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Symmetric Antieigenvalues for Normal Operators

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Abstract

We here study the symmetric antieigenvalues and symmetric antieigenvectors of a normal operator acting on a finite dimensional Hilbert Space using the notion of Lagrange Multipliers. We esimate the symmetric antieigenvalues and symmetric antieigenvectors of a compact normal operator acting on an infinite dimensional Hilbert space. Finally using properties of numerical range we compute the symmetric antieigenvalues of a normal operator.

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1 Introduction

The angle of an operator was introduced in 1968 by Gustafson [3] (see also [7], [10]) while studying the problems in the perturbation theory of semi-group generators. For a bounded linear operator T acting on a complex Hilbert space H the cosine of the angle of T was defined by Gustafson as follows:

$$\cos\phi(T) = \inf_{\|Tf\|\neq 0} \frac{Re(Tf,f)}{\|Tf\|\ \|f\|}.$$

The properties of $\cos\phi\,(T)$ are dependent on the real part of numerical range $W\,(T)$ of T. The quantity $\cos\phi\,(T)$ has another interpretation as the first antieigenvalue of T,

$$\mu_1\,(T) \;=\; \inf_{\|Tf\|\neq 0} \frac{Re(Tf,f)}{\|Tf\|\;\|f\|}.$$

This concept was also introduced by Gustafson [6]. First author along with others [13] introduced the notion of symmetric antieigenvalue and symmetric antieigenvector of an operator $T \in B\,(H)$ as follows:

$$cos\phi_s\,(T) \;=\; \inf_{\|Tf\|\neq 0} \frac{Re\,(Tf,f) + Im\,(Tf,f)}{\sqrt{2}\;\|Tf\|\;\|f\|}.$$

Let

$$\phi_T\,(f) \;=\; \frac{Re\,(Tf,f) + Im\,(Tf,f)}{\sqrt{2}\;\|Tf\|\;\|f\|}, \qquad Tf \neq \theta.$$

Then

$$cos\phi_s\,(T) \;=\; \inf_{Tf\neq\theta} \phi_T\,(f) = \mu_s(say).$$

μ_s is defined as the symmetric antieigenvalue of T and the vectors f for which $\phi_T\,(f)$ attains the infimum (if exists) are called the *symmetric antieigenvectors* of T.

For a self-adjoint operator T with eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$, by the definition of Gustafson [3], $\cos\phi\,(T) = \frac{2\sqrt{\lambda_1\lambda_n}}{\lambda_1+\lambda_n}$, whereas $\cos\phi\,(iT) = 0$ rather abruptly, although iT has the eigenvalues $i\lambda_1,\, i\lambda_2,\, \cdots,\, i\lambda_n$ but from the definition of symmetric antieigenvalues $\cos\phi_s\,(T) = \cos\phi_s\,(iT)$ for a self-adjoint operator T as the definition involves both the real and imaginary part of the numerical range $W\,(T)$, unlike the definition introduced by Gustafson [3], which involves only the real part of $W\,(T)$.

We here explicitly calculate the symmetric antieigenvalues and symmetric antieigenvectors for a normal operator acting on a finite dimensional Hilbert space using the elementary notion of Lagranges Multipliers, this computation is not so easy as the characteristic equation of the symmetric antieigenvector of a bounded linear operator T is non-linear. Using this method Gustafson and Seddighin (see [11], [17]) calculated the antieigenvalues of a normal accretive operator acting on a finite dimensional Hilbert space. We calculate the symmetric antieigenvectors for a normal operator following the method used by Mirman [16] in estmation of antieigenvalues. Mirman [16] used a result of Krein [14] in the estmation of antieigenvalues. We also estimate the symmetric antieigenvalues of a compact normal operator acting on an infinite dimensional Hilbert space in a new method.

2 Computation of symmetric antieigenvalue

We first compute symmetric antieigenvalues of a normal operator T on a finite dimensional Hilbert space H using the notion of Lagrange Multipliers. This method was used by Gustafson and Seddighin [11] in the computation of antieigenvalues of a normal accretive operator.

Theorem 2.1 *Let T be a normal operator on a finite dimensional Hilbert space H with $ReT + ImT > 0$ and $\lambda_j = \alpha_j + i\beta_j$, j=1,2,....,n, be the eigenvalues of T.*
Let

$$\begin{aligned} E &= \left\{ \frac{\alpha_j + \beta_j}{\sqrt{2}|\lambda_j|} : 1 \leq j \leq n \right\} \\ F &= \left\{ \frac{\sqrt{2(\alpha_i - \alpha_j + \beta_i - \beta_j)\left\{(\alpha_j + \beta_j)|\lambda_i|^2 - (\alpha_i + \beta_i)|\lambda_j|^2\right\}}}{(|\lambda_i|^2 - |\lambda_j|^2)} \right| \\ 0 &\leq \frac{(\alpha_j + \beta_j)(|\lambda_i|^2 + |\lambda_j|^2) - 2(\alpha_i + \beta_i)|\lambda_j|^2}{(\alpha_i - \alpha_j + \beta_i - \beta_j)(|\lambda_i|^2 - |\lambda_j|^2)} \leq \quad 1 \quad \Bigg\} \end{aligned}$$

