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The Nonlinear Product of the Bessel Diamond Operator and the Bessel Klein-Gordon Operator Related to the Bessel Biharmonic Equation

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Abstract

In this article, we study the solution of the equation

$$\Diamond_B^k(\Box_B + m^2)^k u(x) = f(x, \triangle_B^{k-1}\Box_B^k(\Box_B + m^2)^k u(x))$$

where $\Diamond_B^k(\Box_B+m^2)^k$ is the product of the Bessel diamond operator and the Bessel Klein-Gordon operator, u is an unknown generalized function, f is a generalized function, m is a positive real number and k is a nonnegative integer. It found that the existence of the solution $u(x)$ of such an equation depends on the condition of f and $\triangle_B^{k-1}\Box_B^k(\Box_B+m^2)^k u(x)$. Moreover such a solution $u(x)$ related to the Bessel biharmonic equation depends on the conditions of p, q and k.

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1 Introduction

The operator $\Diamond_B^k$ was first introduced by Yildirim et al. [9] and is maned the Bessel diamond operator iterated k times defined by

$$\Diamond_B^k = \left[\left(B_{x_1} + B_{x_2} + \cdots + B_{x_p}\right)^2 - \left(B_{x_{p+1}} + \cdots + B_{x_{p+q}}\right)^2\right]^k, \qquad (1)$$

where $p + q = n$ is the dimension of the space $\mathbb{R}_n^+$, $B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}$, $2v_i = 2\alpha_i + 1$, $\alpha_i > -\frac{1}{2}$, $x_i > 0$, $i = 1, 2, \ldots, n$. The Bessel diamond operator can be expressed as the product of the operator $\triangle_B$ and $\Box_B$, that is $\Diamond_B = \Box_B\triangle_B = \triangle_B\Box_B$, where $\triangle_B$ is the Laplace-Bessel operator defined by

$$\triangle_B = B_{x_1} + B_{x_2} + \cdots + B_{x_n}, \tag{2}$$

and $\Box_B$ is the Bessel ultra-hyperbolic operator defined by

$$\Box_B = B_{x_1} + B_{x_2} + \cdots + B_{x_p} - B_{x_{p+1}} - B_{x_{p+2}} - \cdots - B_{x_{p+q}}. \tag{3}$$

In this paper, we study the nonlinear equation of the form

$$\Diamond_B^k(\Box_B + m^2)^k u(x) = f(x, \triangle_B^{k-1}\Box_B^k(\Box_B + m^2)^k u(x)) \tag{4}$$

with f defined and continuous for all $x \in \Omega \cup \partial\Omega$ where Ω ia an open subset of $\mathbb{R}_n^+$ and $\partial\Omega$ denotes the boundary of Ω. We will find the solution $u(x)$ of (4) which is unique under the condition $|f(x, \triangle_B^{k-1}\Box_B^k(\Box_B + m^2)^k u(x))| \leq N$ where N is a constant for all $x \in \Omega$ and the boundary condition $\triangle_B^{k-1}\Box_B^k(\Box_B + m^2)^k u(x) = 0$ for $x \in \partial\Omega$. If we put $k = 1$, $p = n$ and $q = 0$, then we obtain a solution of the inhomogeneous Bessel biharmonic equation.

2 Preliminary Notes

The generalized shift operator T_x^y has the following form [5],

$$T_x^y = C_v^* \int_0^\pi \cdots \int_0^\pi \varphi(s_1, \ldots, s_n) \left(\prod_{i=1}^n \sin^{2v_i - 1}\theta_i \right) d\theta_1 \cdots d\theta_n,$$

where $s_i^2 = x_i^2 + y_i^2 - 2x_iy_i\cos\theta_i,\ x, y \in \mathbb{R}_n^+$ and $C_v^* = \prod_{i=1}^n \frac{\Gamma(v_i+1)}{\Gamma(\frac{1}{2})\Gamma(v_i)}$. We remark that this shift operator is closely connected with the Bessel differential operator [5],

$$\begin{aligned} \frac{d^2\varphi}{dx_i^2} + \frac{2v_i}{x_i}\frac{d\varphi}{dx_i} &= \frac{d^2\varphi}{dy_i^2} + \frac{2v_i}{y_i}\frac{d\varphi}{dy_i}, \\ \varphi(x_i, 0) &= f(x), \\ \varphi_{y_i}(x_i, 0) &= 0, \end{aligned}$$

where $x_i, y_i \in \mathbb{R}_n^+$ for $i = 1, 2, \ldots, n$. The convolution operator determined by the T_x^y is as follows

$$(f * \varphi)(x) = \int_{\mathbb{R}_n^+} f(y) T_x^y \varphi(x) \left(\prod_{i=1}^n y_i^{2v_i} \right) dy. \tag{5}$$

Convolution in (5) is known as a B-convolution. We note the following properties of the B-convolution and the generalized shift operator,

(a) $T_x^y \cdot 1 = 1$.

