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L_p- Approximation by Iterates of Bernstein-Durrmeyer Type Polynomials

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Abstract

The present paper is a study of L_p- approximation by a new type of Bernstein-Durrmeyer type operators. It turns out the order of approximation by these operators is at best $O(n^{-1})$, however smooth the function may be. In order to speed up the rate of convergence by the operators P_n, we apply the technique of iterative combinations given by Micchelli [9] and prove that the order of approximation by these operators is $O(n^{-k})$ for sufficiently smooth functions.

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1 Introduction

The modified Bernstein type polynomial operators

$$P_n(f;t) = n\sum_{k=1}^{n} p_{n,k}(t)\int_0^1 p_{n-1,k-1}(u)f(u)\,du + (1-t)^n f(0),$$

where

$$p_{n,k}(t) = \binom{n}{k} t^k(1-t)^{n-k},\ \ 0 \leqslant t \leqslant 1,$$

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defined on $L_B[0,1]$, the space of Lebesgue integrable functions on $[0,1]$ were introduced by Gupta and Maheshwari [6]wherein they studied the approximation of functions of bounded variation by these operators. In [7] Gupta and Ispir studied the pointwise convergence and Voronovskaja. type asymptotic results in simultaneous approximation. For $f \in L_p[0,1], 1 \leqslant p < \infty$, the operators $P_n(f;t)$ can be expressed as

$$P_n(f;t) = \int_0^1 W_n(u,t) f(u)\, du,$$

where

$$W_n(u,t) = n \sum_{k=1}^{n} p_{n,k}(t)\, p_{n-1,k-1}(u) + (1-t)^n \delta(u),$$

$\delta(u)$ being the Dirac-delta function, is the kernel of the operators.

For $m \in N^0$ (the set of non-negative integers), the $m-$th order moment for the operators P_n is defined as

$$\mu_{n,m}(t) = P_n\left((u-t)^m; t\right).$$

The Iterative combinations $T_{n,k} : L_p[0,1] \to C^\infty[0,1]$ of the operators $P_n(f;t)$ are defined as

$$T_{n,k}(f;t) = \left(I - (I - P_n)^k\right)(f;t) = \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} P_n^r(f;t),\ k \in N,$$

where $P_n^0 = I$ and $P_n^r = P_n(P_n^{r-1})$ for $r \in N$.

In Section 2 of this paper we give some definitions and auxiliary results which will be needed to prove the main results. In Section 3 we obtain an estimate of error in L_p- approximation $(1 \leqslant p < \infty)$ by the iterative combination $T_{n,k}(.;t)$ in terms of L_p- norm of derivatives of the function. From these estimates we obtain a general error estimate in terms of $2k-$ th modulus of smoothness of the function.

In what follows, we suppose that $I_j \subset I_{j-1} \subsetneq (0,1)$ $j = 2,3$ and $I_j = [a_j, b_j]$. Also, $AC[a,b]$ and $BV[a,b]$ denote the classes of absolutely continuous functions and functions of bounded variations respectively in the interval $[a,b]$. Further C is a constant not always the same.

2 Preliminaries

In the sequel we shall need the following results:

Lemma 1. *[2] For the functions $\mu_{n,m}(t)$, we have $\mu_{n,0}(t) = 1$, $\mu_{n,1}(t) = \frac{(-t)}{(n+1)}$, and there holds the recurrence relation*

$(n+m+1)\mu_{n,m+1}(t) = t(1-t)\left\{\mu'_{n,m}(t) + 2m\mu_{n,m-1}(t)\right\} + (m(1-2t) - t)\mu_{n,m}(t)$, *for* $m \geq 1$.

Consequently, we have

(*i*) *$\mu_{n,m}(t)$ is a polynomial in t of degree m;*

(*ii*) *for every $t \in [0,1], \mu_{n,m}(t) = O\left(n^{-[(m+1)/2]}\right)$, where $[\beta]$ is the integer part of β.*

The mth order moment for the operator P_n^p is defined as $\mu_{n,m}^{[p]}(t) = P_n^p\left((u-t)^m; t\right)$, $p \in N$(the set of natural numbers). We denote $\mu_{n,m}^{[1]}(t)$ by $\mu_{n,m}(t)$.

