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On Pairwise Bicontinuous Maps

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Abstract

The purpose of this paper is to introduce the concept of pairwise bicontinuous maps and investigate some of their characterizations.

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1 Introduction

The concept of closure spaces was introduced by E. Čech [3] and then studied by many authors, see e.g. [4, 5, 7, 8]. J.C. Kelly [6] introduced the notion of bitopological spaces. Such spaces are equipped with two arbitrary topologies. Furthermore, Kelly extended some of the standard results of separation axioms in a topological space to a bitopological space. Thereafter, a large number of papers have been written to generalize topological concepts to bitopological setting. In this paper, we introduce the concept of pairwise bicontinuous maps in biclosure spaces and characterize their properties.

2 Preliminaries

A map $u : P(X) \to P(X)$ defined on the power set $P(X)$ of a set X is called a *closure operator* on X and the pair (X, u) is called a *closure space* if the following axioms are satisfied :

(N1) $u\emptyset = \emptyset$,

(N2) $A \subseteq uA$ for every $A \subseteq X$, (N3) $A \subseteq B \Rightarrow uA \subseteq uB$ for all $A, B \subseteq X$.

A closure operator u on a set X is called *additive*(respectively, *idempotent*) if $A, B \subseteq X \Rightarrow u(A \cup B) = uA \cup uB$ (respectively, $A \subseteq X \Rightarrow uuA = uA$). A subset $A \subseteq X$ is *closed* in the closure space (X, u) if $uA = A$ and it is *open* if its complement in X is closed. The empty set and the whole space are both open and closed.

A closure space (Y, v) is said to be a *subspace* of (X, u) if $Y \subseteq X$ and $vA = uA \cap Y$ for each subset $A \subseteq Y$. If Y is closed in (X, u), then the subspace (Y, v) of (X, u) is said to be closed too. Let (X, u) and (Y, v) be closure spaces. A map $f : (X, u) \to (Y, v)$ is said to be *continuous* if $f(uA) \subseteq vf(A)$ for every subset $A \subseteq X$.

One can see that a map $f : (X, u) \to (Y, v)$ is continuous if and only if $uf^{-1}(B) \subseteq f^{-1}(vB)$ for every subset $B \subseteq Y$.

Clearly, if$f : (X, u) \to (Y, v)$ is continuous, then $f^{-1}(F)$ is a closed subset of (X, u) for every closed subset F of (Y, v). Let (X, u) and (Y, v) be closure spaces. A map $f : (X, u) \to (Y, v)$ is said to be *closed* (resp. *open*) if $f(F)$ is a closed (resp. open) subset of (Y, v) whenever F is a closed (resp. open) subset of (X, u).

The *product* of a family $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ of closure spaces, denoted by $\prod\limits_{\alpha \in I} (X_\alpha, u_\alpha)$, is the closure space $\big(\prod\limits_{\alpha \in I} X_\alpha, u\big)$ where $\prod\limits_{\alpha \in I} X_\alpha$ denotes the cartesian product of sets X_α, $\alpha \in I$, and u is the closure operator generated by the projections $\pi_\alpha : \prod\limits_{\alpha \in I} X_\alpha \to X_\alpha$, $\alpha \in I$, i.e., is defined by $uA = \prod\limits_{\alpha \in I} u_\alpha \pi_\alpha(A)$ for each $A \subseteq \prod\limits_{\alpha \in I} X_\alpha$.

Clearly, if $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ is a family of closure spaces, then the projection map $\pi_\beta : \prod\limits_{\alpha \in I} (X_\alpha, u_\alpha) \to (X_\beta, u_\beta)$ is closed and continuous for every $\beta \in I$.

Proposition 2.1. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces and let $\beta \in I$. Then F is a closed subset of (X_β, u_β) if and only if $F \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a closed subset of $\prod\limits_{\alpha \in I} (X_\alpha, u_\alpha)$.*

Proof. Let F be a closed subset of (X_β, u_β). Since π_β is continuous, $\pi_\beta^{-1}(F)$ is a closed subset of $\prod\limits_{\alpha \in I} (X_\alpha, u_\alpha)$. But $\pi_\beta^{-1}(F) = F \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$, hence $F \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a closed subset of $\prod\limits_{\alpha \in I} (X_\alpha, u_\alpha)$.

