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∂-Closed Maps in Biclosure Spaces

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Abstract

In this paper, we introduce the concept of ∂-closed maps, by using ∂-closed sets in biclosure spaces and investigate some of their characterizations.

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1 Introduction

Generalized closed sets, briefly g-closed sets, in a topological space were introduced by N. Levine [8] in order to extend some important properties of closed sets to a larger family of sets. For instance, it was shown that compactness, normality and completeness in a uniform space are inherited by g-closed subsets. J.C. Kelly [7] introduced the notion of bitopological spaces. Such spaces are equipped with two arbitrary topologies. Furthermore, Kelly extended some of the standard results of separation axioms in a topological space to a bitopological space. Closure spaces were introduced by E.Čech [4] and then studied by many authors, see e.g. [5],[6],[9] and [10]. C. Boonpok and J. Khampakdee [1] introduced a new class of closed sets in closure spaces lying, as for generality, between the class of closed sets and the class of generalized closed sets. In this paper, we define the notion of ∂-closed maps and study some of their properties.

2 Preliminaries

A map $u : P(X) \to P(X)$ defined on the power set $P(X)$ of a set X is called a *closure operator* on X and the pair (X, u) is called a *closure space* if the following axioms are satisfied :

(N1) $u\emptyset = \emptyset$,

(N2) $A \subseteq uA$ for every $A \subseteq X$,

(N3) $A \subseteq B \Rightarrow uA \subseteq uB$ for all $A, B \subseteq X$.

A closure operator u on a set X is called *additive* (respectively,*idempotent*) if $A, B \subseteq X \Rightarrow u(A \cup B) = uA \cup uB$ (respectively, $A \subseteq X \Rightarrow uuA = uA$). A subset $A \subseteq X$ is *closed* in the closure space (X, u) if $uA = A$ and it is *open* if its complement is closed. The empty set and the whole space are both open and closed. Let (X, u_1) and (X, u_2) be closure spaces.

A closure space (Y, v) is said to be a *subspace* of (X, u) if $Y \subseteq X$ and $vA = uA \cap Y$ for each subset $A \subseteq Y$. If Y is closed in (X, u), then the subspace (Y, v) of (X, u) is said to be closed too.

Let (Y, v) be a closed subspace of (X, u). If F is a closed subset of (Y, v), then F is a closed subset of (X, u).

Let (X, u) and (Y, v) be closure spaces. A map $f : (X, u) \to (Y, v)$ is said to be *continuous* if $f(uA) \subseteq vf(A)$ for every subset $A \subseteq X$.

One can see that a map $f : (X, u) \to (Y, v)$ is continuous if and only if $uf^{-1}(B) \subseteq f^{-1}(vB)$ for every subset $B \subseteq Y$. Clearly, if $f : (X, u) \to (Y, v)$ is continuous, then $f^{-1}(F)$ is a closed subset of (X, u) for every closed subset F of (Y, v).

Let (X, u) and (Y, v) be closure spaces. A map $f : (X, u) \to (Y, v)$ is said to be *closed* (resp. *open*) if $f(F)$ is a closed (resp. open) subset of (Y, v) whenever F is a closed (resp. open) subset of (X, u).

The *product* of a family $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ of closure spaces, denoted by $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$, is the closure space $\big(\prod_{\alpha\in I} X_\alpha, u\big)$ where $\prod_{\alpha\in I} X_\alpha$ denotes the cartesian product of sets X_α, $\alpha \in I$, and u is the closure operator generated by the projections $\pi_\alpha : \prod_{\alpha\in I}(X_\alpha, u) \to (X_\alpha, u)$, $\alpha \in I$, i.e., is defined by $uA = \prod_{\alpha\in I} u_\alpha\pi_\alpha(A)$ for each $A \subseteq \prod_{\alpha\in I} X_\alpha$.

Clearly, if $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ is a family of closure spaces, then the projection map $\pi_\beta : \prod_{\alpha\in I}(X_\alpha, u_\alpha) \to (X_\beta, u_\beta)$ is closed and continuous for every $\beta \in I$.

Proposition 2.1. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces, let $\beta \in I$ and let $F \subseteq X_\beta$. Then F is a closed subset of (X_β, u_β) if and only if*

$F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ *is a closed subset of* $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$.

Proof. Let F be a closed subset of (X_β, u_β). Since π_β is continuous, $\pi_\beta^{-1}(F)$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$. But $\pi_\beta^{-1}(F) = F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$, hence $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$.

