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Metric Regularity and Fixed Points for Univaled and Set-Valued Operators

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Abstract. The study of fixed points for univaled and set-valued operators has been studied by many authors, among whom we remind M. Gregus [4], J.O. Olaleru [11], W.A. Kirk [6], F.M. Zeyoda [12], R.N. Mukherejee [9], S.B. Nadler [10] and others. The aim of this article is to offer enough conditions upon operators where these have fixed points. The basic hypothesis in this enough condition is the property of metric regularity on point of the graph.

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1. Introduction

Although metric regularity is a local property, we can obtain new results of fixed points in conditions which characterize theorems of fixed points. In general, metric regularity deals with the study of equation of the type $y \in F(x)$, where $y \in X$ is fixed, for a set-valued operator $F : X \rightrightarrows Y$. Many authors have obtained results in the metric regularity field among whom we remind A.L. Dontchev, R.T. Rockafellar [3], A.L. Dontchev, A.S. Lewis, R.T. Rockafellar [1,2,3], A.D. Yoffe [5], L.A. Lyusternik [7] and others. The norm and the radius of metric regularity characterize this property. A point x is an approximate solution of a generalized equation $y \in F(x)$ if the distance from the point y to the set $F(x)$ is small. The metric regularity of the set-valued mapping F means that, locally, a constant multiple of this distance bounds the distance from x to an exact solution. The smallest such constant is the modulus of regularity and it is a measure of the sensitivity or conditioning of the generalized equation. For the set-valued operator $F : X \rightrightarrows Y$ its inverse is denoted by $F^{-1} : Y \rightrightarrows X$ with $x \in F^{-1}(y)$ if and only if $y \in F(x)$ and its graph is defined by $gph\, F = \{(x, y)/y \in F(x)\}$.

Definition 1.1. ([1]). *Let X and Y be complete metric spaces, with the metric d. The set-valued mapping $F : X \rightrightarrows Y$ is said to be metrically regular at $\bar{x}$ for $\bar{y}$ if $\bar{y} \in F(\bar{x})$ and there exists $k \in [0, \infty)$ along with neighborhoods U of $\bar{x}$ and V of $\bar{y}$ so that*

$$d'(x, F^{-1}(y)) \leq k\, d'(y, F(x)), \tag{1.1.1}$$

for all $x \in U$ and $y \in V$. The infimum of the set of values k for which this holds is the modulus of metric regularity, denoted by $reg\, F(\bar{x}, \bar{y})$. Here, $d'(x, A) = \inf_{a \in A} d(x, a)$.

Definition 1.2. ([5]). *Let V be a subset of $X \times Y$, where X and Y are complete metric spaces. We say that F is metrically regular on V if there is $K > 0$ so that*

$$d'(x, F^{-1}(y)) \leq K\, d'(y, F(x)), \tag{1.1.2}$$

for all $(x, y) \in V$. The smallest K for which (1.2) holds will be called the norm of metric regularity of F on V, denoted by $Reg_V F$.

Definition 1.3. ([1]). *For any mapping $G : X \to Y$, where X and Y are Banach spaces, the Lipschitz modulus of a single-valued G at a point $\bar{x}$ is*

$$lip\, G(\bar{x}) = \limsup_{\substack{x, x' \to \bar{x} \\ x \neq x'}} \frac{\|G(x) - G(x')\|}{\|x - x'\|}. \tag{1.1.3}$$

Theorem 1.1. ([1]) (Lyusternik-Graves) *For any continuously Fréchet differentiable mapping $F : X \to Y$, where X and Y are Banach spaces and any $(\bar{x}, \bar{y}) \in gph\, F$, one has*

$$reg\, F((\bar{x}, \bar{y})) = reg\, DF(\bar{x}). \tag{1.1.4}$$

Thus, F is metrically regular at $\bar{x}$ for $\bar{y} = F(\bar{x})$ if and only if $DF(\bar{x})$ is surjective.

Theorem 1.2. ([1]) (Lyusternik-Graves) *Consider a mapping $F : X \rightrightarrows Y$, where X and Y are Banach spaces and any $(\bar{x}, \bar{y}) \in gph\, F$ at which $gph\, F$ is locally closed. If $reg\, F((\bar{x}, \bar{y})) < k < \infty$ and if $G : X \to Y$ is a mapping such that $G(\bar{x}) = 0$ and $lip\, G(\bar{x}) < \lambda < k^{-1}$, then*

$$reg\, (F + G)((\bar{x}, \bar{y})) < (k^{-1} - \lambda)^{-1} = \frac{k}{1 - \lambda k}. \tag{1.1.5}$$

Theorem 1.3. ([10]) (Covitz-Nadler) *Let (X, d) be a complete metric space, and $\mathcal{H}(X)$ the space of all compact subset of X. Suppose that $T : X \rightrightarrows \mathcal{H}(X)$ be a set-valued contraction mapping, i.e.*

$$H(Tx, Ty) \leq c \cdot d(x, y)$$

for all $x, y \in X$ and $c \in [0, 1)$.
Then there exists an $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$.

2. Main results

Next, there are a some results on fixed points.