Conversely, let $F \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ be a closed subset of $\prod\limits_{\alpha \in I} (X_\alpha, u_\alpha)$. Since π_β is closed, $\pi_\beta\Big(F \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha\Big) = F$ is a closed subset of (X_β, u_β). □

The following statement is evident :

Proposition 2.2. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces and let $\beta \in I$. Then G is an open subset of (X_β, u_β) if and only if $G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is an open subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$.*

Definition 2.3. A *biclosure space* is a triple (X, u_1, u_2) where X is a set and u_1, u_2 are two closure operators on X.

Definition 2.4. A subset A of a biclosure space (X, u_1, u_2) is called *closed* if $u_1 u_2 A = A$. The complement of closed set is called *open.*

Clearly, A is a closed subset of a biclosure space (X, u_1, u_2) if and only if A is both a closed subset of (X, u_1) and (X, u_2).

Let A be a closed subset of a biclosure space (X, u_1, u_2) . The following conditions are equivalent

(i) $u_2 u_1 A = A$,

(ii) $u_1 A = A$, $u_2 A = A$.

Definition 2.5. Let (X, u_1, u_2) be a biclosure space. A biclosure space (Y, v_1, v_2) is called a *subspace* of (X, u_1, u_2) if $Y \subseteq X$ and $v_i A = u_i A \cap Y$ for each $i \in \{1, 2\}$ and each subset $A \subseteq Y$.

Proposition 2.6. *Let (X, u_1, u_2) be a biclosure space and let (Y, v_1, v_2) be a closed subspace of (X, u_1, u_2). If F is a closed subset of (Y, v_1, v_2), then F is a closed subset of (X, u_1, u_2).*

Proof. Let F be a closed subset of (Y, v_1, v_2). Then $v_1 F = F$ and $v_2 F = F$. Since Y is both a closed subset of (X, u_1) and (X, u_2), $u_1 F = F$ and $u_2 F = F$. Consequently, F is both a closed subset of (X, u_1) and (X, u_2). Therefore, F is a closed subset of (X, u_1, u_2). □

Proposition 2.7. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces and let $\beta \in I$. Then F is a closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$ if and only if $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a closed subset of $\prod_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha)$.*

Proof. Let $\beta \in I$ and let F be a closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$. Then F is a closed subset of (X_β, u^1_β) and (X_β, u^2_β), respectively. Since $\pi_\beta : \prod_{\alpha \in I} (X_\alpha, u^1_\alpha) \to (X_\beta, u^1_\beta)$ is continuous, $\pi_\beta^{-1}(F) = F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a closed subset of $\prod_{\alpha \in I} (X_\alpha, u^1_\alpha)$.

Similarly, since $\pi_\beta : \prod\limits_{\alpha\in I}(X_\alpha, u^2_\alpha) \to (X_\beta, u^2_\beta)$ is continuous, $\pi^{-1}_\beta(F) = F \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod\limits_{\alpha\in I}(X_\alpha, u^2_\alpha)$. Consequently, $F \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod\limits_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$.

Conversely, let $F \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ be a closed subset of $\prod\limits_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$. Then $F \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod\limits_{\alpha\in I}(X_\alpha, u^1_\alpha)$ and $\prod\limits_{\alpha\in I}(X_\alpha, u^2_\alpha)$, respectively. Since $\pi_\beta : \prod\limits_{\alpha\in I}(X_\alpha, u^1_\alpha) \to (X_\beta, u^1_\beta)$ is closed, $\pi_\beta\Big(F \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big) = F$ is a closed subset of (X_β, u^1_β). Similarly, since $\pi_\beta : \prod\limits_{\alpha\in I}(X_\alpha, u^2_\alpha) \to (X_\beta, u^2_\beta)$ is closed, $\pi_\beta\Big(F \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big) = F$ is a closed subset of (X_β, u^2_β). Consequently, F is a closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$. □

The following statement is evident:

Proposition 2.8. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces and let $\beta \in I$. Then G is an open subset of $(X_\beta, u^1_\beta, u^2_\beta)$ if and only if $G\times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is an open subset of $\prod\limits_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$.*

3 Pairwise Bicontinuous Maps

In this section, we introduce the concept of pairwise bicontinuous maps and study some of their properties.