Conversely, let $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ be a closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$. Since π_β is closed, $\pi_\beta\Big(F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big) = F$ is a closed subset of (X_β, u_β). □

Proposition 2.2. *Let* $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ *be a family of closure spaces, let* $\beta \in I$ *and let* $G \subseteq X_\beta$. *Then* G *is an open subset of* (X_β, u_β) *if and only if* $G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ *is an open subset of* $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$.

Definition 2.3. *A* biclosure space *is a triple* (X, u_1, u_2) *where* X *is a set and* u_1, u_2 *are two closure operators on* X.

Definition 2.4. *A subset* A *of a biclosure space* (X, u_1, u_2) *is called* closed *if* $u_1u_2A = A$. *The complement of closed set is called* open.

Clearly, A is a closed subset of a biclosure space (X, u_1, u_2) if and only if A is both a closed subset of (X, u_1) and (X, u_2).

Let A be a closed subset of a biclosure space (X, u_1, u_2) . The following conditions are equivalent

(i) $u_2u_1A = A$,

(ii) $u_1A = A$, $u_2A = A$.

The following statement is evident:

Proposition 2.5. *Let* $\{(X_\alpha, u_\alpha^1, u_\alpha^2) : \alpha \in I\}$ *be a family of biclosure spaces. For each* $\beta \in I$, *let* $\pi_\beta : \prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2) \to (X_\beta, u_\beta^1, u_\beta^2)$ *be the projection map. If* F *is a closed subset of* $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$, *then* $\pi_\beta(F)$ *is a closed subset of* $(X_\beta, u_\beta^1, u_\beta^2)$.

Definition 2.6. *A subset* A *of a biclosure space* (X, u_1, u_2) *is called* generalized closed *briefly,* g-closed, *if* $u_2A \subseteq G$ *whenever* G *is an open subset of* (X, u_1) *with* $A \subseteq G$. *The complement of a g-closed set called* g-open.

Clearly, if A is a closed subset of a biclosure space (X, u_1, u_2), then A is g-closed. The converse is not true as can be seen from the following example.

Example 2.7. *Let $X = \{a, b\}$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$ and $u_1\{a\} = u_1\{b\} = u_1X = X$. Define a closure operator u_2 on X by $u_2\emptyset = \emptyset$, $u_2\{a\} = \{a\}$ and $u_2\{b\} = u_2X = X$. Then $\{a\}$ is g-closed but it is not closed.*

Proposition 2.8. *Let (X, u_1, u_2) be a biclosure space. Then A is a g-open subset of (X, u_1, u_2) if and only if $F \subseteq X - u_2(X - A)$ for every F is closed subset of (X, u_1) with $F \subseteq A$.*

Proposition 2.9. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces and let $\beta \in I$. Then G is a g-open subset of $(X_\beta, u^1_\beta, u^2_\beta)$ if and only if $G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a g-open subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$.*

Proof. Let F be a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha)$ such that $F \subseteq G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. Then $\pi_\beta(F) \subseteq G$. Since $\pi_\beta(F)$ is a closed subset of (X_β, u^1_β), $\pi_\beta(F) \subseteq X_\beta - u^2_\beta(X_\beta - G)$. Therefore, $F \subseteq \pi^{-1}_\beta(X_\beta - u^2_\beta(X_\beta - G)) = \prod_{\alpha\in I} X_\alpha - \prod_{\alpha\in I} u^2_\alpha\pi_\alpha\Big(\prod_{\alpha\in I} X_\alpha - G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big)$. By Proposition 2.8, $G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a g-open subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$.

Conversely, let F be a closed subset of (X_β, u^1_β) such that $F \subseteq G$. Then $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha \subseteq G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. Since $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is closed and $G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is g-open, $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha \subseteq \prod_{\alpha\in I} X_\alpha - \prod_{\alpha\in I} u^2_\alpha\pi_\alpha\Big(\prod_{\alpha\in I} X_\alpha - G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big)$ by Proposition 2.8. Therefore, $\prod_{\alpha\in I} u^2_\alpha\pi_\alpha\Big((X_\beta - G) \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big) \subseteq \prod_{\alpha\in I} X_\alpha - F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha = (X_\beta - F) \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. Consequently, $u^2_\beta(X_\beta - G) \subseteq X_\beta - F$ implies $F \subseteq X_\beta - u^2_\beta(X_\beta - G)$. Hence, G is a g-open subset of $(X_\beta, u^1_\beta, u^2_\beta)$. □