(b) $T_x^0 \cdot f(x) = f(x)$.

(c) If $f(x), g(x) \in C(\mathbb{R}_n^+)$, $g(x)$ is a bounded function for $x \in \mathbb{R}_n^+$ and

$$\int_{\mathbb{R}_n^+} |f(x)| \left(\prod_{i=1}^{n} x_i^{2v_i} \right) dx < \infty,$$

then

$$\int_{\mathbb{R}_n^+} T_x^y f(x) g(y) \left(\prod_{i=1}^{n} y_i^{2v_i} \right) dy = \int_{\mathbb{R}_n^+} f(y) T_x^y g(x) \left(\prod_{i=1}^{n} y_i^{2v_i} \right) dy.$$

(d) From (c), we have the following equality for $g(x) = 1$,

$$\int_{\mathbb{R}_n^+} T_x^y f(x) \left(\prod_{i=1}^{n} y_i^{2v_i} \right) dy = \int_{\mathbb{R}_n^+} f(y) \left(\prod_{i=1}^{n} y_i^{2v_i} \right) dy.$$

(e) $(f * g)(x) = (g * f)(x)$.

The Fourier-Bessel transformation and its inverse transformation are defined as follows [8],

$$(F_B f)(x) = C_v \int_{\mathbb{R}_n^+} f(y) \left(\prod_{i=1}^{n} J_{v_i - \frac{1}{2}}(x_i y_i) y_i^{2v_i} \right) dy,$$

$$(F_B^{-1} f)(x) = (F_B f)(-x), \ C_v = \left(\prod_{i=1}^{n} 2^{v_i - \frac{1}{2}} \Gamma\left(v_i + \frac{1}{2}\right) \right)^{-1},$$

where $J_{v_i - \frac{1}{2}}(x_i y_i)$ is the normalized Bessel function which is the eigenfunction of the Bessel differential operator. There are following equalities for Fourier-Bessel transformation [8],

$$F_B \delta(x) = 1$$

and

$$F_B(f * g)(x) = F_B f(x) \cdot F_B g(x).$$

Lemma 2.1 *Given the equation* $\triangle_B^k u(x) = \delta(x)$ *for* $x \in \mathbb{R}_n^+$, *where* $\triangle_B^k$ *is the Laplace-Bessel operator iterated k-times defined by* (2). *Then* $u(x) = (-1)^k S_{2k}(x)$ *is an elementary solution of the operator* $\triangle_B^k$, *where*

$$S_{2k}(x) = \frac{2^{n+2|v|-4k} \Gamma(\frac{n+2|v|-2k}{2})}{\Pi_{i=1}^{n} 2^{v_i - \frac{1}{2}} \Gamma(v_i + \frac{1}{2}) \Gamma(k)} |x|^{2k-n-2|v|}. \tag{6}$$

The proof of this Lemma is given in [9].

Lemma 2.2 *Given the equation* $\Box_B^k u(x) = \delta(x)$ *for* $x \in \Gamma_+ = \{x \in \mathbb{R}_n^+ : x_1 > 0, x_2 > 0, \cdots, x_n > 0$ *and* $V > 0\}$, *where* $\Box_B^k$ *is the Bessel-ultra-hyperbolic operator iterated k-times defined by* (3). *Then* $u(x) = R_{2k}(x)$ *is an elementary solution of the operator* $\Box_B^k$, *where*

$$R_{2k}(x) = \frac{V^{\frac{2k-n-2|v|}{2}}}{K_n(2k)} \tag{7}$$

for

$$V = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$$

and

$$K_n(2k) = \frac{\pi^{\frac{n+2|v|-1}{2}}\Gamma(\frac{2+2k-n-2|v|}{2})\Gamma(\frac{1-2k}{2})\Gamma(2k)}{\Gamma(\frac{2+2k-p-2|v|}{2})\Gamma(\frac{p-2k}{2})}$$

The proof of this Lemma is given in [9].