Definition 3.1. Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $i \in \{1, 2\}$. A map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is called *i-continuous* if the map $f : (X, u_i) \to (Y, v_i)$ is continuous. A map f is called *continuous* if f is i-continuous for each $i \in \{1, 2\}$.

Definition 3.2. Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces. A map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is called *bicontinuous* if the map $f : (X, u_1) \to (Y, v_2)$ is continuous.

Definition 3.3. Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces. A map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is called *pairwise bicontinuous* if $f : (X, u_1) \to (Y, v_2)$ and $f : (X, u_2) \to (Y, v_1)$ are continuous.

Remark 1. *Every pairwise bicontinuous map is bicontinuous but the converse is not true as may be seen from the following example.*

Example 3.4. *Let $X = \{a, b\} = Y$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$, $u_1\{a\} = \{a\}$, $u_1\{b\} = \{b\}$ and $u_1X = X$. Define a closure operator u_2 on X by $u_2\emptyset = \emptyset$, $u_2\{a\} = u_2\{b\} = u_2X = X$. Define a closure operator v_1 on Y by $v_1\emptyset = \emptyset$, $v_1\{a\} = \{a\}$, $v_1\{b\} = \{b\}$ and $v_1Y = Y$. Define a closure operator v_2 on Y by $v_2\emptyset = \emptyset$, $v_2\{a\} = \{a\}$, $v_2\{b\} = \{b\}$ and $v_2Y = Y$. Let $\varphi : (X, u_1, u_2) \to (Y, v_1, v_2)$ be the identity map. Then φ is bicontinuous but it is not pairwise bicontinuous because $\varphi : (X, u_2) \to (Y, v_1)$ is not continuous.*

Proposition 3.5. *Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces. Then $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is pairwise bicontinuous if and only if $u_1f^{-1}(B) \subseteq f^{-1}(v_2B)$ and $u_2f^{-1}(B) \subseteq f^{-1}(v_1B)$ for every $B \subseteq Y$.*

Proof. Let $B \subseteq Y$. Then $f^{-1}(B) \subseteq X$. Since f is pairwise bicontinuous, $f(u_1f^{-1}(B)) \subseteq v_2f(f^{-1}(B)) \subseteq v_2B$ and $f(u_2f^{-1}(B)) \subseteq v_1f(f^{-1}(B)) \subseteq v_1B$. Therefore, $u_1f^{-1}(B) \subseteq f^{-1}(v_2B)$ and $u_2f^{-1}(B) \subseteq f^{-1}(v_1B)$.

Conversely, let $A \subseteq X$. Then $f(A) \subseteq Y$. Thus $u_1f^{-1}(f(A)) \subseteq f^{-1}(v_2f(A))$ and $u_2f^{-1}(f(A)) \subseteq f^{-1}(v_1f(A))$. Consequently, $f(u_1A) \subseteq f(u_1f^{-1}(f(A))) \subseteq f(f^{-1}(v_2f(A))) \subseteq v_2f(A)$ and $f(u_2A) \subseteq f(u_2f^{-1}(f(A))) \subseteq f(f^{-1}(v_1f(A))) \subseteq v_1f(A)$. Hence, f is pairwise bicontinuous. □

Proposition 3.6. *Let (X, u_1, u_2), (Y, v_1, v_2) and (Z, w_1, w_2) be biclosure spaces. If $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is pairwise bicontinuous and $g : (Y, v_1, v_2) \to (Z, w_1, w_2)$ is continuous, then $g \circ f : (X, u_1, u_2) \to (Z, w_1, w_2)$ is pairwise bicontinuous.*