Lemma 2. *[4] For the function $p_{n,k}(t)$, there holds the result*

$$t^r(1-t)^r \frac{d^r p_{n,k}(t)}{dt^r} = \sum_{\substack{2i+j \leqslant r \\ i,j \geq 0}} n^i (k-nt)^j q_{i,j,r}(t) p_{n,k}(t),$$

where $q_{i,j,r}(t)$ are certain polynomials in t independent of n and k.

Lemma 3. *[3] There holds the recurrence relation*

$$\mu_{n,m}^{[p+1]}(t) = \sum_{j=0}^{m} \sum_{i=0}^{m-j} \binom{m}{j} \frac{1}{i!} D^i \left(\mu_{n,m-j}^{[p]}(t)\right) \mu_{n,i+j}(t).$$

Lemma 4. *[3] For $k, l \in N$, there holds $T_{n,k}\left((u-t)^l; t\right) = O(n^{-k})$.*

Using Lemma 1 and Lemma 3 we can prove the following:

Lemma 5. *[3] For $p \in N, m \in N^0$ and $t \in [0,1]$ we have*

$$\mu_{n,m}^{[p]}(t) = O\left(n^{-[(m+1)/2]}\right).$$

Let $f \in L_p[a,b]$, $1 \leqslant p < \infty$ and $[a_1, b_1] \subset (a,b)$. Then for sufficiently small $\eta > 0$ the Steklov mean $f_{\eta,m}$ of m th order corresponding to f is defined as follows:

$$f_{\eta,m}(t) = \eta^{-m} \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} \left(f(t) + (-1)^{m-1} \Delta^m_{\sum_{i=1}^m t_i} f(t)\right) \prod_{i=1}^{m} dt_i, \ , t \in I_1,$$

where Δ_h^m is the forward difference operator with step length h.

Lemma 6. *Let $f \in L_p[a,b]$, $1 \leqslant p < \infty$* and *$I_1 = [a_1, b_1] \subset (a,b)$. Then for the function $f_{\eta,m}$, we have* *(a)* $f_{\eta,m}$ *has derivatives up to order* m *over* I_1;

(b) $\|f^{(r)}_{\eta,m}\|_{L_p(I_1)} \leqslant C_r\ \omega_r(f,\eta,[a,b]), r=1,2,...,m;$

(c) $\|f-f_{\eta,m}\|_{L_p(I_1)} \leqslant C_{m+1}\ \omega_m(f,\eta,[a,b]);$

(d) $\|f_{\eta,m}\|_{L_p(I_1)} \leqslant C_{m+2}\ \eta^{-m}\|f\|_{L_p[a,b]};$

(e) $\|f^{(r)}_{\eta,m}\|_{L_p(I_1)} \leqslant C_{m+3}\ \|f\|_{L_p[a,b]},$

where $C_i's$ *are certain constants that depend on* i *but are independent of* f *and* η.

Following [[8], Theorem 18.17] or [[10], pp.163-165], the proof of the above lemma easily follows hence the details are omitted.

Let $f \in L_p[0,a], 1 \leqslant p < \infty$. Then the Hardy-Littlewood majorant $h_f(x)$ of the function f is defined as

$$h_f(x) = \sup_{\xi \neq x} \frac{1}{\xi - x} \int_x^{\xi} f(t)\,dt.$$

Lemma 7. *[11] If* $1 < p < \infty, f \in L_p[0,a]$, *then* $h_f \in L_p[0,a]$ *and*

$$\left\|h_f\right\|_{L_p[0,a]} \leqslant 2^{1/p}\frac{p}{p-1}\|f\|_{L_p[0,a]}.$$

The next lemma gives a bound for the intermediate derivatives of f in terms of the norms of the highest order derivative and the function in L_p-norm.

Lemma 8. *[5] Let* $1 \leqslant p < \infty, f \in L_p[a,b]$. *Suppose* $f^{(k)} \in AC[a,b]$ *and* $f^{(k+1)} \in L_p[a,b]$. *Then*

$$\left\|f^{(j)}\right\|_{L_p[a,b]} \leqslant K_j\left(\left\|f^{(k+1)}\right\|_{L_p[a,b]} + \|f\|_{L_p[a,b]}\right),\ j=1,2,...,k,$$

where K_j *are certain constants independent of* f.