Definition 2.1. *For a set-valued operator $F : X \rightrightarrows Y$ we say that $x \in X$ is a fixed point if $x \in F(x)$.*

We will analyse the existence of fixed points and metric regularity using a some examples of operators.

Theorem 2.1. *We consider the set-valued mapping $T : C \rightrightarrows C$, where $C \subset X$ is closed, so that*
(i) $d'(x, T^{-1}(y)) \leq k\, d'(y, Tx)$, for any $(x, y) \in C \times C$
(ii) for any $u \in C$, $T^{-1}(u)$ is nonempty
(iii) $k \in (0, 1)$.
Then T has fixed points.

Proof. The (i) condition states the metric regularity on $V = C \times C$ (see Definition 1.2) with $Reg_V T \in (0, 1)$. From the condition (ii), for any $u \in C$, the set $T^{-1}(u)$ is nonempty. For $(u, v) \in C \times C$ and $x \in T^{-1}(u)$, we have: $d'(x, T^{-1}(v)) \leq k \cdot d'(v, T(x)) \leq k \cdot d'(v, u)$, because $u \in T(x)$. So, $\sup\limits_{x \in T^{-1}(u)} d'(x, T^{-1}(v)) = e(T^{-1}(u), T^{-1}(v)) \leq k \cdot d'(v, u) = k \cdot d(v, u)$, $k = Reg_V T$. Analogously, for any $v \in C$, the set $T^{-1}(v)$ is nonempty. For $y \in T^{-1}(v)$, we have $d'(y, T^{-1}(u)) \leq k \cdot d'(u, T(y)) \leq k \cdot d'(u, v)$, because $v \in T(y)$. So, $\sup\limits_{y \in T^{-1}(v)} d'(y, T^{-1}(u)) = e(T^{-1}(v), T^{-1}(u)) \leq k \cdot d'(u, v) = k \cdot d(u, v)$. It results that $H(T^{-1}(u), T^{-1}(v)) = max\{e(T^{-1}(v), T^{-1}(u)),$
$e(T^{-1}(u), T^{-1}(v))\} \leq k \cdot d(u, v)$, where H is the Hausdorff metrig generated by $e(Y, Z) = sup\{d'(y, Z), y \in Y\}$. From the Covitz-Nadler theorem for set-valued contractive operators [10], the existence of the fixed point for the operator T^{-1}, $x_0 \in T^{-1}(x_0)$ or $x_0 \in T(x_0)$ is assured. □

Theorem 2.2. *We consider the set-valued mapping $F : C \rightrightarrows C$, where $C \subset X$ and X is complete metric space, C is closed and $V \subset C \times C$, closed, for which we have*
(i) $d'(x, F^{-1}(y)) \leq k\, d'(y, Fx)$, for any $(x, y) \in V$ and $k \in \left(0, \frac{1}{2}\right)$
(ii) for any $a \in C$ there exists $t \in C$ such that $t \in F^{-1}(a)$
(iii) $Reg_V F \in \left(0, \frac{1}{2}\right)$
(iv) $(x, y) \in V$ if and only if $(y, x) \in V$.
Then T has a fixed point.

Proof. Conditions (i) and (iii) express the metric regularity of F on V with the norm of regularity from the interval $\left(0, \frac{1}{2}\right)$. For $y \in pr_2 V$, we have

$\sup\{d'(x, F^{-1}(y)) : x \in pr_1V\} \leq k\, d'(y, F(x)) \leq k\, d(y,t)$, because $t \in F(x)$. From (ii) we have the existence of $t \in C$ for which $x \in F^{-1}(t) \Leftrightarrow t \in F(x)$ is assured. We can write: $\sup\{d'(x, F^{-1}(y)) : x \in pr_1V\} = e(F^{-1}(y), F^{-1}(t)) \leq Reg_V F \cdot d(y,t)$. In an analogical way, from $(y,x) \in V$ we have

$$e(F^{-1}(t), F^{-1}(y)) \leq Reg_V F \cdot d(y,t).$$

From the two inequalities we have $H(F^{-1}(y), F^{-1}(t)) \leq e(F^{-1}(t), F^{-1}(y)) + e(F^{-1}(y), F^{-1}(t)) \leq 2Reg_V F \cdot d(y,t)$ and from (iii), $2Reg_V F \in (0,1)$. From Theorem 2 ([10]), we obtain the existence of a point $x_0 \in C$ for which $x_0 \in F^{-1}(x_0)$, or $x_0 \in F(x_0)$. If for $a \in C$, there exists $t = a$ with $a \in F^{-1}(a)$, then we have the fixed point a, but in general $t \neq a$. □

Theorem 2.3. *We consider the complete metric space X and $C \subset X$ closed. Let set-valued mapping $F : C \rightrightarrows C$ such that*
(i) for any $(x,y) \in C \times C$, $d'(x, F^{-1}(y)) \leq k \cdot d'(y, Fx)$
(ii) for any $(\bar{x}, \bar{y}) \in gph\, F$ at which $gph\, F$ is locally closed, $reg\, F(\bar{x}, \bar{y}) < k < \infty$
(iii) $G(\bar{x}) = 0$ and $lip\, G(\bar{x}) < \lambda < \frac{1}{k} - 1$
(iv) $(F+G)^{-1}(u)$ is nonempty, for any $u \in C$.
Then $F + G$ has fixed points.