Lemma 2.3 *The functions* $S_{2k}(x)$ *and* $R_{2k}(x)$ *are homogeneous distributions of order* $(2k - n - 2|v|)$ *for* $Re(2k) < n + 2|v|$. *In particular, the B-convolution* $S_{2k}(x) * R_{2k}(x)$ *exists and is a tempered distribution.*

The proof of this Lemma is given in [9].

Lemma 2.4 The B-convolutions of tempered distributions

(a) *Let* $S_{2k}(x)$ *and* $S_{2m}(x)$ *be defined by* (6), *then* $S_{2k}(x)*S_{2m}(x) = S_{2k+2m}(x)$, *where k and m are nonnegative integer,*

(b) *Let* $R_{2k}(x)$ *and* $R_{2m}(x)$ *be defined by* (7), *then* $R_{2k}(x)*R_{2m}(x) = R_{2k+2m}(x)$, *where k and m are nonnegative integer,*

(c) *Let* $S_{2k}(x)$ *and* $S_{2m}(x)$ *be defined by* (6) *and if* $S_{2k}(x) * S_{2m}(x) = \delta(x)$, *then* $S_{2k}(x)$ *is an inverse of* $S_{2m}(x)$ *in the B-convolution algebra, denoted by* $S_{2k}(x) = S_{2m}^{*-1}(x)$, *moreover* $S_{2m}^{*-1}(x)$ *is unique.*

The proof of this Lemma is given in [7].

Lemma 2.5 *Given the equation* $\Diamond_B^k u(x) = \delta(x)$ *for* $x \in \mathbb{R}_n^+$, *where* $\Diamond_B^k$ *is the Bessel diamond operator iterated k-times defined by* (1). *Then* $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ *is an elementary solution of the operator* $\Diamond_B^k$.

The proof of this Lemma is given in [9].

Lemma 2.6 (The elementary solution of Bessel Klein-Gordon operator)
Given the equation $(\Box_B + m^2)^k u(x) = \delta(x)$ *for* $x \in \mathbb{R}_n^+$, *where* $\Box_B$ *is defined by* (3). *Then* $u(x) = W_{2k}(x, m)$ *is an elementary solution of the operator* $(\Box_B + m^2)^k$ *where*

$$W_{2k}(x, m) = \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r R_{2k+2r}(x) \tag{8}$$

for $R_{2k+2r}(x)$ *is defined by* (7).

Proof. Since the operator $\Box_B$ is a linearly continuous and have $1-1$ mapping, it has an inverse. By Lemma 2.2, we obtain

$$W_{2k}(x, m) = \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r \Box_B^{-k-r} \delta(x) = (\Box_B + m^2)^{-k} \delta(x),$$

where $(\Box_B + m^2)^{-k}$ is the inverse operator of the operator $(\Box_B + m^2)^k$. By applying the operator $(\Box_B + m^2)^k$ to both sides of the above equation, we have

$$(\Box_B + m^2)^k W_{2k}(x, m) = (\Box_B + m^2)^k (\Box_B + m^2)^{-k} \delta(x) = \delta(x).$$

This completes the proof. $\Box$

Lemma 2.7 *Given the equation*

$$\triangle_B^k u(x) = 0 \quad for \quad x \in \mathbb{R}_n^+, \tag{9}$$

where $\triangle_B^k$ *is the Laplace-Bessel operator iterated* k*-times defined by* (2). *Then* $u(x) = (-1)^k \left(S_{2(k-1)}(x)\right)^{(l)}$ *is a solution of* (9), *where* l *is a nonnegative integer with* $l = \frac{n+2|v|-4}{2}$, $n + 2|v| \geq 4$, n *is even and* $\left(S_{2(k-1)}(x)\right)^{(l)}$ *is a function defined by* (6) *with* l *derivatives.*

The proof of this Lemma is given in [6].

Lemma 2.8 *Given the equation*

$$\triangle_B^k u(x) = f(x, u(x)) \quad for \quad x \in \mathbb{R}_n^+, \tag{10}$$

where $\triangle_B^k$ *is the Laplace-Bessel operator iterated* k*-times defined by* (2), f *is defined and has continuous first derivative for all* $x \in \Omega \cup \partial\Omega$, Ω *ia an open subset of* $\mathbb{R}_n^+$ *and* $\partial\Omega$ *is the boundary of* Ω, *assume that* f *is bounded, that is* $|f(x, u(x))| \leq N$ *and the boundary condition* $u(x) = 0$ *for* $x \in \partial\Omega$. *Then we obtain* $u(x)$ *as an unique solution of* (10).

The proof of this Lemma is given in [6].

3 Main Results

These are the main results of the paper.