Lemma 9. *Let* $f \in L_p[0,1]$, $1 \leqslant p \leqslant \infty$ *and* $[a,b] \subset (0,1)$. *Then, for* n *sufficiently large we have*

$$\|P_n(f)\|_{L_p[a,b]} \leqslant C\,\|f\|_{L_p[0,1]}.$$

Proof. First, we consider the case $p = 1$. Let φ be the characteristic function of the interval $[a, b]$. Then, we have

$$\begin{aligned}
\|P_n(f)\|_{L_p[a,b]} &= \int_a^b \left| \int_0^1 W_n(u,t) f(u)\, du \right| dt \\
&= \int_a^b \left| \int_0^1 W_n(u,t)\varphi(u) f(u)\, du \right| dt \\
&\qquad + \int_a^b \left| \int_0^1 W_n(u,t)(1-\varphi(u)) f(u)\, du \right| dt \\
&= F_1 + F_2 \ \text{ say.}
\end{aligned}$$

$$\begin{aligned}
F_1 &= \int_a^b \left(\int_0^1 W_n(u,t)\varphi(u) |f(u)|\, du \right) dt \\
&\leqslant \int_a^b \left(\int_a^b W_n(u,t)\, dt \right) |f(u)|\, du \\
&\leqslant C \int_a^b |f(u)|\, du \\
&\leqslant C\, \|f\|_{L_1[a,b]}.
\end{aligned}$$

And

$$\begin{aligned}
F_2 &= \int_a^b \left(\int_0^1 W_n(u,t)(1-\varphi(u)) |f(u)|\, du \right) dt \\
&\leqslant C\,\delta^{-2m} \int_0^1 \left(\int_a^b W_n(u,t)(u-t)^{2m}\, dt \right) |f(u)|\, du \\
&\leqslant C\,\delta^{-2m} n^{-m} \|f\|_{L_1[0,1]} \to 0 \ \text{ as } n \to \infty.
\end{aligned}$$

Next, for $p = \infty$, we have

$$\begin{aligned}
\|P_n(f)\|_\infty &= \left\| \int_0^1 W_n(u,t) f(u)\, du \right\| \\
&\leqslant \|f\|_\infty \int_0^1 W_n(u,t)\, du \\
&\leqslant \|f\|_\infty.
\end{aligned}$$

Thus, the lemma is established for the values $p = 1$ and $p = \infty$. Therefore, in view of the Riesz-Thorin interpolation theorem the Lemma is proved for $1 \leqslant p \leqslant \infty$ □

Corollary 1. *Using an induction on $r \in N$, it follows that*

$$\|P_n^r(f)\|_{L_p[a,b]} \leqslant C\,\|f\|_{L_p[a,b]}$$

for all $1 \leqslant p \leqslant \infty$.
Consequently,

$$\|T_{n,k}(f)\|_{L_p[a,b]} \leqslant C\,\|P_n^r(f)\|_{L_p[a,b]} \leqslant C\,\|P_n(f)\|_{L_p[a,b]} \leqslant C\,\|f\|_{L_p[a,b]}.$$

3 Main Result

In this section we obtain an error estimate in terms of L_p- norm. The proof of the case $p > 1$ makes use of Lemma 7 regarding Hardy-Littlewood majorant and Lemma 8, while for $p = 1$ we require only Lemma 8.

Theorem 1. *If $p > 1, f \in L_p[0,1], f$ has derivatives of order $2k$ on I_1 with $f^{(2k-1)} \in AC(I_1)$ and $f^{(2k)} \in L_p(I_1)$, then for sufficiently large n*

$$\|T_{n,k}(f;\cdot) - f(\cdot)\|_{L_p(I_2)} \leqslant C_1 n^{-k}\left[\left\|f^{(2k)}\right\|_{L_p(I_1)} + \|f\|_{L_p[0,1]}\right]. \qquad (3.3.3.1)$$

Moreover, if $f \in L_1[0,1], f$ has derivatives up to the order $(2k-1)$ on I_1 with $f^{(2k-2)} \in AC(I_1)$ and $f^{(2k-1)} \in BV(I_1)$, then for sufficiently large n there holds

$$\|T_{n,k}(f;\cdot) - f(\cdot)\|_{L_1(I_2)} \leqslant C_2 n^{-k}\left[\left\|f^{(2k-1)}\right\|_{BV(I_1)} + \left\|f^{(2k-2)}\right\|_{L_1(I_2)} + \|f\|_{L_1(I_2)}\right], \qquad (3.3.3.2)$$

where C_1 and C_2 are certain constants independent of f and n.

