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$T^*_{\frac{1}{2}}$-Spaces and $T^{**}_{\frac{1}{2}}$-Spaces

Chawalit Boonpok

Department of Mathematics
Faculty of Science
Mahasarakham University
Mahasarakham 44150, Thailand
chawalit.b@msu.ac.th

Abstract

In this paper, we introduce the concepts of $T^*_{\frac{1}{2}}$-spaces and $T^{**}_{\frac{1}{2}}$-spaces and study some of their properties.

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1 Introduction

Generalized closed sets, briefly g-closed sets, in a topological space were introduced by N. Levine [8] in order to extend some important properties of closed sets to a larger family of sets. For instance, it was shown that compactness, normality and completeness in a uniform space are inherited by g-closed subsets. J.C. Kelly [7] introduced the notion of bitopological spaces. Such spaces are equipped with two arbitrary topologies. Furthermore, Kelly extended some of the standard results of separation axioms in a topological space to a bitopological space. Closure spaces were introduced by E.Čech [4] and then studied by many authors, see e.g. [5], [6], [9] and [10]. C. Boonpok and J. Khampakdee [1] introduced a new class of closed sets in closure spaces lying, as for generality, between the class of closed sets and the class of generalized closed sets. Using the concept of ∂-closed sets, introduced two new kinds of spaces, namely $T'_{\frac{1}{2}}$-spaces and a $T''_{\frac{1}{2}}$-spaces. The two kinds of spaces are investigated. In this paper, we define the notions of $T^*_{\frac{1}{2}}$-spaces and $T^{**}_{\frac{1}{2}}$-spaces and study their fundamental properties.

2 Preliminaries

A map $u : P(X) \to P(X)$ defined on the power set $P(X)$ of a set X is called a *closure operator* on X and the pair (X, u) is called a *closure space* if the following axioms are satisfied :

(N1) $u\emptyset = \emptyset$,

(N2) $A \subseteq uA$ for every $A \subseteq X$,

(N3) $A \subseteq B \Rightarrow uA \subseteq uB$ for all $A, B \subseteq X$.

A closure operator u on a set X is called *additive* (respectively,*idempotent*) if $A, B \subseteq X \Rightarrow u(A \cup B) = uA \cup uB$ (respectively, $A \subseteq X \Rightarrow uuA = uA$). A subset $A \subseteq X$ is *closed* in the closure space (X, u) if $uA = A$ and it is *open* if its complement is closed. The empty set and the whole space are both open and closed. Let (X, u_1) and (X, u_2) be closure spaces.

A closure space (Y, v) is said to be a *subspace* of (X, u) if $Y \subseteq X$ and $vA = uA \cap Y$ for each subset $A \subseteq Y$. If Y is closed in (X, u), then the subspace (Y, v) of (X, u) is said to be closed too.

Let (Y, v) be a closed subspace of (X, u). If F is a closed subset of (Y, v), then F is a closed subset of (X, u).

Let (X, u) and (Y, v) be closure spaces. A map $f : (X, u) \to (Y, v)$ is said to be *continuous* if $f(uA) \subseteq vf(A)$ for every subset $A \subseteq X$.

One can see that a map $f : (X, u) \to (Y, v)$ is continuous if and only if $uf^{-1}(B) \subseteq f^{-1}(vB)$ for every subset $B \subseteq Y$. Clearly, if $f : (X, u) \to (Y, v)$ is continuous, then $f^{-1}(F)$ is a closed subset of (X, u) for every closed subset F of (Y, v).

Let (X, u) and (Y, v) be closure spaces. A map $f : (X, u) \to (Y, v)$ is said to be *closed* (resp. *open*) if $f(F)$ is a closed (resp. open) subset of (Y, v) whenever F is a closed (resp. open) subset of (X, u).

The *product* of a family $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ of closure spaces, denoted by $\prod\limits_{\alpha\in I}(X_\alpha, u_\alpha)$, is the closure space $\big(\prod\limits_{\alpha\in I} X_\alpha, u\big)$ where $\prod\limits_{\alpha\in I} X_\alpha$ denotes the cartesian product of sets X_α, $\alpha \in I$, and u is the closure operator generated by the projections $\pi_\alpha : \prod\limits_{\alpha\in I}(X_\alpha, u) \to (X_\alpha, u)$, $\alpha \in I$, i.e., is defined by $uA = \prod\limits_{\alpha\in I} u_\alpha\pi_\alpha(A)$ for each $A \subseteq \prod\limits_{\alpha\in I} X_\alpha$.