Proof. Let $A \subseteq X$. Since $g \circ f(u_1A) = g(f(u_1A))$, $g \circ f(u_2A) = g(f(u_2A))$ and f is pairwise bicontinuous, $g(f(u_1A)) \subseteq g(v_2f(A))$ and $g(f(u_2A)) \subseteq g(v_1f(A))$. Since g is continuous, $g(v_2f(A)) \subseteq w_2g(f(A))$ and $g(v_1f(A)) \subseteq w_1g(f(A))$. Thus $g \circ f(u_1A) \subseteq w_2g \circ f(A)$ and $g \circ f(u_2A) \subseteq w_1g \circ f(A)$. Consequently, $g \circ f$ is pairwise bicontinuous. □

Proposition 3.7. *Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let (Z, w_1, w_2) be a closed subspaces of (X, u_1, u_2). If $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is pairwise bicontinuous, then $f|_Z : (Z, w_1, w_2) \to (Y, v_1, v_2)$ is pairwise bicontinuous.*

Proof. Let f be pairwise bicontinuous. If $B \subseteq Z$, then $f|_Z(w_1(B)) = f|_Z(u_1B \cap Z = f|_Z(u_1B) = f(u_1B) \subseteq v_2f(B) = v_2f|_Z(B)$ and $f|_Z(w_2(B)) = f|_Z(u_2B \cap Z = f|_Z(u_2B) = f(u_2B) \subseteq v_1f(B) = v_1f|_Z(B)$. Consequently, $f|_Z$ is pairwise bicontinuous. □

Definition 3.8. Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $i \in \{1, 2\}$. A map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is called *i-closed* (resp. *i-open*) if the map $f : (X, u_i) \to (Y, v_i)$ is closed (resp. open). A map f is called *closed* (resp. *open*) if f is i-closed (resp. i-open) for each $i \in \{1, 2\}$.

Definition 3.9. Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces. A map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is called *biclosed* (resp. *biopen*) if the map $f : (X, u_1) \to (Y, v_2)$ is closed (resp. open).

Definition 3.10. Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces. A map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is called *pairwise biclosed* (resp. *pairwise biopen*) if $f : (X, u_1) \to (Y, v_2)$ and $f : (X, u_2) \to (Y, v_1)$ are closed (resp. open).

Remark 2. *Every pairwise biclosed map is biclosed but the converse is not true as may be seen from the following example.*

Example 3.11. *Let $X = \{a, b\} = Y$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$, $u_1\{a\} = \{a\}$, $u_1\{b\} = \{b\}$ and $u_1X = X$. Define a closure operator u_2 on X by $u_2\emptyset = \emptyset$, $u_2\{a\} = \{a\}$ and $u_2\{b\} = u_2X = X$. Define a closure operator v_1 on Y by $v_1\emptyset = \emptyset$ and $v_1\{a\} = v_1\{b\} = v_1Y = Y$. Define a closure operator v_2 on Y by $v_2\emptyset = \emptyset$, $v_2\{a\} = \{a\}$, $v_2\{b\} = \{b\}$ and $v_2Y = Y$. Let $\varphi : (X, u_1, u_2) \to (Y, v_1, v_2)$ be the identity map. Then φ is biclosed but it is not pairwise biclosed because $\varphi : (X, u_2) \to (Y, v_1)$ is not closed.*

Remark 3. *Every pairwise biopen map is biopen but the converse is not true as may be seen from the following example.*

Example 3.12. *Let $X = \{a, b\} = Y$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$, $u_1\{a\} = \{a\}$, $u_1\{b\} = \{b\}$ and $u_1X = X$. Define a closure operator u_2 on X by $u_2\emptyset = \emptyset$, $u_2\{a\} = \{a\}$ and $u_2\{b\} = u_2X = X$. Define a closure operator v_1 on Y by $v_1\emptyset = \emptyset$, $v_1\{b\} = \{b\}$ and $v_1\{a\} = v_1Y = Y$. Define a closure operator v_2 on Y by $v_2\emptyset = \emptyset$, $v_2\{a\} = \{a\}$, $v_2\{b\} = \{b\}$ and $v_2Y = Y$. Let $\varphi : (X, u_1, u_2) \to (Y, v_1, v_2)$ be the identity map. Then φ is biopen but it is not pairwise biopen because $\varphi : (X, u_2) \to (Y, v_1)$ is not open.*