Proof. Let $p > 1$, then for all $u \in I_1$ and $t \in I_2$, we can write

$$f(u) - f(t) = \sum_{j=1}^{2k-1} \frac{f^{(j)}(t)}{j!}(u-t)^j + \frac{1}{(2k-1)!}\int_t^u \varphi(u)(u-v)^{2k-1} f^{(2k)}(v)\,dv + F(u,t)(1-\varphi(u)), \qquad (3.3.3.3)$$

where $\varphi(u)$ is the characteristic function of the interval I_1 and

$$F(u,t) = f(u) - \sum_{j=0}^{2k-1} \frac{f^{(j)}(t)}{j!}(u-t)^j.$$

Therefore ,operating by $T_{n,k}$ on both sides of (3.3.3.3), we obtain three terms, say E_1, E_2 and E_3 corresponding to the three terms in the right hand side of (3.3.3.3).

In view of Lemma 4 and Lemma 8, we get

$$\|E_1\|_{L_p(I_2)} \leqslant C\, n^{-k}\left(\left\|f^{(2k)}\right\|_{L_p(I_2)} + \|f\|_{L_p(I_2)}\right).$$

Let $h_{f^{(2k)}}$ be the Hardy-Littleood majorant of $f^{(2k)}$ on I_1. Now, we find an estimate for E_2 as follows:

$$\begin{aligned}
\|E_2\|^p_{L_P(I_2)} &= \frac{1}{(2k-1)!}\int_{a_2}^{b_2}\left|\int_0^1 W_n(u,t)\Big(\int_t^u \varphi(u)(u-v)^{2k-1}f^{(2k)}(v)\,dv\Big)\,du\right|^p dt\\
&\leqslant C\int_{a_2}^{b_2}\left|\int_{a_1}^{b_1} W_n(u,t)(u-t)^{2k}h_{f^{(2k)}}(u)\,du\right|^p dt\\
&\leqslant Cn^{-kp}\int_{a_2}^{b_2}\left(\int_{a_1}^{b_1} W_n(u,t)|h_{f(2k)}(u)|^p\,du\right)dt.
\end{aligned}$$

Using Fubini's theorem and Lemma 7 , we get

$$\|E_2\|^p_{L_P(I_2)} \leqslant Cn^{-kp}\|h_{f^{(2k)}}\|^p_{L_p[a,b]}$$

$$\|E_2\|^p_{L_P(I_2)} \leqslant Cn^{-kp}\|f^{(2k)}\|^p_{L_p(I_1)}$$

$$\|E_2\|_{L_p(I_2)} \leqslant Cn^{-k}\|f^{(2k)}\|_{L_p(I_1)}.$$

In order to estimate E_3, it is sufficient to consider $I = \left|P_n\Big(F(u,t)(1-\varphi(u));t\Big)\right|$

For $u\in[0,1]\setminus[a_1,b_1], t\in I_2$ we can find a $\delta>0$ such that $|u-t|\geq\delta$. Thus

$$\begin{aligned}
I &= \left|P_n\Big(F(u,t)(1-\varphi(u));t\Big)\right|\\
&= \left|P_n\Big(\Big(f(u)-\sum_{j=0}^{2k-1}\frac{f^{(j)}(t)}{j!}(u-t)^j\Big)\big(1-\varphi(u)\big);t\Big)\right|\\
&\leqslant \delta^{-2k}P_n\Big(|f(u)|(u-t)^{2k};t\Big)+\delta^{-2k}\sum_{j=0}^{2k-1}\frac{\left|f^{(j)}(t)\right|}{j!}P_n\Big(|u-t|^{2k+j};t\Big)\\
&= J_2+J_3,\ \text{say}.
\end{aligned}$$