Then μ_S is exactly equal to the smallest number in $E \cup F$.
Furthermore, if

$$\mu_S = \frac{\sqrt{2\left(\alpha_i - \alpha_j + \beta_i - \beta_j\right)\left\{\left(\alpha_j + \beta_j\right)\left|\lambda_i\right|^2 - \left(\alpha_i + \beta_i\right)\left|\lambda_j\right|^2\right\}}}{\left(\left|\lambda_i\right|^2 - \left|\lambda_j\right|^2\right)},$$

then

$$\mu_S = \frac{Re\,(Tz, z) + Im\,(Tz, z)}{\sqrt{2}\|Tz\|\|z\|}$$

for some

$$z = (z_1, z_2, ..., z_n)$$

with

$$\begin{aligned} |z_i|^2 &= \frac{\left(\alpha_j + \beta_j\right)\left(|\lambda_i|^2 + |\lambda_j|^2\right) - 2\left(\alpha_i + \beta_i\right)|\lambda_j|^2}{\left(\alpha_i - \alpha_j + \beta_i - \beta_j\right)\left(|\lambda_i|^2 - |\lambda_j|^2\right)} \\ |z_j|^2 &= \frac{\left(\alpha_i + \beta_i\right)\left(|\lambda_i|^2 + |\lambda_j|^2\right) - 2\left(\alpha_j + \beta_j\right)|\lambda_i|^2}{\left(\alpha_i - \alpha_j + \beta_i - \beta_j\right)\left(|\lambda_i|^2 - |\lambda_j|^2\right)} \\ |z_k| &= 0 \quad for \quad k \neq i, \quad k \neq j. \end{aligned}$$

Proof. Suppose, $z = (z_1, z_2, ..., z_n)$ be a unit vector in H, where $z_j = x_j + iy_j$, where j = 1, 2, ... , n.

$$\Phi_T(z) = \frac{Re(Tz, z) + Im(Tz, z)}{\sqrt{2}\|Tz\|} = \frac{\sum_{i=1}^{n}(\alpha_i + \beta_i)\,|z_i|^2}{\sqrt{2}\sqrt{\sum_{i=1}^{n}|\lambda_i|^2\,|z_i|^2}} \quad (1)$$

Now, we find the $min\Phi_T(z)$ over the sphere $\|z\| = 1$ such that $\|Tz\| \neq 0$.

Suppose, $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$.
Consider the function

$$J(x, y) = \frac{\sum_{i=1}^{n}(\alpha_i + \beta_i)(x_i^2 + y_i^2)}{\sqrt{\sum_{i=1}^{n}|\lambda_i|^2\,(x_i^2 + y_i^2)}} \quad (2)$$

J(x,y) is a function of 2n variables defined on the sphere $\|z\| = 1$ i.e. $\sum_{i=1}^{n}(x_i^2 + y_i^2) = 1$. Now, we minimize the function J(x,y) subject to the condition $\sum_{i=1}^{n}(x_i^2 + y_i^2) = 1$.

Since J(x,y) is a continuous function defined on a compact set $\|z\| = 1$, then the minimum exists on the sphere $\|z\| = 1$. If J attains its minimum at z on the unit sphere $\|z\| = 1$, then gradients of J and the unit sphere $\|z\| = 1$ at the point z will be parallel.
Then

$$\frac{\partial J}{\partial x_i} = 2\xi x_i \quad for\ i = 1, 2, ..., n$$

and

$$\frac{\partial J}{\partial y_i} = 2\xi y_i \quad for\ i = 1, 2, ..., n$$

So we get

$$2(\alpha_k + \beta_k)x_k(\sum_{i=1}^{n}|\lambda_i|^2\,|z_i|^2) - 2\,|\lambda_k|^2\,x_k\sum_{i=1}^{n}(\alpha_i + \beta_i)\,|z_i|^2 = 2\xi x_k(\sum_{i=1}^{n}|\lambda_i|^2\,|z_i|^2)^{\frac{3}{2}}$$

$$2(\alpha_k + \beta_k)y_k(\sum_{i=1}^{n}|\lambda_i|^2\,|z_i|^2) - 2\,|\lambda_k|^2\,y_k\sum_{i=1}^{n}(\alpha_i + \beta_i)\,|z_i|^2 = 2\xi y_k(\sum_{i=1}^{n}|\lambda_i|^2\,|z_i|^2)^{\frac{3}{2}} \quad (3)$$

for k=1,2,....,n.
Together with $\sum_{i=1}^{n}(x_i^2 + y_i^2) = 1$, the system of equations (3) forms a system of 2n+1 equations.
Suppose $z = (z_1, z_2, ..., z_n)$ is a solution of the system of equations (3) i.e. $z = (z_1, z_2, ..., z_n)$ is a symmetric antieigenvector of T.
We consider the following cases:-

CaseI: All components of z, except one of them (say z_1), are zero.
Since z_1 is the only non-zero component of z, then

$$\mu_S(T) = \frac{\alpha_1 + \beta_1}{\sqrt{2}\,|\lambda_1|}.$$

CaseII: Only two components of z (say, z_1 and z_2)are non-zero and rest are zero.
Then we have,

$$2(\alpha_1+\beta_1)[|\lambda_1|^2|z_1|^2+|\lambda_2|^2|z_2|^2]-|\lambda_1|^2[(\alpha_1+\beta_1)|z_1|^2+(\alpha_2+\beta_2)|z_2|^2] = 2\xi[|\lambda_1|^2|z_1|^2+|\lambda_2|^2|z_2|^2]^{\frac{3}{2}} \quad (4)$$

$$2(\alpha_2+\beta_2)[|\lambda_1|^2|z_1|^2+|\lambda_2|^2|z_2|^2]-|\lambda_2|^2[(\alpha_1+\beta_1)|z_1|^2+(\alpha_2+\beta_2)|z_2|^2] = 2\xi[|\lambda_1|^2|z_1|^2+|\lambda_2|^2|z_2|^2]^{\frac{3}{2}} \quad (5)$$