Proposition 3.13. *Let (X, u_1, u_2), (Y, v_1, v_2) and (Z, w_1, w_2) be biclosure spaces. If $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is closed and $g : (Y, v_1, v_2) \to (Z, w_1, w_2)$ is pairwise biclosed, then $g \circ f : (X, u_1, u_2) \to (Z, w_1, w_2)$ is pairwise biclosed.*

Proof. Let F be a closed subset of (X, u_1) and let F' be a closed subset of (X, u_2). Since f is closed, $f(F)$ is a closed subset of (Y, v_1) and $f(F')$ is a closed subset of (Y, v_2). Since g is pairwise biclosed, $g(f(F))$ is a closed subset of (Z, w_2) and $g(f(F'))$ is a closed subset of (Z, w_1). Hence, $g \circ f(F)$ is a closed subset of (Z, w_2) and $g \circ f(F')$ is a closed subset of (Z, w_1). Consequently, $g \circ f$ is pairwise biclosed. □

Proposition 3.14. *Let (X, u_1, u_2), (Y, v_1, v_2) and (Z, w_1, w_2) be biclosure spaces. If $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is open and $g : (Y, v_1, v_2) \to (Z, w_1, w_2)$ is pairwise biopen, then $g \circ f : (X, u_1, u_2) \to (Z, w_1, w_2)$ is pairwise biopen.*

Proposition 3.15. *Let (X, u_1, u_2), (Y, v_1, v_2) and (Z, w_1, w_2) be biclosure spaces. Let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ and $g : (Y, v_1, v_2) \to (Z, w_1, w_2)$ be maps. Then*

(i) If $g \circ f$ is pairwise biclosed and f is surjective and continuous, then g is pairwise biclosed.

(ii) If $g \circ f$ is pairwise biclosed and g is injective and continuous, then f is pairwise biclosed.

Proof. (i) Let F be a closed subset of (Y, v_1) and let F' be a closed subset of (Y, v_2). Since f is continuous, $f^{-1}(F)$ is a closed subset of (X, u_1) and $f^{-1}(F')$ is a closed subset of (X, u_2). Since $g \circ f$ is pairwise biclosed and f is surjective, $g \circ f(f^{-1}(F)) = g(F)$ is a closed subset of (Z, w_2) and $g \circ f(f^{-1}(F')) = g(F)$ is a closed subset of (Z, w_1). Hence, g is pairwise biclosed.

(ii) Let F be a closed subset of (X, u_1) and let F' be a closed subset of (X, u_2). Since $g \circ f$ is pairwise biclosed, $g \circ f(F)$ is a closed subset of (Z, w_2) and $g \circ f(F')$ is a closed subset of (Z, w_1). Since g is continuous and injective, $g^{-1}(g \circ f(F)) = f(F)$ is a closed subset of (Y, v_2) and $g^{-1}(g \circ f(F')) = f(F')$ is a closed subset of (Y, v_1). Therefore, f is pairwise biclosed. □

Proposition 3.16. *Let (X, u_1, u_2), (Y, v_1, v_2) and (Z, w_1, w_2) be biclosure spaces. Let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ and $g : (Y, v_1, v_2) \to (Z, w_1, w_2)$ be maps. Then*

(i) If $g \circ f$ is pairwise biopen and f is surjective and continuous, then g is pairwise biopen.

(ii) If $g \circ f$ is pairwise biopen and g is injective and continuous, then f is pairwise biopen.