Clearly, if $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ is a family of closure spaces, then the projection map $\pi_\beta : \prod\limits_{\alpha\in I}(X_\alpha, u_\alpha) \to (X_\beta, u_\beta)$ is closed and continuous for every $\beta \in I$.

Proposition 2.1. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces, let $\beta \in I$ and let $F \subseteq X_\beta$. Then F is a closed subset of (X_β, u_β) if and only if*

$F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a closed subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$.

Proof. Let F be a closed subset of (X_β, u_β). Since π_β is continuous, $\pi_\beta^{-1}(F)$ is a closed subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$. But $\pi_\beta^{-1}(F) = F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$, hence $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a closed subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$.

Conversely, let $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ be a closed subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$. Since π_β is closed, $\pi_\beta\Big(F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha\Big) = F$ is a closed subset of (X_β, u_β). □

Proposition 2.2. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces, let $\beta \in I$ and let $G \subseteq X_\beta$. Then G is an open subset of (X_β, u_β) if and only if $G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is an open subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$.*

Definition 2.3. *A biclosure space is a triple (X, u_1, u_2) where X is a set and u_1, u_2 are two closure operators on X.*

Definition 2.4. *A subset A of a biclosure space (X, u_1, u_2) is called closed if $u_1u_2A = A$. The complement of closed set is called open.*

Clearly, A is a closed subset of a biclosure space (X, u_1, u_2) if and only if A is both a closed subset of (X, u_1) and (X, u_2).

Let A be a closed subset of a biclosure space (X, u_1, u_2) . The following conditions are equivalent

(i) $u_2u_1A = A$,

(ii) $u_1A = A$, $u_2A = A$.

Definition 2.5. *A subset A of a biclosure space (X, u_1, u_2) is called generalized closed briefly, g-closed, if $u_2A \subseteq G$ whenever G is an open subset of (X, u_1) with $A \subseteq G$. The complement of a g-closed set called g-open.*

Clearly, if A is a closed subset of a biclosure space (X, u_1, u_2), then A is g-closed. The converse is not true as can be seen from the following example.

Example 2.6. *Let $X = \{a, b\}$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$ and $u_1\{a\} = u_1\{b\} = u_1X = X$. Define a closure operator u_2 on X by $u_2\emptyset = \emptyset$, $u_2\{a\} = \{a\}$ and $u_2\{b\} = u_2X = X$. Then $\{a\}$ is g-closed but it is not closed.*

Proposition 2.7. *Let (X, u_1, u_2) be a biclosure space. Then A is a g-open subset of (X, u_1, u_2) if and only if $F \subseteq X - u_2(X - A)$ for every F is closed subset of (X, u_1) with $F \subseteq A$.*

Proposition 2.8. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces and let $\beta \in I$. Then G is a g-open subset of $(X_\beta, u^1_\beta, u^2_\beta)$ if and only if $G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a g-open subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$.*

Proof. Let F be a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha)$ such that $F \subseteq G\times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. Then $\pi_\beta(F) \subseteq G$. Since $\pi_\beta(F)$ is a closed subset of (X_β, u^1_β), $\pi_\beta(F) \subseteq X_\beta - u^2_\beta(X_\beta - G)$. Therefore, $F \subseteq \pi^{-1}_\beta(X_\beta - u^2_\beta(X_\beta - G)) = \prod_{\alpha\in I} X_\alpha - \prod_{\alpha\in I} u^2_\alpha\pi_\alpha\Big(\prod_{\alpha\in I} X_\alpha - G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big)$. By Proposition 2.7, $G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a g-open subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$.