and

$$|z_1|^2+|z_2|^2=1 \quad (6)$$

Solving the equations (4), (5) and (6), we get

$$|z_1|^2 = \frac{(\alpha_2+\beta_2)(|\lambda_1|^2+|\lambda_2|^2)-2(\alpha_1+\beta_1)|\lambda_2|^2)}{(\alpha_1-\alpha_2+\beta_1-\beta_2)(|\lambda_1|^2-|\lambda_2|^2)}$$

$$|z_2|^2 = \frac{(\alpha_1+\beta_1)(|\lambda_1|^2+|\lambda_2|^2)-2(\alpha_2+\beta_2)|\lambda_1|^2}{(\alpha_1-\alpha_2+\beta_1-\beta_2)(|\lambda_1|^2-|\lambda_2|^2)}$$

As the point z is on the unit sphere so we have

$$0 \le |z_i|^2 \le 1, \quad i=1,2.$$

i.e.

$$0 \le \frac{(\alpha_j+\beta_j)\left(|\lambda_i|^2+|\lambda_j|^2\right)-2\left(\alpha_i+\beta_i\right)|\lambda_j|^2}{(\alpha_i-\alpha_j+\beta_i-\beta_j)\left(|\lambda_i|^2-|\lambda_j|^2\right)} \le 1$$

where i=1,2 and j=1,2 and $i \neq j$.
Now,

$$(\alpha_1+\beta_1)|z_1|^2+(\alpha_2+\beta_2)|z_2|^2 = \frac{2[(\alpha_2+\beta_2)|\lambda_1|^2-(\alpha_1+\beta_1)|\lambda_2|^2]}{|\lambda_1|^2-|\lambda_2|^2}$$

and

$$|\lambda_1|^2 |z_1|^2 + |\lambda_2|^2 |z_2|^2 = \frac{[(\alpha_2 + \beta_2) |\lambda_1|^2 - (\alpha_1 + \beta_1) |\lambda_2|^2]}{\alpha_1 - \alpha_2 - \beta_1 - \beta_2}$$

Then,

$$J(x, y) = \frac{2\sqrt{(\alpha_1 - \alpha_2 + \beta_1 - \beta_2)\{(\alpha_2 + \beta_2) |\lambda_1|^2 - (\alpha_1 + \beta_1) |\lambda_2|^2\}}}{|\lambda_1|^2 - |\lambda_2|^2}$$

Therefore,

$$\mu_S = \frac{\sqrt{2(\alpha_1 - \alpha_2 + \beta_1 - \beta_2)\{(\alpha_2 + \beta_2) |\lambda_1|^2 - (\alpha_1 + \beta_1) |\lambda_2|^2\}}}{(|\lambda_1|^2 - |\lambda_2|^2)}.$$

Hence, in general,

$$\mu_S(T) = \frac{\sqrt{2 (\alpha_i - \alpha_j + \beta_i - \beta_j) \left\{(\alpha_j + \beta_j) |\lambda_i|^2 - (\alpha_i + \beta_i) |\lambda_j|^2\right\}}}{\left(|\lambda_i|^2 - |\lambda_j|^2\right)}$$

when the unit vector $z = (z_1, z_2, ..., z_n)$ has only two non-zero components and

$$0 \leq \frac{(\alpha_j + \beta_j) \left(|\lambda_i|^2 + |\lambda_j|^2\right) - 2 (\alpha_i + \beta_i) |\lambda_j|^2}{(\alpha_i - \alpha_j + \beta_i - \beta_j) \left(|\lambda_i|^2 - |\lambda_j|^2\right)} \leq 1$$

for i=1,2,....,n and j=1,2,....,n and $i \neq j$.

Case III: Suppose $\Phi_T(z)$ attains its minimum at z_0, where z_0 has three non-zero components, say z_i, z_j and z_k and rest are equal to zero i.e.

$$z_0 = \overbrace{(0, ..., z_i, 0, .., 0, z_j, 0, ..., 0, z_k, 0, ..., 0)}^{n-components}.$$

Now,

$$\Phi_T(z_0) = \frac{(\alpha_i + \beta_i) |z_i|^2 + (\alpha_j + \beta_j) |z_j|^2 + (\alpha_k + \beta_k) |z_k|^2}{\sqrt{2}\sqrt{|\lambda_i|^2 |z_i|^2 + |\lambda_j|^2 |z_j|^2 + |\lambda_k|^2 |z_k|^2}}$$

Let $|z_i|^2 = u$, $|z_j|^2 = v$ and $|z_k|^2 = w$ and consider the simplex

$$V : u + v + w = 1 \quad \text{and} \quad 0 \leq u \leq 1, 0 \leq v \leq 1, 0 \leq w \leq 1.$$

Let $f(u, v, w) = \Phi_T(z_0)$. Clearly, f is a positive function defined on the simplex V. Since, V is compact set and f is continuous on V, then f attains its minimum

in V. Let R be the minimum of f over V.
Suppose f attains its minimum R at some interior point (u_0, v_0, w_0) of V. By the convexity property of f, we get that the local minimum of f attained at any interior point of V is greater than the local minimum of f attained at the boundary point of V. By the convexity of the function f, we can say that f attains its minimum at some point of the boundary of V, i.e. the minimum value of the function f is attained at a point which lies on some co-ordinate plane, say u-v plane.
Let,

$$\inf_{u+v=1,0\leq u\leq 1,0\leq v\leq 1} \frac{(\alpha_i+\beta_i)u+(\alpha_j+\beta_j)v}{\sqrt{2}\sqrt{|\lambda_i|^2 u+|\lambda_j|^2 v}} = R_1(\text{say})$$