The following statement is evident:

Lemma 3.17. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces. Then for each $\beta \in I$, the projection map $\pi_\beta : \prod_{\alpha \in I}(X_\alpha, u^1_\alpha, u^2_\alpha) \to (X_\beta, u^1_\beta, u^2_\beta)$ is closed.*

Proposition 3.18. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ and $\{(Y_\alpha, v^1_\alpha, v^2_\alpha) : \alpha \in I\}$ be families of biclosure spaces. For each $\alpha \in I$, let $f_\alpha : X_\alpha \to Y_\alpha$ be a surjection and let $f : \prod_{\alpha \in I} X_\alpha \to \prod_{\alpha \in I} Y_\alpha$ be defined by $f((x_\alpha)_{\alpha \in I}) = (f_\alpha(x_\alpha))_{\alpha \in I}$. Then $f : \prod_{\alpha \in I}(X_\alpha, u^1_\alpha, u^2_\alpha) \to \prod_{\alpha \in I}(Y_\alpha, v^1_\alpha, v^2_\alpha)$ is pairwise biclosed if and only if $f_\alpha : (X_\alpha, u^1_\alpha, u^2_\alpha) \to (Y_\alpha, v^1_\alpha, v^2_\alpha)$ is pairwise biclosed for each $\alpha \in I$.*

Proof. Let $\beta \in I$ and let F be a closed subset of (X_β, u^1_β) and F' be a closed subset of (X_β, u^2_β). Then $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a closed subset of $\prod_{\alpha \in I}(X_\alpha, u^1_\alpha)$ and

$F' \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^2_\alpha)$. Since f is pairwise biclosed, $f\Big(F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big)$ is a closed subset of $\prod_{\alpha\in I}(Y_\alpha, v^2_\alpha)$ and $f\Big(F' \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big)$ is a closed subset of $\prod_{\alpha\in I}(Y_\alpha, v^1_\alpha)$. But $f\Big(F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big) = f_\beta(F) \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} Y_\alpha$ and $f\Big(F' \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big) = f_\beta(F') \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} Y_\alpha$, hence $f_\beta(F) \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} Y_\alpha$ is a closed subset of $\prod_{\alpha\in I}(Y_\alpha, v^2_\alpha)$ and $f_\beta(F') \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} Y_\alpha$ is a closed subset of $\prod_{\alpha\in I}(Y_\alpha, v^1_\alpha)$. By Proposition 2.1, $f_\beta(F)$ is a closed subset of (Y_β, v^2_β) and $f_\beta(F')$ is a closed subset of (Y_β, v^1_β). Hence, f_β is pairwise biclosed.

Conversely, let f_β be pairwise biclosed for each $\beta \in I$. Suppose that f is not pairwise biclosed. Therefore, $f : \prod_{\alpha\in I}(X_\alpha, u^1_\alpha) \to \prod_{\alpha\in I}(Y_\alpha, v^2_\alpha)$ is not closed or $f : \prod_{\alpha\in I}(X_\alpha, u^2_\alpha) \to \prod_{\alpha\in I}(Y_\alpha, v^1_\alpha)$ is not closed. If $f : \prod_{\alpha\in I}(X_\alpha, u^1_\alpha) \to \prod_{\alpha\in I}(Y_\alpha, v^2_\alpha)$ is not closed. Then there exists a closed subset F of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha)$ such that $\prod_{\beta\in I} v^2_\beta \pi_\beta(f(F)) \nsubseteq f(F)$. Therefore, there exists $\beta \in I$ such that $v^2_\beta f_\beta(\pi_\beta(F)) \nsubseteq f_\beta(\pi_\beta(F))$. By Lemma 3.17, $\pi_\beta(F)$ is a closed subset of (X_β, u^1_β). Since f_β is pairwise biclosed, $f_\beta(\pi_\beta(F))$ is a closed subset of (Y_β, v^2_β). This is a contradiction. If $f : \prod_{\alpha\in I}(X_\alpha, u^2_\alpha) \to \prod_{\alpha\in I}(Y_\alpha, v^1_\alpha)$ is not closed. Then there exists a closed subset F' of $\prod_{\alpha\in I}(X_\alpha, u^2_\alpha)$ such that $\prod_{\beta\in I} v^1_\beta \pi_\beta(f(F')) \nsubseteq f(F')$. Therefore, there exists $\beta \in I$ such that $v^1_\beta f_\beta(\pi_\beta(F')) \nsubseteq f_\beta(\pi_\beta(F'))$. By Lemma 3.17, $\pi_\beta(F')$ is a closed subset of (X_β, u^2_β). Since f_β is pairwise biclosed, $f_\beta(\pi_\beta(F'))$ is a closed subset of (Y_β, v^1_β). This is a contradiction. □