Conversely, let F be a closed subset of (X_β, u^1_β) such that $F \subseteq G$. Then $F\times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha \subseteq G\times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. Since $F\times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is closed and $G\times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is g-open, $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha \subseteq \prod_{\alpha\in I} X_\alpha - \prod_{\alpha\in I} u^2_\alpha\pi_\alpha\Big(\prod_{\alpha\in I} X_\alpha - G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big)$ by Proposition 2.7. Therefore, $\prod_{\alpha\in I} u^2_\alpha\pi_\alpha\Big((X_\beta - G) \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big) \subseteq \prod_{\alpha\in I} X_\alpha - F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha = (X_\beta - F) \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. Consequently, $u^2_\beta(X_\beta - G) \subseteq X_\beta - F$ implies $F \subseteq X_\beta - u^2_\beta(X_\beta - G)$. Hence, G is a g-open subset of $(X_\beta, u^1_\beta, u^2_\beta)$. □

Proposition 2.9. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces and let $\beta \in I$. Then F is a g-closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$ if and only if $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a g-closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$.*

Proposition 2.10. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces. For each $\beta \in I$, let $\pi_\beta : \prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha) \to (X_\beta, u^1_\beta, u^2_\beta)$ be the projection map. If F is a g-closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$, then $\pi_\beta(F)$ is a g-closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$.*

Proof. Let F be a g-closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ and let G be an open subset of (X_β, u^1_β) such that $\pi_\beta(F) \subseteq G$. Then $F \subseteq \pi^{-1}_\beta(G) = G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. Since $G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is an open subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha)$, $\prod_{\alpha\in I} u^2_\alpha\pi_\alpha(F) \subseteq G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. Consequently, $u^2_\beta\pi_\beta(F) \subseteq G$. Hence, $\pi_\beta(F)$ is a g-closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$. □

3 ∂-Closed Sets

Definition 3.1. *A subset A of a biclosure space (X, u_1, u_2) is called a* ∂-closed *if $u_2A \subseteq G$ whenever G is a g-open subset of (X, u_1) with $A \subseteq G$. The complement of a ∂-closed set is called* ∂-open *.*

Remark 1. *For a subset A of a biclosure space (X, u_1, u_2), the following implications hold :*

$$A \text{ is closed} \Rightarrow A \text{ is } \partial\text{-closed} \Rightarrow A \text{ is g-closed}$$

The implication is not reversible as shown by the following example.

Example 3.2. *Let $X = \{a, b\}$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$ and $u_1\{a\} = u_1\{b\} = u_1X = X$. Define a closure operator u_2 on X by $u_2\emptyset = \emptyset$, $u_2\{a\} = \{a\}$ and $u_2\{b\} = u_2X = X$. Then $\{a\}$ is ∂-closed but it is not closed.*

Example 3.3. *Let $X = \{1, 2\}$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$ and $u_1\{1\} = u_1\{2\} = u_1X = X$. Define a closure operator u_2 on X by $u_2\emptyset = \emptyset$ and $u_2\{1\} = u_2\{2\} = u_2X = X$. Then $\{1\}$ is g-closed but it is not ∂-closed.*

Proposition 3.4. *Let (X, u_1, u_2) be a biclosure space and let $A \subseteq X$. Then A is ∂-open if and only if $F \subseteq X - u_2(X - A)$ for every F is a g-closed subset of (X, u_1) with $F \subseteq A$.*

Proof. Assume that A is ∂-open and let $F \subseteq A$ be a g-closed subset of (X, u_1). Then $X - A \subseteq X - F$. Since $X - A$ is ∂-closed and $X - F$ is g-open subset of (X, u_1), $u_2(X - A) \subseteq X - F$. Hence $F \subseteq X - u_2(X - A)$.

Conversely, let U be a g-open subset of (X, u_1) such that $X - A \subseteq U$. Then $X - U \subseteq A$. Since $X - U$ is g-closed subset of (X, u_1), $X - U \subseteq X - u_2(X - A)$. Consequently, $u_2(X - A) \subseteq U$. Hence, $X - A$ is ∂-closed and so A is ∂-open. □

Proposition 3.5. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces and let $\beta \in I$. Then G is a ∂-open subset of $(X_\beta, u^1_\beta, u^2_\beta)$ if and only if $G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a ∂-open subset of $\prod_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha)$.*

Proof. Let F be a g-closed subset of $\prod_{\alpha \in I} (X_\alpha, u^1_\alpha)$ such that $F \subseteq G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$. Then $\pi_\beta(F) \subseteq G$. Since $\pi_\beta(F)$ is g-closed in (X_β, u^1_β), $\pi_\beta(F) \subseteq X_\beta - u^2_\beta(X_\beta - G)$. Therefore, $F \subseteq \pi^{-1}_\beta(X_\beta - u^2_\beta(X_\beta - G)) = \prod_{\alpha \in I} X_\alpha - \prod_{\alpha \in I} u^2_\alpha \pi_\alpha \Big(\prod_{\alpha \in I} X_\alpha - G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha \Big)$. By Proposition 3.4, $G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a ∂-open subset of $\prod_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha)$.