Then,

$$\frac{(\alpha_i+\beta_i)u+(\alpha_j+\beta_j)v+(\alpha_k+\beta_k)w}{\sqrt{2}\sqrt{|\lambda_i|^2 u+|\lambda_j|^2 v+|\lambda_k|^2 w}} \geq R_1$$

for all $(u, v, w) \in V$.
This shows that in this case, for an antieigenvector there is no need to consider a vector with more than two non-zero components.
Similarly, vectors with four and higher number of non-zero components may be ruled out.
Hence proof is complete. □

Corollary 2.2 *If T is a strictly positive self-adjoint operator with $m \leq (Tf, f) \leq M$ $\forall f, \|f\| = 1$, then*

$$\mu_s = \frac{\sqrt{2mM}}{(m+M)}.$$

Proof. We observe that μ_s is stationary value of $\phi_T(f)$ and $\frac{\sqrt{\lambda_i\lambda_j}}{\lambda_i+\lambda_j}$ assumes the least value for greatest $\left(\frac{\lambda_i}{\lambda_j}\right)$, which in this case is $\frac{M}{m}$.□

The Theorem 2.1 is the necessary condition for a vector $z = (z_1, z_2, ..., z_n)$ to be a symmetric antieigenvector. The critical points obtained in the above theorem will be a symmetric antieigenvalue , if the following condition is satisfied.

Theorem 2.3 *Let S be the set of all critical points*

$$\begin{aligned} z &= (z_1, z_2, ..., z_n) \\ &= (x_1+iy_1, x_2+iy_2, ..., x_n+iy_n) \end{aligned}$$

found in the Theorem 2.1. Let $J(x,y)$ be the function defined in (2). Consider the Hessian of J i.e. the $2n \times 2n$ matrix defined by

$$\frac{\partial^2 J}{\partial x_k \partial x_j} = -\frac{2[(\alpha_j+\beta_j)|\lambda_k|^2 + (\alpha_k+\beta_k)|\lambda_j|^2]x_j x_k}{[\sum_{i=1}^n |\lambda_i|^2 (x_i^2+y_i^2)]^{\frac{3}{2}}} + \frac{3x_j x_k |\lambda_j|^2 |\lambda_k|^2 [\sum_{i=1}^n (\alpha_i+\beta_i)(x_i^2+y_i^2)]}{(\sum_{i=1}^n |\lambda_i|^2 (x_i^2+y_i^2))^{\frac{5}{2}}}$$

$$\frac{\partial^2 J}{\partial y_k \partial y_j} = -\frac{2[(\alpha_j+\beta_j)|\lambda_k|^2 + (\alpha_k+\beta_k)|\lambda_j|^2]y_j y_k}{[\sum_{i=1}^n |\lambda_i|^2 (x_i^2+y_i^2)]^{\frac{3}{2}}} + \frac{3y_j y_k |\lambda_j|^2 |\lambda_k|^2 [\sum_{i=1}^n (\alpha_i+\beta_i)(x_i^2+y_i^2)]}{(\sum_{i=1}^n |\lambda_i|^2 (x_i^2+y_i^2))^{\frac{5}{2}}}$$

$$\begin{aligned}\frac{\partial^2 J}{\partial x_j^2} =& \frac{2(\alpha_j+\beta_j)(\sum_{i=1}^n |\lambda_i|^2 (x_i^2+y_i^2)) - |\lambda_j|^2 (\sum_{i=1}^n (\alpha_i+\beta_i)(x_i^2+y_i^2))}{(\sum_{i=1}^n |\lambda_i|^2 (x_i^2+y_i^2))^{\frac{3}{2}}} \\ &+ x_j^2 \left(\frac{-2|\lambda_j|^2 (\alpha_j+\beta_j)(\sum_{i=1}^n |\lambda_i|^2 (x_i^2+y_i^2))}{(\sum_{i=1}^n |\lambda_i|^2 (x_i^2+y_i^2))^{\frac{5}{2}}}\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 J}{\partial y_j^2} =& \frac{2(\alpha_j+\beta_j)(\sum_{i=1}^n |\lambda_i|^2 (x_i^2+y_i^2)) - |\lambda_j|^2 (\sum_{i=1}^n (\alpha_i+\beta_i)(x_i^2+y_i^2))}{(\sum_{i=1}^n |\lambda_i|^2 (x_i^2+y_i^2))^{\frac{3}{2}}} \\ &+ y_j^2 \left(\frac{-2|\lambda_j|^2 (\alpha_j+\beta_j)(\sum_{i=1}^n |\lambda_i|^2 (x_i^2+y_i^2)) + 3|\lambda_j|^4 (\sum_{i=1}^n (\alpha_i+\beta_i)(x_i^2+y_i^2))}{(\sum_{i=1}^n |\lambda_i|^2 (x_i^2+y_i^2))^{\frac{5}{2}}}\right)\end{aligned}$$

$$\frac{\partial^2 J}{\partial x_k \partial y_j} = \frac{-2x_k y_j((\alpha_k+\beta_k)|\lambda_j|^2 + (\alpha_j+\beta_j)|\lambda_k|^2)}{(\sum_{i=1}^n |\lambda_i|^2 (x_i^2+y_i^2))^{\frac{3}{2}}} + \frac{3x_k y_j |\lambda_j|^2 |\lambda_k|^2 (\sum_{i=1}^n (\alpha_i+\beta_i)(x_i^2+y_i^2))}{(\sum_{i=1}^n |\lambda_i|^2 (x_i^2+y_i^2))^{\frac{5}{2}}}$$

Let S_1 be the set of all elements in S for which the Hessian matrix of J is positive definite. Then all the antieigenvectors of T are in S_1.