Proposition 3.19. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ and $\{(Y_\alpha, v^1_\alpha, v^2_\alpha) : \alpha \in I\}$ be families of biclosure spaces. For each $\alpha \in I$, let $f_\alpha : X_\alpha \to Y_\alpha$ be a surjection and let $f : \prod_{\alpha\in I} X_\alpha \to \prod_{\alpha\in I} Y_\alpha$ be defined by $f((x_\alpha)_{\alpha\in I}) = (f_\alpha(x_\alpha))_{\alpha\in I}$. If $f : \prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha) \to \prod_{\alpha\in I}(Y_\alpha, v^1_\alpha, v^2_\alpha)$ is pairwise biopen, then $f_\alpha : (X_\alpha, u^1_\alpha, u^2_\alpha) \to (Y_\alpha, v^1_\alpha, v^2_\alpha)$ is pairwise biopen for each $\alpha \in I$.*

The following statement is evident:

Proposition 3.20. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces. Then for each $\beta \in I$, the projection map $\pi_\beta : \prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha) \to (X_\beta, u^1_\beta, u^2_\beta)$ is continuous.*

Proposition 3.21. *Let (X, u_1, u_2) be a biclosure space, $\{(Y_\alpha, v^1_\alpha, v^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces and $f : X \to \prod_{\alpha\in I} Y_\alpha$ be a map. Then f :*

$(X,u_1,u_2) \to \prod_{\alpha\in I}(Y_\alpha,v_\alpha^1,v_\alpha^2)$ *is pairwise bicontinuous if and only if* $\pi_\alpha \circ f$: $(X,u_1,u_2) \to (Y_\alpha,v_\alpha^1,v_\alpha^2)$ *is pairwise bicontinuous for each* $\alpha \in I$.

Proof. Let f be pairwise bicontinuous. Since π_α is continuous for each $\alpha \in I$, $\pi_\alpha \circ f$ is pairwise bicontinuous for each $\alpha \in I$.

Conversely, let $\pi_\alpha \circ f$ be pairwise bicontinuous for each $\alpha \in I$. Suppose that f is not pairwise bicontinuous. Consequently, $f : (X,u_1,) \to \prod_{\alpha\in I}(Y_\alpha,v_\alpha^2)$ is not bicontinuous or $f : (X,u_2) \to \prod_{\alpha\in I}(Y_\alpha,v_\alpha^1)$ is not bicontinuous. If f : $(X,u_1) \to \prod_{\alpha\in I}(Y_\alpha,v_\alpha^2)$ is not bicontinuous. Then there exists a subset A of X such that $f(u_1A) \nsubseteq \prod_{\alpha\in I} v_\alpha^2\pi_\alpha(f(A))$. Therefore, there exists $\beta \in I$ such that $\pi_\beta(f(u_1A)) \nsubseteq v_\beta^2\pi_\beta(f(A))$. This is contradicts the bicontinuity of $\pi_\beta \circ f$. If $f : (X,u_2) \to \prod_{\alpha\in I}(Y_\alpha,v_\alpha^1)$ is not bicontinuous. Then there exists a subset A of X such that $f(u_2A) \nsubseteq \prod_{\alpha\in I} v_\alpha^1\pi_\alpha(f(A))$. Therefore, there exists $\beta \in I$ such that $\pi_\beta(f(u_2A)) \nsubseteq v_\beta^1\pi_\beta(f(A))$. This is contradicts the bicontinuity of $\pi_\beta \circ f$. Hence, f is pairwise bicontinuous. □