Conversely, let F be a g-closed subset of (X_β, u^1_β) such that $F \subseteq G$. Then $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha \subseteq G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. Since $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is g-closed and $G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is ∂-open in $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha)$, $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha \subseteq \prod_{\alpha\in I} X_\alpha - \prod_{\alpha\in I} u^2_\alpha\pi_\alpha\Big(\prod_{\alpha\in I} X_\alpha - G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big)$ by Proposition 3.4. Therefore, $\prod_{\alpha\in I} u^2_\alpha\pi_\alpha\Big((X_\beta - G) \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big) \subseteq \prod_{\alpha\in I} X_\alpha - F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha = (X_\beta - F) \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. Consequently, $u^2_\beta(X_\beta - G) \subseteq X_\beta - F$ implies $F \subseteq X_\beta - u^2_\beta(X_\beta - G)$. Hence, G is a ∂-open subset of $(X_\beta, u^1_\beta, u^2_\beta)$. □

Proposition 3.6. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces and let $\beta \in I$. Then F is a ∂-closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$ if and only if $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a ∂-closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$.*

Proof. Let F be a ∂-closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$. Then $X_\beta - F$ is a ∂-open subset of $(X_\beta, u^1_\beta, u^2_\beta)$. By Proposition 3.5, $(X_\beta - F) \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha = \prod_{\alpha\in I} X_\alpha - F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a ∂-open subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$. Hence, $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a ∂-closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$.

Conversely, let G be a g-open subset of (X_β, u^1_β) such that $F \subseteq G$. Then $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha \subseteq G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. Since $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a ∂-closed and $G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is g-open in $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha)$, $\prod_{\alpha\in I} u^2_\alpha\pi_\alpha\Big(F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\beta\Big) \subseteq G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. Consequently, $u^2_\beta F \subseteq G$. Therefore, F is a ∂-closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$. □

Proposition 3.7. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces. For each $\beta \in I$, let $\pi_\beta : \prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha) \to (X_\beta, u^1_\beta, u^2_\beta)$ be the projection map. Then*

(i) If F is a ∂-closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$, then $\pi_\beta(F)$ is a ∂-closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$.

(ii) If F is a ∂-closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$, then $\pi^{-1}_\beta(F)$ is a ∂-closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$.

Proof. (i) Let F be a ∂-closed subset of $\prod_{\alpha\in I} X_\alpha, u^1_\alpha, u^2_\alpha)$ and let G be a g-open subset of (X_β, u^1_β) such that $\pi_\beta(F) \subseteq G$. Then $F \subseteq \pi^{-1}_\beta(G) = G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$.

Since $G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a g-open subset of $\prod_{\alpha \in I}(X_\alpha, u^1_\alpha)$, $\prod_{\alpha \in I} u^2_\alpha \pi_\alpha(F) \subseteq G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$. Consequently, $u^2_\beta \pi_\beta(F) \subseteq G$. Hence, $\pi_\beta(F)$ is a ∂-closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$.

(ii) Let F be a ∂-closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$. Then $\pi_\beta^{-1}(F) = F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$. By Proposition 3.6, $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a ∂-closed subset of $\prod_{\alpha \in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$. Hence, $\pi_\beta^{-1}(F)$ is a ∂-closed subset of $\prod_{\alpha \in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$. □

4 $T^*_{\frac{1}{2}}$-Spaces and $T^{**}_{\frac{1}{2}}$-Spaces

As applications of ∂-closed sets, two new kinds of spaces, namely $T^*_{\frac{1}{2}}$-spaces and $T^{**}_{\frac{1}{2}}$-spaces, are introduce.