Proposition 2.10. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces and let $\beta \in I$. Then F is a g-closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$ if and only if $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a g-closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$.*

Proposition 2.11. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces. For each $\beta \in I$, let $\pi_\beta : \prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha) \to (X_\beta, u^1_\beta, u^2_\beta)$ be the projection map. If F is a g-closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$, then $\pi_\beta(F)$ is a g-closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$.*

Proof. Let F be a g-closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$ and let G be an open subset of (X_β, u_β^1) such that $\pi_\beta(F) \subseteq G$. Then $F \subseteq \pi_\beta^{-1}(G) = G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. Since $G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is an open subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1)$, $\prod_{\alpha\in I} u_\alpha^2\pi_\alpha(F) \subseteq G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. Consequently, $u_\beta^2\pi_\beta(F) \subseteq G$. Hence, $\pi_\beta(F)$ is a g-closed subset of $(X_\beta, u_\beta^1, u_\beta^2)$. □

3 ∂-Closed Sets

Definition 3.1. *A subset A of a biclosure space (X, u_1, u_2) is called a ∂-*closed *if $u_2A \subseteq G$ whenever G is a g-open subset of (X, u_1) with $A \subseteq G$. The complement of a ∂-closed set is called ∂-*open *.*

Remark 1. *For a subset A of a biclosure space (X, u_1, u_2), the following implications hold :*

$$A \text{ is closed} \Rightarrow A \text{ is } \partial\text{-closed} \Rightarrow A \text{ is g-closed}$$

The implication is not reversible as shown by the following example.

Example 3.2. *Let $X = \{a, b\}$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$ and $u_1\{a\} = u_1\{b\} = u_1X = X$. Define a closure operator u_2 on X by $u_2\emptyset = \emptyset$, $u_2\{a\} = \{a\}$ and $u_2\{b\} = u_2X = X$. Then $\{a\}$ is ∂-closed but it is not closed.*

Example 3.3. *Let $X = \{1, 2\}$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$ and $u_1\{1\} = u_1\{2\} = u_1X = X$. Define a closure operator u_2 on X by $u_2\emptyset = \emptyset$ and $u_2\{1\} = u_2\{2\} = u_2X = X$. Then $\{1\}$ is g-closed but it is not ∂-closed.*

Proposition 3.4. *Let (X, u_1, u_2) be a biclosure space and let $A \subseteq X$. Then A is ∂-open if and only if $F \subseteq X - u_2(X - A)$ for every F is a g-closed subset of (X, u_1) with $F \subseteq A$.*

Proof. Assume that A is ∂-open and let $F \subseteq A$ be a g-closed subset of (X, u_1). Then $X - A \subseteq X - F$. Since $X - A$ is ∂-closed and $X - F$ is g-open subset of (X, u_1), $u_2(X - A) \subseteq X - F$. Hence $F \subseteq X - u_2(X - A)$.

Conversely, let U be a g-open subset of (X, u_1) such that $X - A \subseteq U$. Then $X - U \subseteq A$. Since $X - U$ is g-closed subset of (X, u_1), $X - U \subseteq X - u_2(X - A)$. Consequently, $u_2(X - A) \subseteq U$. Hence, $X - A$ is ∂-closed and so A is ∂-open. □

Proposition 3.5. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces and let $\beta \in I$. Then G is a ∂-open subset of $(X_\beta, u^1_\beta, u^2_\beta)$ if and only if $G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a ∂-open subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$.*

Proof. Let F be a g-closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha)$ such that $F \subseteq G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. Then $\pi_\beta(F) \subseteq G$. Since $\pi_\beta(F)$ is g-closed in (X_β, u^1_β), $\pi_\beta(F) \subseteq X_\beta - u^2_\beta(X_\beta - G)$. Therefore, $F \subseteq \pi^{-1}_\beta(X_\beta - u^2_\beta(X_\beta - G)) = \prod_{\alpha\in I} X_\alpha - \prod_{\alpha\in I} u^2_\alpha\pi_\alpha\Big(\prod_{\alpha\in I} X_\alpha - G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big)$. By Proposition 3.4, $G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a ∂-open subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$.