Proof. We know that the critical points of a function found by the Lagrange Multipliers Method is local minimum if the Hessian matrix of that function is positive definite. The proof of the above theorem follows from this fact. □

The sufficient condition in Theorem 2.3 is only for local minimum, not for the global minimum. Hence, Theorem 2.3 is helpful to construct the set of 'local' symmetric antieigenvectors and corresponding 'local' symmetric antieigenvalues.

Now, we give examples to estimate the symmetric antieigenvalues and symmetric antieigenvectors using the method described in Theorem 2.1.

Example 2.4 *Let* $T = \begin{pmatrix} 1+3i & 2 \\ 2 & 1+3i \end{pmatrix}$.

Then T is a 2×2 normal matrix with eigenvalues $\lambda_1 = \alpha_1 + i\beta_1 = 3+3i$ and $\lambda_2 = \alpha_2 + i\beta_2 = -1+3i$.

Then, $\frac{\alpha_1+\beta_1}{\sqrt{2}|\lambda_1|} = 1$ and $\frac{\alpha_2+\beta_2}{\sqrt{2}|\lambda_2|} = \frac{2}{\sqrt{20}}$. Therefore, $E = \{1, \frac{1}{\sqrt{5}}\}$.

Now,

$$\frac{\sqrt{2}(\alpha_1-\alpha_2+\beta_1-\beta_2)\{(\alpha_1+\beta_1)|\lambda_2|^2 - (\alpha_2+\beta_2)|\lambda_1|^2\}}{|\lambda_1|^2 - |\lambda_2|^2} = \sqrt{3}.$$

Therefore, $F = \{\sqrt{3}\}$. *Now,* $\mu_S = \min E \cup F = \min\{1, \frac{1}{\sqrt{5}}, \sqrt{3}\} = \frac{1}{\sqrt{5}}$
The eigenvector corresponding to the eigenvalue λ_2 *is the symmetric antieigenvalue of T.*

In the Theorem 2.1, we computed μ_S for a finite dimensional normal operator T on a Hilbert space H such that $ReT + ImT$ is a strictly positive operator and to prove the Theorem 2.1 we use some geometrical arguments on the convex function on the unit sphere.

Now we compute the symmetric antieigenvalue for a compact normal operator on a complex Hilbert space. Here we use spectral decomposition theorem of a normal operator on an infinite dimensional Hilbert space. We prove next theorem using only algebraic arguments.

Theorem 2.5 *Let T be compact normal operator on a complex Hilbert space H and* $\lambda_j = \alpha_j + i\beta_j$ *are the eigenvalues of T. Let*

$$\begin{aligned} E &= \left\{\frac{\alpha_j + \beta_j}{\sqrt{2}|\lambda_j|} : 1 \leq j \leq n\right\} \\ F &= \left\{\frac{\sqrt{2(\alpha_i - \alpha_j + \beta_i - \beta_j)\left\{(\alpha_j + \beta_j)|\lambda_i|^2 - (\alpha_i + \beta_i)|\lambda_j|^2\right\}}}{(|\lambda_i|^2 - |\lambda_j|^2)} \,\right| \\ 0 &\leq \frac{(\alpha_j + \beta_j)(|\lambda_i|^2 + |\lambda_j|^2) - 2(\alpha_i + \beta_i)|\lambda_j|^2}{(\alpha_i - \alpha_j + \beta_i - \beta_j)(|\lambda_i|^2 - |\lambda_j|^2)} \leq 1 \left.\vphantom{\frac{a}{b}}\right\} \end{aligned}$$

Then μ_S *is exactly equal to the smallest number in* $E \cup F$.
Furthermore, if

$$\mu_S = \frac{\sqrt{2\left(\alpha_i - \alpha_j + \beta_i - \beta_j\right)\left\{\left(\alpha_j + \beta_j\right)\left|\lambda_i\right|^2 - \left(\alpha_i + \beta_i\right)\left|\lambda_j\right|^2\right\}}}{\left(\left|\lambda_i\right|^2 - \left|\lambda_j\right|^2\right)},$$

then

$$\mu_S = \frac{Re\,(Tz, z) + Im\,(Tz, z)}{\sqrt{2}\|Tz\|\|z\|}$$

for some

$$z = \sum z_j$$

with

$$\begin{aligned}
\|z_i\|^2 &= \frac{(\alpha_j+\beta_j)\left(|\lambda_i|^2+|\lambda_j|^2\right)-2\left(\alpha_i+\beta_i\right)|\lambda_j|^2}{(\alpha_i-\alpha_j+\beta_i-\beta_j)\left(|\lambda_i|^2-|\lambda_j|^2\right)}\\
\|z_j\|^2 &= \frac{(\alpha_i+\beta_i)\left(|\lambda_i|^2+|\lambda_j|^2\right)-2\left(\alpha_j+\beta_j\right)|\lambda_i|^2}{(\alpha_i-\alpha_j+\beta_i-\beta_j)\left(|\lambda_i|^2-|\lambda_j|^2\right)}\\
\|z_k\| &= 0 \quad for \quad k\neq i, \quad k\neq j.
\end{aligned}$$