Definition 4.1. *A biclosure space (X, u_1, u_2) is said to be a $T^*_{\frac{1}{2}}$-space if every ∂-closed subset of (X, u_1, u_2) is a closed subset of (X, u_2).*

Definition 4.2. *A biclosure space (X, u_1, u_2) is said to be a $T^{**}_{\frac{1}{2}}$-space if every g-closed subset of (X, u_1, u_1) is a ∂-closed subset of (X, u_2).*

We note that the concepts of a $T^*_{\frac{1}{2}}$-space and a $T^{**}_{\frac{1}{2}}$-space are independent as shown in the following examples.

Example 4.3. *Let $X = \{a, b, c\}$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$, $u_1\{a\} = \{a, c\}$, $u_1\{b\} = \{b, c\}$, $u_1\{c\} = \{c\}$ and $u_1\{a, b\} = u_1\{a, c\} = u_1\{b, c\} = u_1X = X$. Define a closure operator u_2 on X by $u_1\emptyset = \emptyset$, $u_2\{a\} = \{a, c\}$, $u_2\{b\} = \{b, c\}$, $u_2\{c\} = \{c\}$ and $u_2\{a, b\} = u_2\{a, c\} = u_2\{b, c\} = u_2X = X$.Then (X, u_1, u_2) is a $T^{**}_{\frac{1}{2}}$-space. But (X, u_1, u_2) is not a $T^*_{\frac{1}{2}}$-space since $\{a, c\}$ is a ∂-closed subset of (X, u_2) but it is not closed.*

Example 4.4. *Let $X = \{a, b, c\}$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$ and $u_1\{a\} = u_1\{b\} = u_1\{c\} = u_1\{a, b\} = u_1\{a, c\} = u_1\{b, c\} = u_1X = X$. Define a closure operator u_1 on X by $u_1\emptyset = \emptyset$ and $u_2\{a\} = u_2\{b\} = u_2\{c\} = u_2\{a, b\} = u_2\{a, c\} = u_2\{b, c\} = u_2X = X$. Then (X, u_1, u_2) is not a $T^{**}_{\frac{1}{2}}$-space since $\{c\}$ is a g-closed subset of (X, u_2) but it is not ∂-closed. However, (X, u_1, u_2) is a $T^*_{\frac{1}{2}}$-space.*

Example 4.5. *Let $X = \{p, q\}$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$, $u_1\{p\} = \{p\}$ and $u_1\{q\} = u_1X = X$. Define a closure operator u_2 on X by $u_2\emptyset = \emptyset$, $u_2\{p\} = \{p\}$ and $u_2\{q\} = u_2X = X$. Then (X, u_1, u_2) is both a $T^*_{\frac{1}{2}}$-space and a $T^{**}_{\frac{1}{2}}$-space.*

Proposition 4.6. *Let (X, u_1, u_2) be a biclosure space. Then*

(i) If (X, u_1, u_2) is a $T^*_{\frac{1}{2}}$-space, then every singleton subset of X is either a g-closed subset of (X, u_1) or an open subset of (X, u_2).

(ii) If every singleton subset of X is a g-closed subset of (X, u_1, u_2), then (X, u_1, u_2) is a $T^*_{\frac{1}{2}}$-space.

Proof. (i) Suppose that $(X, u_1, U-2)$ is a $T^*_{\frac{1}{2}}$-space. Let $x \in X$ and assume that $\{x\}$ is not g-closed. Then $X - \{x\}$ is not g-open. This implies $X - \{x\}$ is ∂-closed since X is the only g-open set which contains $X - \{x\}$. Since (X, u) is a $T^*_{\frac{1}{2}}$-space, $X - \{x\}$ is a closed subset of (X, u_2) or equivalently, $\{x\}$ is an open subset of (X, u_2).

(ii) Let A be a ∂-closed subset of (X, u_1, u_2). Suppose that $x \notin A$. Then $\{x\} \subseteq X - A$ and we have $A \subseteq X - \{x\}$. Since A is a ∂-closed subset of (X, u_1, u_2) and $X - \{x\}$ is an g-open subset of (X, u_1), $uA \subseteq X - \{x\}$, i.e., $\{x\} \subseteq X - uA$. Hence, $x \notin uA$ and thus $uA \subseteq A$. Therefore, A is a closed subset of (X, u_2). Hence, (X, u_1, u_2) is a $T^*_{\frac{1}{2}}$-space. □