Theorem 3.1 *Given the nonlinear equation*

$$\Diamond_B^k(\Box_B + m^2)^k u(x) = f(x, \triangle_B^{k-1}\Box_B^k(\Box_B + m^2)^k u(x)) \tag{11}$$

where $\Diamond_B^k$ *is the Bessel diamond operator iterated* k *times defined by* (1), $\triangle_B^{k-1}$ *is the Laplace-Bessel operator iterated* $k-1$ *times defined by* (2), $(\Box_B + m^2)^k$ *is the Bessel Klein-Gordon operator iterated* k *times,* $\Box_B^k$ *is the Bessel ultra-hyperbolic operator iterated* k *times defined by* (3) *and* m *is a positive real number. Let* f *be defined and having continuous first derivative for all* $x \in \Omega \cup \partial\Omega$ *where* Ω *ia an open subset of* $\mathbb{R}_n^+$ *and* $\partial\Omega$ *denotes the boundary of* Ω *and* n *is even with* $n + 2|v| \geq 4$. *Assume that* f *is bounded, that is for all* $x \in \Omega$

$$|f(x, \triangle_B^{k-1}\Box_B^k(\Box_B + m^2)^k u(x))| \leq N \tag{12}$$

where N *is a constant and the boundary condition for* $x \in \partial\Omega$

$$\triangle_B^{k-1}\Box_B^k(\Box_B + m^2)^k u(x) = 0. \tag{13}$$

Then we obtain

$$u(x) = (-1)^{k-1} S_{2(k-1)}(x) * R_{2k}(x) * W_{2k}(x, m) * U(x) \tag{14}$$

as a solution of (11) *with the boundary condition*

$$u(x) = (-1)^{k-2} \left(S_{2(k-2)}(x)\right)^{(l)} * R_{2k}(x) * W_{2k}(x, m), \tag{15}$$

where $U(x)$ *is a continuous function for* $x \in \Omega \cup \partial\Omega$, $l = \frac{n+2|v|-4}{2}$, $k = 2, 3, \ldots$, $S_{2(k-2)}(x)$, $R_{2k}(x)$ *and* $W_{2k}(x, m)$ *are given by* (6), (7) *and* (8) *respectively. Moreover* $V(x) = (-1)^{1-k} S_{2(1-k)}(x) * u(x)$ *is a solution of the equation*

$$\Box_B^k(\Box_B + m^2)^k V(x) = U(x). \tag{16}$$

Furthermore, if we put $k = 1$, $p = n$ *and* $q = 0$, *then* $V(x)$ *is a solution of the inhomogeneous Bessel biharmonic equation.*

Proof. We have

$$\begin{aligned} \Diamond_B^k(\Box_B + m^2)^k u(x) &= \triangle_B\left(\triangle_B^{k-1}\Box_B^k(\Box_B + m^2)^k u(x)\right) \\ &= f(x, \triangle_B^{k-1}\Box_B^k(\Box_B + m^2)^k u(x)). \end{aligned} \tag{17}$$

Since $u(x)$ has continuous derivatives up to order $6k$ for $k = 1, 2, \ldots$, we may assume that for all $x \in \Omega$

$$\triangle_B^{k-1} \Box_B^k (\Box_B + m^2)^k u(x) = U(x). \tag{18}$$

Thus (17) can be written in the form

$$\Diamond_B^k (\Box_B + m^2)^k u(x) = \triangle_B U(x) = f(x, U(x)), \tag{19}$$

by (12), we have for $x \in \Omega$

$$|f(x, U(x))| \leq N \tag{20}$$

and by (13), we obtain for $x \in \Omega$, $U(x) = 0$ or

$$\triangle_B^{k-1} \Box_B^k (\Box_B + m^2)^k u(x) = 0. \tag{21}$$

Then by Lemma 2.8, there exists an unique solution $U(x)$ of (19) which satisfies (20). Now consider the equation (18), we have

$$\triangle_B^{k-1} (-1)^{k-1} S_{2(k-1)}(x) = \delta, \; \Box_B^k R_{2k}(x) = \delta \;\text{ and }\; (\Box_B + m^2)^k W_{2k}(x, m) = \delta$$