Proposition 3.22. *Let* $\{(X_\alpha,u_\alpha^1,u_\alpha^2) : \alpha \in I\}$ *and* $\{(Y_\alpha,v_\alpha^1,v_\alpha^2) : \alpha \in I\}$ *be families of biclosure spaces. For each* $\alpha \in I$, *let* $f_\alpha : X_\alpha \to Y_\alpha$ *be a map and let* $f : \prod_{\alpha\in I} X_\alpha \to \prod_{\alpha\in I} Y_\alpha$ *be defined by* $f((x_\alpha)_{\alpha\in I}) = (f_\alpha(x_\alpha))_{\alpha\in I}$. *Then* $f : \prod_{\alpha\in I}(X_\alpha,u_\alpha^1,u_\alpha^2) \to \prod_{\alpha\in I}(Y_\alpha,v_\alpha^1,v_\alpha^2)$ *is pairwise bicontinuous if and only if* $f_\alpha : (X_\alpha,u_\alpha^1,u_\alpha^2) \to (Y_\alpha,v_\alpha^1,v_\alpha^2)$ *is pairwise bicontinuous for each* $\alpha \in I$.

Proof. Let f be pairwise bicontinuous, let $\beta \in I$ and let $A \subseteq X_\beta$. Then $f_\beta(u_\beta^1A) = \pi_\beta\Big(f_\beta(u_\beta^1A)\times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} f_\alpha(u_\alpha^1X_\alpha)) = \pi_\beta(f(u_\beta^1A\times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} u_\alpha^1X_\alpha\Big)) = \pi_\beta(f(\prod_{\alpha\in I} u_\alpha^1\pi_\alpha(A\times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha))) \subseteq \pi_\beta(\prod_{\alpha\in I} v_\alpha^2\pi_\alpha(f(A\times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha))) = \pi_\beta(\prod_{\alpha\in I} v_\alpha^2\pi_\alpha(f_\beta(A)\times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} f_\alpha(X_\alpha))) = \pi_\beta(v_\beta^2f_\beta(A)\times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} v_\alpha^2f_\alpha(X_\alpha)) = v_\beta^2f_\beta(A)$ and $f_\beta(u_\beta^2A) = \pi_\beta\Big(f_\beta(u_\beta^2A)\times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} f_\alpha(u_\alpha^2X_\alpha)) = \pi_\beta(f(u_\beta^2A \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} u_\alpha^2X_\alpha\Big)) = \pi_\beta(f(\prod_{\alpha\in I} u_\alpha^2\pi_\alpha(A \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha))) \subseteq \pi_\beta(\prod_{\alpha\in I} v_\alpha^1\pi_\alpha(f(A \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha))) = \pi_\beta(\prod_{\alpha\in I} v_\alpha^1\pi_\alpha(f_\beta(A)\times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} f_\alpha(X_\alpha))) = \pi_\beta(v_\beta^1f_\beta(A)\times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} v_\alpha^1f_\alpha(X_\alpha)) = v_\beta^1f_\beta(A)$. Hence, f_β is pairwise bicontinuous.

Conversely, let f_α be pairwise bicontinuous for each $\alpha \in I$ and let $A \subseteq \prod_{\alpha\in I} X_\alpha$. Then $f(\prod_{\alpha\in I} u_\alpha^1\pi_\alpha(A)) = \prod_{\alpha\in I} f_\alpha(\prod_{\alpha\in I} u_\alpha^1\pi_\alpha(A)) = \prod_{\alpha\in I} f_\alpha(u_\alpha^1\pi_\alpha(A)) \subseteq \prod_{\alpha\in I} v_\alpha^2f_\alpha(\pi_\alpha(A)) = \prod_{\alpha\in I} v_\alpha^2\pi_\alpha(f(A))$ and $f(\prod_{\alpha\in I} u_\alpha^2\pi_\alpha(A)) = \prod_{\alpha\in I} f_\alpha(\prod_{\alpha\in I} u_\alpha^2\pi_\alpha(A)) =$

$\prod_{\alpha\in I} f_\alpha(u^2_\alpha\pi_\alpha(A)) \subseteq \prod_{\alpha\in I} v^1_\alpha f_\alpha(\pi_\alpha(A)) = \prod_{\alpha\in I} v^1_\alpha\pi_\alpha(f(A))$. Therefore f is pairwise bicontinuous. □

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