Conversely, let F be a g-closed subset of (X_β, u^1_β) such that $F \subseteq G$. Then $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha \subseteq G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. Since $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is g-closed and $G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is ∂-open in $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha)$, $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha \subseteq \prod_{\alpha\in I} X_\alpha - \prod_{\alpha\in I} u^2_\alpha\pi_\alpha\Big(\prod_{\alpha\in I} X_\alpha - G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big)$ by Proposition 3.4. Therefore, $\prod_{\alpha\in I} u^2_\alpha\pi_\alpha\Big((X_\beta - G) \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big) \subseteq \prod_{\alpha\in I} X_\alpha - F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha = (X_\beta - F) \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. Consequently, $u^2_\beta(X_\beta - G) \subseteq X_\beta - F$ implies $F \subseteq X_\beta - u^2_\beta(X_\beta - G)$. Hence, G is a ∂-open subset of $(X_\beta, u^1_\beta, u^2_\beta)$. □

Proposition 3.6. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces and let $\beta \in I$. Then F is a ∂-closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$ if and only if $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a ∂-closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$.*

Proof. Let F be a ∂-closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$. Then $X_\beta - F$ is a ∂-open subset of $(X_\beta, u^1_\beta, u^2_\beta)$. By Proposition 3.5, $(X_\beta - F) \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha = \prod_{\alpha\in I} X_\alpha - F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a ∂-open subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$. Hence, $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a ∂-closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$.

Conversely, let G be a g-open subset of (X_β, u^1_β) such that $F \subseteq G$. Then $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha \subseteq G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. Since $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a ∂-closed and $G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is g-open in $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha)$, $\prod_{\alpha\in I} u^2_\alpha\pi_\alpha\Big(F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\beta\Big) \subseteq G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. Consequently, $u^2_\beta F \subseteq G$. Therefore, F is a ∂-closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$. □

Proposition 3.7. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces. For each $\beta \in I$, let $\pi_\beta : \prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha) \to (X_\beta, u^1_\beta, u^2_\beta)$ be the projection map.*

Then

(i) If F is a ∂-closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$, then $\pi_\beta(F)$ is a ∂-closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$.

(ii) If F is a ∂-closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$, then $\pi^{-1}_\beta(F)$ is a ∂-closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$.

Proof. (i) Let F be a ∂-closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ and let G be a g-open subset of (X_β, u^1_β) such that $\pi_\beta(F) \subseteq G$. Then $F \subseteq \pi^{-1}_\beta(G) = G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. Since $G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a g-open subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha)$, $\prod_{\alpha\in I} u^2_\alpha \pi_\alpha(F) \subseteq G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. Consequently, $u^2_\beta \pi_\beta(F) \subseteq G$. Hence, $\pi_\beta(F)$ is a ∂-closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$.

(ii) Let F be a ∂-closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$. Then $\pi^{-1}_\beta(F) = F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. By Proposition 3.6, $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a ∂-closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$. Hence, $\pi^{-1}_\beta(F)$ is a ∂-closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$. □

4 ∂-Closed Maps

In this section, we introduce the notion of ∂-closed maps and study some of their properties.

Definition 4.1. *Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces. A map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is called* closed *if $f(F)$ is a closed subset of (Y, v_1, v_2) for every closed subset F of (X, u_1, u_2).*

Definition 4.2. *Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces. A map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is called* ∂-closed *if $f(F)$ is a ∂-closed subset of (Y, v_1, v_2) for every closed subset F of (X, u_1, u_2).*

Every closed map is ∂-closed but the converse is not true as may be seen from the following example.

Example 4.3. *Let $X = \{1, 2, 3, 4\} = Y$ and define closure operators u_1 and u_2 on X by $u_1\emptyset = \emptyset$, $u_1\{1, 3, 4\} = \{1, 3, 4\}$ and $u_1\{1\} = u_1\{2\} = u_1\{3\} = u_1\{4\} = u_1\{1, 2\} = u_1\{1, 3\} = u_1\{1, 4\} = u_1\{2, 3\} = u_1\{2, 4\} = u_1\{3, 4\} = u_1\{1, 2, 3\} = u_1\{1, 2, 4\} = u_1\{1, 3, 4\} = u_1\{2, 3, 4\} = u_1X = X$ $u_2\emptyset = \emptyset$, $u_2\{1, 3, 4\} = \{1, 3, 4\}$ and $u_2\{1\} = u_2\{2\} = u_2\{3\} = u_2\{4\} = u_2\{1, 2\} = u_2\{1, 3\} = u_2\{1, 4\} = u_2\{2, 3\} = u_2\{2, 4\} = u_2\{3, 4\} = u_2\{1, 2, 3\} = u_2\{1, 2, 4\} =$*