On an application of Hölder's inequality, Lemma 1 and Fubini's theorem we get

$$\left\|J_2\right\|_{L_p(I_2)} \leqslant C\,n^{-k}\left(\int_{a_2}^{b_2}\int_0^1 \left|f(u)\right|^p W_n(t,u)\,dt\,du\right)^{1/p} \leqslant C\,n^{-k}\|f\|_{L_p[0,1]}.$$

Now in view of Lemma 1 and Lemma 8, we have the inequality

$$\left\|J_3\right\|_{L_p(I_2)} \leqslant C\, n^{-k} \left(\|f\|_{L_p(I_2)} + \left\|f^{(2k)}\right\|_{L_p(I_2)}\right).$$

Therefore,

$$E_3 \leqslant C \quad n^{-k} \left(\|f\|_{L_p[0,1]} + \left\|f^{(2k)}\right\|_{L_p(I_2)}\right).$$

Combining the estimates for $E_1 - E_3$, (3.3.3.1) follows.

Now, let $p = 1$. Then we can expand $f(u)$ for almost all $t \in I_2$ and for all $u \in I_1$, as

$$\begin{aligned} f(u) - f(t) &= \sum_{j=1}^{2k-1} \frac{f^{(j)}(t)}{j!}(u-t)^j + \frac{1}{(2k-1)!} \int_t^u \varphi(u)(u-v)^{2k-1}\, df^{(2k-1)}(v) \\ &+ F(u,t)(1-\varphi(u)), \end{aligned} \tag{3.3.3.4}$$

where $\varphi(u)$ and $F(u,t)$ are defined as above.
Therefore, operating by $T_{n,k}$ on both sides of (3.3.3.4), we obtain three terms E_4, E_5 and E_6, say corresponding to the three terms in the right hand side of (3.3.3.4).

Now, proceeding as in the case of the estimate of E_1, we have

$$\left\|E_4\right\|_{L_1(I_2)} \leqslant C\ n^{-k} \left(\|f\|_{L_1(I_2)} + \left\|f^{(2k-1)}\right\|_{L_1(I_2)}\right). \tag{3.3.3.5}$$

In order to get an estimate for E_5, using Cor. 1, it is sufficient to consider the estimate for K :

$$\begin{aligned} K &= \left\| P_n \left(\int_t^u \varphi(u)|u-v|^{2k-1}\, df^{(2k-1)}(v); t \right) \right\|_{L_1(I_2)} \\ &\leqslant \int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} W_n(t,u)|u-t|^{2k-1} \left| \int_t^u \left| df^{2k-1}(v) \right| \right| du \right) dt. \end{aligned}$$

For each n there exists the integer $r = r(n)$ s.t. $r/\sqrt{n} \leqslant \max\,(b_1 - a_2, b_2 - a_1) \leqslant (r+1)/\sqrt{n}$.
Therefore,

$$K \leqslant \sum_{l=0}^{r} \int_{a_2}^{b_2} \Bigg\{ \int_{t+l/\sqrt{n}}^{t+(l+1)/\sqrt{n}} W_n(t,u)|u-t|^{2k-1} \int_t^{t+(l+1)/\sqrt{n}} \varphi(v) \left| df^{(2k-1)}(v) \right| du$$

$$+ \int_{t-(l+1)/\sqrt{n}}^{t-l/\sqrt{n}} W_n(t,u)|u-t|^{2k-1} \int_{t-(l+1)/\sqrt{n}}^{t} \varphi(v)\left|df^{(2k-1)}(v)\right| du\Big\} dt$$

$$\leqslant 2\sum_{l=1}^{r} \frac{n^2}{l^4}\frac{l+1}{\sqrt{n}} \int_{a_1}^{b_1} W_n(t,u)|u-t|^{2k+3}\, du \left\|f^{(2k-1)}\right\|_{BV(I_1)}$$

$$+ \frac{2}{\sqrt{n}} \int_{a_1}^{b_1} W_n(t,u)|u-t|^{2k-1}\, du \left\|f^{(2k-1)}\right\|_{BV(I_1)}$$

$$\leqslant M\ n^{-k}\left\|f^{(2k-1)}\right\|_{BV(I_1).}\text{(Applying Fubini's theorem).}$$

