^{\infty} \left| \int\limits_{-\infty}^{\infty} \sum_{k=1}^{\infty} p_{k,\lambda}\left(f(x + \alpha_{k,\lambda} t) - f(x)\right) K_\lambda(t)\,dt \right| dx \\
\leq\ & \int\limits_{-\infty}^{\infty} \left(\int\limits_{-\infty}^{\infty} \left| \sum_{k=1}^{\infty} p_{k,\lambda}\left(f(x + \alpha_{k,\lambda} t) - f(x)\right) \right| dx \right) K_\lambda(t)\,dt \\
\leq\ & \int\limits_{-\infty}^{-\delta} \int\limits_{-\infty}^{\infty} \left| \sum_{k=1}^{\infty} p_{k,\lambda}\left(f(x + \alpha_{k,\lambda} t) - f(x)\right) dx \right| K_\lambda(t)\,dt \\
& + \int\limits_{-\delta}^{\delta} \int\limits_{-\infty}^{\infty} \left| \sum_{k=1}^{\infty} p_{k,\lambda}\left(f(x + \alpha_{k,\lambda} t) - f(x)\right) dx \right| K_\lambda(t)\,dt \\
& + \int\limits_{\delta}^{\infty} \int\limits_{-\infty}^{\infty} \left| \sum_{k=1}^{\infty} p_{k,\lambda}\left(f(x + \alpha_{k,\lambda} t) - f(x)\right) dx \right| K_\lambda(t)\,dt \\
=\ & I_1(t) + I_2(t) + I_3(t)
\end{aligned}$$

First let us take $I_2(t)$ integral.

$$\begin{aligned} I_2(t) &= \int_{-\delta}^{\delta}\left(\int_{-\infty}^{\infty}\left|\sum_{k=1}^{\infty} p_{k,\lambda}\left(f\left(x+\alpha_{k,\lambda}t\right)-f\left(x\right)\right)\right| dx\right) K_\lambda(t)\,dt \\ &\leq \int_{-\delta}^{\delta}\left(\sup_{\left|\alpha_{k,\lambda}t\right|\leq\delta}\int_{-\infty}^{\infty}\left|\sum_{k=1}^{\infty} p_{k,\lambda}\left(f\left(x+\alpha_{k,\lambda}t\right)-f\left(x\right)\right)\right| dx\right) K_\lambda(t)\,dt \\ &\leq \int_{-\delta}^{\delta}\omega_{L_1}(f,\delta)\,K_\lambda(t)\,dt = \omega_{L_1}(f,\delta)\int_{-\delta}^{\delta}K_\lambda(t)\,dt \leq \omega_{L_1}(f,\delta) \end{aligned}$$

where

$$I_2(t) \leq \omega_{L_1}(f,\delta).$$

If we take limits of both sides first for $\lambda\to\infty$ and then for $\delta\to 0$, then

$$\lim_{\lambda\to\infty} I_2(t) = 0.$$

Now, let us take $I_3(t)$ integral.

$$\begin{aligned} I_3(t) &= \int_{\delta}^{\infty}\left(\int_{-\infty}^{\infty}\left|\sum_{k=1}^{\infty} p_{k,\lambda}\left(f\left(x+\alpha_{k,\lambda}t\right)-f\left(x\right)\right)\right| dx\right) K_\lambda(t)\,dt \\ &\leq \int_{\delta}^{\infty}\left(\int_{-\infty}^{\infty}\left|\sum_{k=1}^{\infty} p_{k,\lambda} f\left(x+\alpha_{k,\lambda}t\right)\right| dx + \int_{-\infty}^{\infty}\left|f(x)\right| dx\right) K_\lambda(t)\,dt \\ &\leq \int_{\delta}^{\infty}\left(\int_{-\infty}^{\infty}\sum_{k=1}^{\infty}\left|p_{k,\lambda}\right|\frac{f\left(x+\alpha_{k,\lambda}t\right)}{\varphi\left(x+\alpha_{k,\lambda}t\right)}\frac{\varphi\left(x+\alpha_{k,\lambda}t\right)}{\varphi(x)}\varphi(x)\,dx\right. \\ &\quad\left.+\int_{-\infty}^{\infty}\left|f(x)\right| dx\right) K_\lambda(t)\,dt \\ &\leq \int_{\delta}^{\infty}\left(\int_{-\infty}^{\infty} M\mu\left(\alpha^{*}t\right)\varphi(x)\,dx + \int_{-\infty}^{\infty}\left|f(x)\right| dx\right) K_\lambda(t)\,dt \\ &= M\left\|\varphi\right\|_{L_1}\int_{\delta}^{\infty}\mu\left(\alpha^{*}t\right)K_\lambda(t)\,dt + \left\|f\right\|_{L_1}\int_{\delta}^{\infty}K_\lambda(t)\,dt \end{aligned}$$

in consequence, when

$$I_3(t) \leq M \|\varphi\|_{L_1} \int_{\delta}^{\infty} \mu(\alpha^* t) K_\lambda(t)\, dt + \|f\|_{L_1} \int_{\delta}^{\infty} K_\lambda(t)\, dt.$$

If the limit of both sides of the inequality above is taken for $\lambda \to \infty$,

$$\lim_{\lambda\to\infty} I_3(t) = 0$$

is obtained.

By using similar methods,

$$\lim_{\lambda\to\infty} I_1(t) = 0$$

equality is obtained for $I_1(t)$.

Hence, for $\lambda \to \infty$ we get

$$\|L_\lambda f - f\|_{L_1} \to 0.$$

The proof is completed.

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