$u_2\{1,3,4\} = u_2\{2,3,4\} = u_2X = X$. Define closure operators v_1 and v_1 on Y by $v_1\emptyset = \emptyset$, $v_1\{1\} = \{1,3\}$, $v_1\{2\} = \{2,3\}$, $v_1\{3\} = v_1\{4\} = v_1\{3,4\} = \{3,4\}$ and $v_1\{1,2\} = v_1\{1,3\} = v_1\{1,4\} = v_1\{2,3\} = v_1\{2,4\} = v_1\{1,2,3\} = v_1\{1,2,4\} = v_1\{1,3,4\} = v_1\{2,3,4\} = v_1Y = Y$ and $v_2\emptyset = \emptyset$, $v_2\{1\} = \{1,3\}$, $v_2\{2\} = \{2,3\}$, $v_2\{3\} = v_2\{4\} = v_2\{3,4\} = \{3,4\}$ and $v_2\{1,2\} = v_2\{1,3\} = v_2\{1,4\} = v_2\{2,3\} = v_2\{2,4\} = v_2\{1,2,3\} = v_2\{1,2,4\} = v_2\{1,3,4\} = v_2\{2,3,4\} = v_2Y = Y$. Let $f : (X,u_1,u_2) \to (Y,v_1,v_2)$ be the identity map. Then f is ∂-closed but it is not closed because $\{1,3,4\}$ is a closed subset of (X,u_1,u_2) but $f(\{1,3,4\}) = \{1,3,4\}$ is not a closed subset of (Y,v_1,v_2).

The following statement is evident :

Proposition 4.4. *Let (X,u_1,u_2), (Y,v_1,v_2) and (Z,w_1,w_2) be biclosure spaces. If $f : (X,u_1,u_2) \to (Y,v_1,v_2)$ is closed and $g : (Y,v_1,v_2) \to (Z,w_1,w_2)$ is ∂-closed, then $g \circ f : (X,u_1,u_2) \to (Z,w_1,w_2)$ is ∂-closed.*

Proposition 4.5. *Let $\{(X_\alpha,u^1_\alpha,u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces. Then for each $\beta \in I$, the projection map $\pi_\beta : \prod\limits_{\alpha\in I}(X_\alpha,u^1_\alpha,u^2_\alpha) \to (X_\beta,u^1_\beta,u^2_\beta)$ is ∂-closed.*

Proof. Let $\beta \in I$. Let F be a closed subset of $\prod\limits_{\alpha\in I}(X_\alpha,u^1_\alpha,u^2_\alpha)$ and let G be a g-open subset of (X_β,u^1_β) such that $\pi_\beta(F) \subseteq G$. Then $F \subseteq \pi_\beta^{-1}(G) = G \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. Since $G \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a g-open subset of $\prod\limits_{\alpha\in I}(X_\alpha,u^1_\alpha)$ and F is a ∂-closed subset of $\prod\limits_{\alpha\in I}(X_\alpha,u^1_\alpha,u^2_\alpha)$, $\prod\limits_{\alpha\in I} u^2_\alpha\pi_\alpha(F) \subseteq G \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. Consequently, $u^2_\beta\pi_\beta(F) \subseteq G$. Therefore, $\pi_\beta(F)$ is a ∂-closed subset of $(X_\beta,u^1_\beta,u^2_\beta)$. Hence, π_β is ∂-closed. □

Proposition 4.6. *Let (X,u_1,u_2) be a biclosure space, $\{(Y_\alpha,v^1_\alpha,v^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces and $f : X \to \prod\limits_{\alpha\in I} Y_\alpha$ be a map. Then $f : (X,u_1,u_2) \to \prod\limits_{\alpha\in I}(Y_\alpha,v^1_\alpha,v^2_\alpha)$ is ∂-closed if and only if $\pi_\alpha \circ f : (X,u_1,u_2) \to (Y_\alpha,v^1_\alpha,v^2_\alpha)$ is ∂-closed for each $\alpha \in I$.*