To estimate E_6, for all $u \in [0,1]\setminus[a_1,b_1], t \in I_2$, we can choose a $\delta > 0$ such that $|u-t| \geq \delta$. Therefore, we get the inequality

$$\begin{aligned}\left\|E_4\right\|_{L_1(I_2)} &\leqslant C \int_{a_2}^{b_2}\int_0^1 W_n(t,u)|f(u)(1-\varphi(u))|\,du\,dt \\ &+ \sum_{i=0}^{2k-1}\frac{1}{i!}\int_{a_2}^{b_2}\int_0^1 W_n(t,u)|f^{(i)}(t)(u-t)^i(1-\varphi(u))|\,du\,dt = S_1+S_2, \text{ say.}\end{aligned}$$

Since, $W_n(t,u)$ is symmetric in t and u, there follows

$$S_1 \leqslant C\ \|f\|_{L_1(I_2)}.$$

In view of Lemma 8, we obtain

$$S_2 \leqslant C\ n^{-k}\left(\|f\|_{L_1(I_2)} + \left\|f^{(2k-1)}\right\|_{L_1(I_2)}\right).$$

From these estimates of S_1, S_2, E_4, E_5 and the definition of $T_{n,k}$, we get (3.3.3.2). □

Theorem 2. *If $p \geq 1, f \in L_p[0,1]$. Then for all n sufficiently large there holds*

$$\|T_{n,k}(f;.)-f\|_{L_p(I_2)} \leqslant C_k\ \left(\omega_{2k}\Big(f,\frac{1}{\sqrt{n}},p,I_1.\Big) + n^{-k}\|f\|_{L_p[0,1]}\right), \quad (3.3.3.6)$$

where C_k is a constant independent of f and n.

Proof. We take points x_1, y_1 such that $a_1 < x_1 < a_2 < y_1 < b_2 < b_1$ and denote the interval $[x_1,y_1]$ by I_2 . In order to prove the theorem it is sufficient to prove it for the function fg, where $g \in C_0^{\infty}$ be such that supp $g \subset [a_1,b_1]$ and $g = 1$

in $[a_2, b_2]$. Let for convenience $\hat{f} = fg$. Let $\hat{f}_{\eta,2k}$ be the Steklov mean of order $2k$ corresponding to the function $\hat{f}$,where $\eta > 0$ is sufficiently small. Then we have

$$\begin{aligned}\|T_{n,k}(\hat{f};.) - \hat{f}\|_{L_p(I_2)} &\leqslant \|T_{n,k}(\hat{f} - \hat{f}_{\eta,2k};.)\|_{L_p(I_2)} + \|T_{n,k}(\hat{f}_{\eta,2k};.) - \hat{f}_{\eta,2k}\|_{L_p(I_2)} \\ &+ \|\hat{f}_{\eta,2k} - \hat{f}\|_{L_p(I_2)} = J_1 + J_2 + J_3,\ \text{say.}\end{aligned}$$

In view of Cor. 1 and property (c) of Steklov mean, we get

$$J_1 \leqslant C\ \omega_{2k}\big(\hat{f}, \eta, p, I_1\big).$$

Now using Theorem 1 , and properties (b) and (d), we obtain

$$\begin{aligned}J_2 &\leqslant C\ n^{-k}\left(\|\hat{f}_{\eta,2k}\|_{L_p(I_2')} + \|\hat{f}^{(2k)}_{\eta,2k}\|_{L_p(I_2')}\right) \\ &\leqslant C\ n^{-k}\left(\|f\|_{L_p(I_1)} + \eta^{-2k}\omega_{2k}\big(\hat{f}, \eta, p, I_1\big)\right).\end{aligned}$$

By using property (c) of Steklov means we get the inequality

$$J_3 \leqslant C\ \omega_{2k}\big(\hat{f}, \eta, p, I_1\big).$$

Choosing $\eta = 1/\sqrt{n}$, the result follows from the estimates of $J_1 - J_3$. □

Remark 1. *Similar results can be obtained for the operators $M_{n,\alpha,\beta}(f, x)$ and the operators $B_n(f, x)$ defined in [7]and 420[1] respectively.*

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