Proof. Let E_j be the eigenspace corresponding to the eigenvalue λ_j and P_j be the orthogonal projection on E_j. Then by the spectral decomposition theorem for a compcat normal operator we have $z=\sum_{j=1}^{j=\infty} z_j$ where $z_j \in E_j$.
For each vector $z\in H$ with $\|z\|=1$

$$\Phi_T(z)=\frac{\sum_{j=1}^{\infty}(\alpha_j+\beta_j)\|z_j\|^2}{\sqrt{2\sum_{j=1}^{\infty}|\lambda_j|^2\|z_j\|^2}} \quad and \quad \mu_S=\inf_{\|z\|=1}\Phi_T(z).$$

Let $z=\sum_{j=1}^{i=\infty} z_j$ be a symmetric antieigenvector of T. Now we consider the following cases:-

Case I: Suppose $z=z_j$ for some j. Clearly $Tz\neq\theta$. Then $\mu_S=\frac{\alpha_j+\beta_j}{\sqrt{2}|\lambda_j|}$.

Case II: Suppose $z=z_i+z_j$ for some i, j, $i\neq j$.
Then

$$\Phi_T(z)=\frac{(\alpha_i+\beta_i)\|z_i\|^2+(\alpha_j+\beta_j)\|z_j\|^2}{\sqrt{2}\sqrt{|\lambda_i|^2\|z_i\|^2+|\lambda_j|^2\|z_j\|^2}}$$

where $\|z_i\|^2+\|z_j\|^2=1$.
Let, $\|z_i\|^2=t$. Then, $\|z_j\|^2=1-\|z_i\|^2=1-t$.
Then.

$$\begin{aligned}
\Phi_T(z) &= \frac{(\alpha_i-\alpha_j+\beta_i-\beta_j)t+(\alpha_j+\beta_j)}{\sqrt{2}\sqrt{(|\lambda_i|^2-|\lambda_j|^2)t+|\lambda_j|^2}}\\
&= \frac{at+b}{\sqrt{ct+d}} = f(t)(\text{say}).
\end{aligned}$$

where

$$\begin{aligned}
&a=\alpha_i-\alpha_j+\beta_i-\beta_j\\
&b=\alpha_j+\beta_j\\
&c=2(|\lambda_i|^2-|\lambda_j|^2)\\
&d=2|\lambda_j|^2.
\end{aligned}$$

and f is a real valued function defined on the interval (0,1).
We find the minimum of f on the interval (0,1). Differentiating f(t) with respect to t, we get,

$$f'(t) = \frac{act + 2ad - bc}{(ct + d)^{\frac{3}{2}}}$$

Now, $f'(\tilde{t}) = 0 \Rightarrow \tilde{t} = \frac{bc-2ad}{ac}$.
Therefore,

$$\begin{aligned}\tilde{t} = \|x_i\|^2 &= \frac{(\alpha_j + \beta_j)(|\lambda_i|^2 - |\lambda_j|^2) - 2(\alpha_i - \alpha_j + \beta_i - \beta_j)\,|\lambda_j|^2}{(\alpha_i - \alpha_j + \beta_i - \beta_j)(|\lambda_i|^2 - |\lambda_j|^2)} \\ &= \frac{(\alpha_j + \beta_j)(|\lambda_i|^2 + |\lambda_j|^2) - 2(\alpha_i + \beta_i)\,|\lambda_j|^2}{(\alpha_i - \alpha_j + \beta_i - \beta_j)(|\lambda_i|^2 - |\lambda_j|^2)}\end{aligned}$$

and

$$\begin{aligned}\|z_j\|^2 &= 1 - \|z_i\|^2 \\ &= \frac{(\alpha_i + \beta_i)(|\lambda_i|^2 + |\lambda_j|^2) - 2(\alpha_j + \beta_j)\,|\lambda_i|^2}{(\alpha_i - \alpha_j + \beta_i - \beta_j)(|\lambda_i|^2 - |\lambda_j|^2)}.\end{aligned}$$

Now

$$f''(t) = \frac{-ac^2t - 4acd + 3bc^2}{(ct + d)^{\frac{5}{2}}}.$$

Since,

$$0 < \frac{(\alpha_j + \beta_j)(|\lambda_i|^2 + |\lambda_j|^2) - 2(\alpha_i + \beta_i)\,|\lambda_j|^2}{(\alpha_i - \alpha_j + \beta_i - \beta_j)(|\lambda_i|^2 - |\lambda_j|^2)} < 1$$

and

$$0 < \frac{(\alpha_i + \beta_i)(|\lambda_i|^2 + |\lambda_j|^2) - 2(\alpha_j + \beta_j)\,|\lambda_i|^2}{(\alpha_i - \alpha_j + \beta_i - \beta_j)(|\lambda_i|^2 - |\lambda_j|^2)} < 1$$

then $f''(\tilde{t}) > 0$.
Therefore f(t) attains its minimum value at $t = \tilde{t}$ and

$$f(\tilde{t}) = \frac{\sqrt{2a(bc - ad)}}{c}.$$

Substituting the values of a, b, c, d, we get,

$$\mu_S = \frac{\sqrt{2\,(\alpha_i - \alpha_j + \beta_i - \beta_j)\left\{(\alpha_j + \beta_j)\,|\lambda_i|^2 - (\alpha_i + \beta_i)\,|\lambda_j|^2\right\}}}{\left(|\lambda_i|^2 - |\lambda_j|^2\right)}.$$