Proof. Let $\beta \in I$. Let F be a closed subset of (X,u_1,u_2) and let G be a g-open subset of (Y_β,u^1_β) such that $\pi_\beta\circ f(F) = \pi_\beta(f(F)) \subseteq G$. Then $f(F) \subseteq \pi_\beta^{-1}(G) = G \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} Y_\alpha$. Since $G \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} Y_\alpha$ is a g-open subset of $\prod\limits_{\alpha\in I}(Y_\alpha,v^1_\alpha)$ and $f(F)$ is a ∂-closed subset of $\prod\limits_{\alpha\in I}(Y_\alpha,v^1_\alpha,v^2_\alpha)$, $\prod\limits_{\alpha\in I} v^2_\alpha\pi_\alpha f(F) \subseteq G \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} Y_\alpha$. Consequently, $v^2_\beta\pi_\beta f(F) \subseteq G$. Therefore, $\pi_\beta f(F)$ is a ∂-closed subset of $(Y_\beta,v^1_\beta,v^2_\beta)$. Hence, $\pi_\beta \circ f$ is ∂-closed.

Conversely, let $\pi_\alpha \circ f$ be ∂-closed for each $\alpha \in I$. Suppose that f is not ∂-closed. Then there exists a closed subset F of (X, u_1, u_2) such that $f(F)$ is not ∂-closed subset of $\prod_{\alpha \in I}(Y_\alpha, v^1_\alpha, v^2_\alpha)$. Therefore, there exists $\beta \in I$ such that $\pi_\beta(f(F))$ is not ∂-closed subset of $(Y_\beta, v^1_\beta, v^2_\beta)$. But $\pi_\beta \circ f$ is ∂-closed, hence $\pi_\beta(f(F))$ is a ∂-closed subset of $(Y_\beta, v^1_\beta, v^2_\beta)$. This is a contradiction. □

Proposition 4.7. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ and $\{(Y_\alpha, v^1_\alpha, v^2_\alpha) : \alpha \in I\}$ be families of biclosure spaces. For each $\alpha \in I$, let $f_\alpha : X_\alpha \to Y_\alpha$ be a surjection and let $f : \prod_{\alpha \in I} X_\alpha \to \prod_{\alpha \in I} Y_\alpha$ be defined by $f\big((x_\alpha)_{\alpha \in I}\big) = \big(f_\alpha(x_\alpha)\big)_{\alpha \in I}$. Then $f : \prod_{\alpha \in I}(X_\alpha, u^1_\alpha, u^2_\alpha) \to \prod_{\alpha \in I}(Y_\alpha, v^1_\alpha, v^2_\alpha)$ is ∂-closed if and only if $f_\alpha : (X_\alpha, u^1_\alpha, u^2_\alpha) \to (Y_\alpha, v^1_\alpha, v^2_\alpha)$ is ∂-closed for each $\alpha \in I$.*

Proof. Let $\beta \in I$ and let F be a closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$. Then $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a closed subset of $\prod_{\alpha \in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$. Since f is ∂-closed, $f\Big(F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha\Big)$ is a ∂-closed subset of $\prod_{\alpha \in I}(Y_\alpha, v^1_\alpha, v^2_\alpha)$. But $f\Big(F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha\Big) = f_\beta(F) \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} Y_\alpha$, hence $f_\beta(F) \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} Y_\alpha$ is a ∂-closed subset of $\prod_{\alpha \in I}(Y_\alpha, v^1_\alpha, v^2_\alpha)$. By Proposition 3.6, $f_\beta(F)$ is a ∂-closed subset of $(Y_\beta, v^1_\beta, v^2_\beta)$. Hence, f_β is ∂-closed.

Conversely, let f_β be ∂-closed for each $\beta \in I$. Suppose that f is not ∂-closed. Then there exists a closed subset F of $\prod_{\alpha \in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ such that $f(F)$ is not a ∂-closed subset of $\prod_{\alpha \in I}(Y_\alpha, v^1_\alpha, v^2_\alpha)$. Therefore, there exists $\beta \in I$ such that $f_\beta(\pi_\beta(F))$ is not a ∂-closed subset of $(Y_\beta, v^1_\beta, v^2_\beta)$. But $\pi_\beta(F)$ is a closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$ and f_β is ∂-closed, $f_\beta(\pi_\beta(F))$ is a ∂-closed subset of $(Y_\beta, v^1_\beta, v^2_\beta)$. This is a contradiction. □

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