Proposition 4.7. *Let (X, u_1, u_2) be a biclosure space. If (X, u) is a $T^{**}_{\frac{1}{2}}$-space, then every singleton subset of X is either a ∂-open subset of (X, u_2) or a closed subset of (X, u_1).*

Proof. It follows from Proposition 4.6(i). □

Proposition 4.8. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces. Then $\prod_{\alpha \in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ is a $T^*_{\frac{1}{2}}$-space if and only if $(X_\alpha, u^1_\alpha, u^2_\alpha)$ is a $T^*_{\frac{1}{2}}$-space for each $\alpha \in I$.*

Proof. Suppose that $\prod_{\alpha \in I}(X_\alpha, u_\alpha)$ is a $T^*_{\frac{1}{2}}$-space. Let $\beta \in I$ and let F be a ∂-closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$. Then $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a ∂-closed subset of $\prod_{\alpha \in I}(X_\alpha, u^1_\alpha, u^2_\beta)$. Since $\prod_{\alpha \in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ is a $T^*_{\frac{1}{2}}$-space, $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a closed subset of $\prod_{\alpha \in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$. Consequently, F is a closed subset of (X_β, u^2_β). Hence, $(X_\beta, u^1_\beta, u^2_\beta)$ is a $T^*_{\frac{1}{2}}$-space.

Conversely, suppose that $(X_\alpha, u^1_\alpha, u^2_\alpha)$ is a $T^*_{\frac{1}{2}}$-space for each $\alpha \in I$. Let F be a ∂-closed subset of $\prod_{\alpha \in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ and let $(x_\alpha)_{\alpha \in I} \notin F$. Then there exists $\beta \in I$ such that $x_\beta \notin \pi_\beta(F)$. Since $\pi_\beta(F)$ is ∂-closed and $(X_\beta, u^1_\beta, u^2_\beta)$ is a $T^*_{\frac{1}{2}}$-space, $\pi_\beta(F)$ is a closed subset of (X_β, u^2_β). Thus, $x_\beta \notin u^2_\beta \pi_\beta(F)$ implies

$(x_\alpha)_{\alpha\in I} \notin \prod\limits_{\alpha\in I} u^2_\alpha \pi_\alpha(F)$. Therefore, F is a closed subset of $\prod\limits_{\alpha\in I}(X_\alpha, u^2_\alpha)$. Hence, $\prod\limits_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ is a $T^*_{\frac{1}{2}}$-space. □

Proposition 4.9. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces. If $\prod\limits_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ is a $T^{**}_{\frac{1}{2}}$-space, then $(X_\alpha, u^1_\alpha, u^2_\alpha)$ is a $T^{**}_{\frac{1}{2}}$-space for each $\alpha \in I$.*

Proof. Suppose that $\prod\limits_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ is a $T^{**}_{\frac{1}{2}}$-space. Let $\beta \in I$ and let F be a g-closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$. Then $F \times \prod\limits_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is a g-closed subset of $\prod\limits_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$. Since $\prod\limits_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ is a $T^{**}_{\frac{1}{2}}$-space, $F \times \prod\limits_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is a ∂-closed subset of $\prod\limits_{\alpha\in I}(X_\alpha, u^1_\alpha, u\alpha^2)$. Then F is a ∂-closed subset of (X_β, u^2_β). Hence, $(X_\beta, u^1_\beta, u^2_\beta)$ is a $T^{**}_{\frac{1}{2}}$-space.

Conversely, let $(X_\alpha, u^1_\alpha, u^2_\alpha)$ be a $T^{**}_{\frac{1}{2}}$-space for each $\alpha \in I$. Suppose that $\prod\limits_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ is not a $T^{**}_{\frac{1}{2}}$-space. Therefore, there exists a g-closed subset F of $\prod\limits_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ but it is not ∂-closed subset of $\prod\limits_{\alpha\in I}(X_\alpha, u^2_\alpha)$. Then there exists $\beta \in I$ such that $\pi_\beta(F)$ is not ∂-closed subset of (X_β, u^2_β). But $(X_\beta, u^1_\beta, u^2_\beta)$ is a $T^{**}_{\frac{1}{2}}$-space and $\pi_\beta(F)$ is a g-closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$, hence $\pi_\beta(F)$ is a ∂-closed subset of (X_β, u^2_β). This is a contradiction. □

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