Case III: Next suppose that z is a linear combination of more than two eigenvectors.
Let $\|z_i\|^2 = t_i$ for all i and

$$h(\mathbf{t}) = \frac{\sum_{j=1}^{\infty}(\alpha_j + \beta_j)t_j}{\sqrt{2\sum_{j=1}^{\infty}|\lambda_j|^2 t_j \sum_{j=1}^{\infty} t_j}}$$

where $\mathbf{t} = (t_1, t_2, ...)$ and $\sum_{j=1}^{\infty} t_j = 1$.
Since h attains its minimum value at $\mathbf{t}$ then $\frac{\partial h}{\partial t_i} = 0$ for all i.

$$\Longrightarrow 2(\alpha_i + \beta_i)a^2 - b|\lambda_i|^2 - a^2 b = 0 \forall i = 1, 2, .. \tag{7}$$

where

$$\begin{aligned} a^2 &= (Tz, Tz) \quad \textit{and} \\ b &= Re(Tz, Tz) + Im(Tz, Tz) \end{aligned}$$

Now we show that there exists a unit vector g which is a linear combination of two eigenvectors corresponding to distinct eigenvalues, say, $\lambda_i = \alpha_i + i\beta_i$ and $\lambda_j = \alpha_j + i\beta_j$ such that

$$\Phi_T(z) = \Phi_T(g)$$

and g is a symmetric antieigenvector.
Let e_i and e_j be unit eigenvectors corresponding to the eigenvalues λ_i and λ_j respectively. We claim that there exist scalars μ_i and μ_j such that

$$g = \mu_i e_i + \mu_j e_j$$

$$|\mu_i|^2 + |\mu_j|^2 = 1 \tag{8}$$

$$Re(Tg, g) + Im(Tg, g) = Re(Tz, z) + Im(Tz, z) \tag{9}$$

$$(Tg, Tg) = (Tz, Tz) \tag{10}$$

and g is symmetric antieigenvector.
Since λ_i and λ_j are eigenvalues of T, then λ_i and λ_j satisfy the equation (7) . Therefore

$$2(\alpha_i + \beta_i)a^2 - b|\lambda_i|^2 - a^2 b = 0 \tag{11}$$

$$2(\alpha_j + \beta_j)a^2 - b|\lambda_j|^2 - a^2 b = 0 \tag{12}$$

Solving the equations (8), (9) and (10), we get

$$|\mu_i|^2 = \frac{b - (\alpha_j + \beta_j)}{\alpha_i + \beta_i - \alpha_j - \beta_j}$$

and also

$$|\mu_i|^2 = \frac{a^2 - |\lambda_j|^2}{|\lambda_i|^2 - |\lambda_j|^2}.$$

Using the equations (11) and (12), we get the value of $|\mu_i|^2$ is same. Hence a g is obtained.
A simple calculation shows that

$$\begin{aligned} \mu_S &= \frac{b}{\sqrt{2}a} \\ &= \frac{\sqrt{2\left(\alpha_i - \alpha_j + \beta_i - \beta_j\right)\left\{\left(\alpha_j + \beta_j\right)|\lambda_i|^2 - \left(\alpha_i + \beta_i\right)|\lambda_j|^2\right\}}}{\left(|\lambda_i|^2 - |\lambda_j|^2\right)} \end{aligned}$$

Hence the proof is complete. □

We now give an example to illustrate the above theorem.

Example 2.6 *Suppose the normal operator T is $Tz = (1+i)(z, e_1)e_1 + (2+i)(z, e_2)e_2 + (1+2i)(z, e_3)e_3$.*
Clearly,

$$\begin{aligned} p &= |(z, e_1)|^2 + 2\,|(z, e_2)|^2 + |(z, e_3)|^2 = 1 + |(z, e_3)|^2 \\ q &= |(z, e_1)|^2 + |(z, e_2)|^2 + 2\,|(z, e_3)|^2 = 1 + |(z, e_3)|^2 \\ r^2 &= 2\,|(z, e_1)|^2 + 5\,|(z, e_1)|^2 + 5\,|(z, e_1)|^2 = 2 + \{|(z, e_2)|^2 + |(z, e_3)|^2\} \end{aligned}$$

Now,

$$\begin{aligned} \Phi_T(z) &= \frac{p+q}{\sqrt{2}r} \\ &= \frac{2+t}{\sqrt{2(2+3t)}} \end{aligned}$$

where $t = |(z, e_2)|^2 + |(z, e_3)|^2$ and $0 \leq t \leq 1$.
For the minimum value, we must have $t = \frac{2}{3}$. When $t = \frac{2}{3}$, $\mu_S = \frac{2\sqrt{2}}{3}$.
If we take $|(z, e_1)|^2 = |(z, e_2)|^2 = \frac{1}{3}$, then $|(z, e_3)|^2 = \frac{1}{3}$ and the unit vector $z = \frac{1}{\sqrt{3}}\left(e_1 + e_2 + e_3\right)$ is a symmetric antieigenvector of T.
Now we show that it is possible to construct a linear combination of only two eigenvectors corresponding to two eigenvalues for which minimum of $\Phi_T(z)$ attained.
Set $|(z, e_1)|^2 = \frac{1}{3}$ and $|(z, e_2)|^2 = \frac{2}{3}$ and $|(z, e_1)|^2 = 0$. Then $z = \frac{1}{\sqrt{3}}\left(e_1 + \sqrt{2}e_2\right)$ is the required symmetric antieigenvector.

In Theorem 2.5 we have computed symmetric antieigenvector for a compact normal operator. Now, we find symmetric antieigenvalue of a normal operator using the convexity property of numerical range.