where δ is the Dirac delta distribution, that is the functions $(-1)^{k-1} S_{2(k-1)}(x)$, $R_{2k}(x)$ and $W_{2k}(x, m)$ are the elementary solution of the operators $\triangle_B^{k-1}$, $\Box_B^k$ and $(\Box_B + m^2)^k$ respectively. The functions $(-1)^{k-1} S_{2(k-1)}(x)$, $R_{2k}(x)$ and $W_{2k}(x, m)$ are defined by (6), (7) and (8) respectively. Hence B-convolving both sides of (18) by $(-1)^{k-1} S_{2(k-1)}(x) * R_{2k}(x) * W_{2k}(x, m)$, we obtain

$$\begin{aligned} U(x) &* \left[(-1)^{k-1} S_{2(k-1)}(x) * R_{2k}(x) * W_{2k}(x, m)\right] \\ &= \triangle_B^{k-1} \Box_B^k (\Box_B + m^2)^k u(x) * \left[(-1)^{k-1} S_{2(k-1)}(x) * R_{2k}(x) * W_{2k}(x, m)\right] \\ &= \left[\triangle_B^{k-1} (-1)^{k-1} S_{2(k-1)}(x)\right] * \left[\Box_B^k R_{2k}(x)\right] * \left[(\Box_B + m^2)^k W_{2k}(x, m)\right] * u(x) \\ &= \delta * \delta * \delta * u(x). \end{aligned}$$

Then

$$u(x) = (-1)^{k-1} S_{2(k-1)}(x) * R_{2k}(x) * W_{2k}(x, m) * U(x) \tag{22}$$

as required. We consider for $x \in \partial\Omega$

$$\triangle_B^{k-1} \Box_B^k (\Box_B + m^2)^k u(x) = 0.$$

By Lemma 2.7, we obtain

$$\Box_B^k (\Box_B + m^2)^k u(x) = (-1)^{k-2} \left(S_{2(k-2)}(x)\right)^{(l)}.$$

B-convolving both sides of the above equation by $R_{2k}(x) * W_{2k}(x,m)$, we have

$$\begin{aligned}
[R_{2k}(x) * W_{2k}(x,m)] * (-1)^{k-2}\left(S_{2(k-2)}(x)\right)^{(l)} \\
&= [R_{2k}(x) * W_{2k}(x,m)] * \square_B^k(\square_B + m^2)^k u(x) \\
&= \left[\square_B^k R_{2k}(x)\right] * \left[(\square_B + m^2)^k W_{2k}(x,m)\right] * u(x) \\
&= \delta * \delta * u(x).
\end{aligned}$$

Then for $x \in \partial\Omega$ and $k = 2, 3, \ldots$

$$u(x) = R_{2k}(x) * W_{2k}(x,m) * (-1)^{k-2}\left(S_{2(k-2)}(x)\right)^{(l)} \tag{23}$$

as required. Now B-convolving both sides of (22) by $(-1)^{1-k}S_{2(1-k)}(x)$, we obtain

$$\begin{aligned}
(-1)^{1-k}S_{2(1-k)}(x) * u(x) \\
&= (-1)^{1-k}S_{2(1-k)}(x) * (-1)^{k-1}S_{2(k-1)}(x) * R_{2k}(x) * W_{2k}(x,m) * U(x) \\
&= S_0(x) * R_{2k}(x) * W_{2k}(x,m) * U(x) \\
&= \delta(x) * R_{2k}(x) * W_{2k}(x,m) * U(x) \\
&= R_{2k}(x) * W_{2k}(x,m) * U(x).
\end{aligned}$$

Let $V(x) = (-1)^{1-k}S_{2(1-k)}(x) * u(x)$ be given, then $u(x) = (-1)^{k-1}S_{2(k-1)}(x) * V(x)$, and by (18) we have

$$\begin{aligned}
U(x) &= \triangle_B^{k-1}\square_B^k(\square_B + m^2)^k(-1)^{k-1}S_{2(k-1)}(x) * V(x) \\
&= \triangle_B^{k-1}(-1)^{k-1}S_{2(k-1)}(x) * \square_B^k(\square_B + m^2)^k V(x) \\
&= \delta(x) * \square_B^k(\square_B + m^2)^k V(x) \\
&= \square_B^k(\square_B + m^2)^k V(x).
\end{aligned} \tag{24}$$

If we put $k = 1$, $p = n$ and $q = 0$, then the operators $\square_B^k$ and $(\square_B + m^2)^k$ reduced to $\triangle_B$ and $\triangle_B + m^2$ respectively. Then $V(x)$ is the solution of the inhomogeneous Bessel biharmonic equation

$$\triangle_B^2 V(x) = g(x, \triangle_B V(x))$$

where $g(x, \triangle_B V(x)) = U(x) - m^2\triangle_B V(x)$. □

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