Theorem 2.7 *Let T be normal operator such that* $(ReT + ImT)$ *is a strictly positive operator and* $S = \frac{1}{\sqrt{2}}(ReT + ImT) + iT^*T$. *Let the numerical range* $W(S)$ *of* S *is closed.*
Then either

$$\mu_S = \frac{\alpha_j + \beta_j}{\sqrt{2}\,|\lambda_j|} \text{ for some } \lambda_j = \alpha_j + i\beta_j$$

or

$$\mu_S = \frac{\sqrt{2\,(\alpha_i - \alpha_j + \beta_i - \beta_j)\left\{(\alpha_j + \beta_j)\,|\lambda_i|^2 - (\alpha_i + \beta_i)\,|\lambda_j|^2\right\}}}{\left(|\lambda_i|^2 - |\lambda_j|^2\right)}$$

for some $\lambda_i = \alpha_i + i\beta_i$ *and* $\lambda_j = \alpha_j + i\beta_j$ *where* $\lambda_i, \lambda_j \in \sigma(T)$, *where,* $\sigma(T)$ *is the spectrum of* T.

Proof. By Krein[14],

$$\mu_S^2 = \inf\{\frac{x^2}{y}|x + iy \in W(S)\}$$

Since T is a normal operator, then S is also normal. Since $W(S)$ is closed, then $W(S)$ is the convex hull of spectrum $\sigma(S)$ of S.
Suppopse the minimum of the function $p(x,y) = \frac{x^2}{y}$ is attained at $x_0 + iy_0 \in W(S)$. Then $x_0 + iy_0$ is a point lies on the polygon whose vertices are contained in the spectrum $\sigma(S)$ of S. Therefore, minimum of the function $p(x,y) = \frac{x^2}{y}$ is at the point where a member of the family of parabolas $y = x^2/k$ touches one point of this polygon. The point where a member of the family of parabolas $y = x^2/k$ touches the polygon is either a vertex of the polygon or on the straight line joining two vertices of the polygon.

Case I: The minimum of the function $p(x,y) = x^2/y$ is attained at some vertex of the polygon.
By the spectral mapping theorem of a normal operator, the vertices of the polygon are $\frac{\alpha_j+\beta_j}{\sqrt{2}} + i\,|\lambda_j|^2$, where $\lambda_j = \alpha_j + i\beta_j \in \sigma(T)$.
Suppose the infimum of $p(x,y) = x^2/y$ is attained at $\frac{\alpha_j+\beta_j}{\sqrt{2}} + i\,|\lambda_j|^2$, then

$$\mu_S = \frac{\alpha_j + \beta_j}{\sqrt{2}\,|\lambda_j|}.$$

Case II: Suppose the minimum of the function $p(x,y) = x^2/y$ is attained at some interior point of the line segment joining two vertices$A : \frac{\alpha_i+\beta_i}{\sqrt{2}} + i\,|\lambda_i|^2$ and $B : \frac{\alpha_j+\beta_j}{\sqrt{2}} + i\,|\lambda_j|^2$
Then the slope of the line segment joining the vertices A,B is

$$m = \frac{2(|\lambda_i|^2 - |\lambda_j|^2)}{\alpha_i - \alpha_j + \beta_i - \beta_j} \tag{13}$$

Therefore the equation of the line segment joining the vertices A,B is

$$q(x,y) = y - |\lambda_j|^2 - m(x - \frac{\alpha_j + \beta_j}{\sqrt{2}\,|\lambda_j|} \tag{14}$$

Clearly, m is a non-zero finite number, since no member of this family of parabolas cannot touch the straight line which is parallel to x-axis or y-axis. Therefore, $|\lambda_i| \neq |\lambda_j|$
Now we use the Lagranges multiplier's method to find the minimum of the function

$$p(x,y) = x^2/y$$

on the line segment $q(x,y) = 0$.
Therefore at the point where the minimum of $p(x,y)$ is attained,

$$\frac{\partial p}{\partial x} = \xi \frac{\partial q}{\partial x}$$
$$and$$
$$\frac{\partial p}{\partial y} = \xi \frac{\partial q}{\partial y}$$

for some ξ.

Therefore,

$$\frac{2x}{y} = -\xi m \tag{15}$$
$$and$$
$$-\frac{x^2}{y^2} = \xi \tag{16}$$

Eliminating ξ from (15) and (16), we get

$$y = \frac{mx}{2}.$$

Substituting $y = \frac{mx}{2}$ in (14), we get

$$x = \frac{\sqrt{2}\{(\alpha_j + \beta_j)\,|\lambda_i|^2 - (\alpha_i + \beta_i)\,|\lambda_j|^2\}}{|\lambda_i|^2 - |\lambda_j|^2}$$

and

$$y = \frac{(\alpha_j + \beta_j)\,|\lambda_i|^2 - (\alpha_i + \beta_i)\,|\lambda_j|^2}{\alpha_i - \alpha_j + \beta_i - \beta_j}$$

Therefore, the minimum of

$$p(x,y) = \frac{2\,(\alpha_i - \alpha_j + \beta_i - \beta_j)\left\{(\alpha_j + \beta_j)\,|\lambda_i|^2 - (\alpha_i + \beta_i)\,|\lambda_j|^2\right\}}{\left(|\lambda_i|^2 - |\lambda_j|^2\right)^2}$$

.

Hence,

$$\mu_S = \frac{\sqrt{2\,(\alpha_i - \alpha_j + \beta_i - \beta_j)\left\{(\alpha_j + \beta_j)\,|\lambda_i|^2 - (\alpha_i + \beta_i)\,|\lambda_j|^2\right\}}}{\left(|\lambda_i|^2 - |\lambda_j|^2\right)}.$$

□

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