Proof. The (i) condition states the metric regularity for any $(x,y) \in C \times C$. Because $reg\, F(\bar{x}, \bar{y}) < k < \infty$ and G satisfies the (iii) condition, we have $reg\,(F+G)(\bar{x}, \bar{y}) < \frac{k}{1-\lambda k} < 1$ (see Theorem 1.2). For $\lambda < \frac{1}{k} - 1$, we have $\frac{\lambda}{1-\lambda k} < 1$ so that $\lambda < \frac{1-k}{k} < \frac{1}{k}$. So set-valued mapping $F + G : C \rightrightarrows C$ has the metric regularity property, and $reg\,(F+G)(\bar{x}, \bar{y}) < 1$. We apply Theorem 2.1, so that $F + G$ has fixed points. □

Further, for the polynomial $P(x) = \sum_0^n a_i x^i$, $a_i \in \mathbb{R}$, $i \in \{0, 1, \ldots, n\}$, $a_n \neq 0$, we will determinate another polynomial G such that $F + G$ has fixed points.

Theorem 2.4. *We consider the polynomial $P : I \to J$, $P(x) = \sum_0^n a_i x^i$, $a_i \in \mathbb{R}$, $i \in \{0, 1, \ldots, n\}$, $a_n \neq 0$, $(\bar{x}, \bar{y}) \in gph\, P$, where I, J are closed real intervals and the polynomial $G : I \to \mathbb{R}$ so that:*
(i) $P'(\bar{x}) \neq 0$ and $\frac{1}{|P'(\bar{x})|} \leq k$, for $\bar{x} \in I$ and $k \in (0,1)$
(ii) $d'(x, P^{-1}(y)) \leq k \cdot d'(y, P(x))$, for any $(x,y) \in I \times J$, with $P'(x) \neq 0$, for all $x \in I$
(iii) $G(\bar{x}) = 0$ and $G'(\bar{x}) \neq 0$, $|G'(\bar{x})| < \lambda < \frac{1}{k} - 1$
(iv) $(P+G)^{-1}(u)$ is nonempty, for any $u \in \mathbb{R}$.
Then $P + G$ has fixed points.

Proof. From $P'(\bar{x}) \neq 0$, P has the metric regularity property and from Theorem 1.1 we have the relation

$$reg\, P(\bar{x}, \bar{y}) = reg\, DP(\bar{x}).$$

The polynomial P is continuous, Fréchet differentiable, so $DP(\bar{x})$ is surjective operator. From the Remark 1.3.1 ([1]), we have

$$\begin{aligned} reg\, P(\bar{x}, \bar{y}) = \inf\{k > 0 | k \cdot F(\mathbb{B}_{\mathbb{R}}) \supset \mathbb{B}_{\mathbb{R}}\} = \\ = \sup\{d'(0, P^{-1}(y)) | y \in \mathbb{B}_{\mathbb{R}}\} = reg\, DP(\bar{x}) = \\ = \sup\{d'(O, DP(\bar{x})^{-1})(y) | y \in \mathbb{B}_{\mathbb{R}}\}, \end{aligned}$$

where $\mathbb{B}_{\mathbb{R}} = [-1, 1]$. The last equality takes place because $DP(\bar{x})$ is linear operator. From $|P'(\bar{x})x| = |P'(\bar{x})| \cdot |x| \leq 1$, we deduce that $|x| \leq \frac{1}{|P'(\bar{x})|}$, $\forall\, x \in \{DP(\bar{x})^{-1}(y) | y \in \mathbb{B}_{\mathbb{R}}\}$. We can apply Theorem 1.2, so $reg\,(P + G)(\bar{x}, \bar{y}) < \frac{1}{1-\lambda k} < 1$. Theorem 2.3 assure the existence of fixed points. □

Definition 2.2. *A mapping $F : X \rightrightarrows Y$ is sublinear when:*
(i) $0 \in F(0)$
(ii) $F(\lambda\, x) \supset \lambda F(x)$*, for any* $\lambda > 0$ *and* $x \in X$
(iii) $F(x + x') \supset F(x) + F(x')$*, for any* $x, x' \in X$.

Definition 2.3. *A mapping $F : X \rightrightarrows Y$ has $\mathcal{F}$-property when F is sublinear and for any $x \in X$ we have $x \in F(x) + F^{-1}(x)$ and $F^{-1}(x)$ is nonempty.*

Next, we will analyse a some examples emphasizing the $\mathcal{F}$-property of multivalent operators and fixed points.

Example 2.3. $F : \mathbb{R} \rightrightarrows \mathbb{R}$, $F(x) = [|x|, \infty)$, for which we have: $0 \in F(0)$, $F(\lambda\, x) = [\lambda |x|, \infty) \supseteq \lambda F(x)$, $F(x + y) \supset F(x) + F(y)$, $x \in F(x) + F^{-1}(x)$ and for $x \in \mathbb{R}$, $x < 0$, $F^{-1}(x) = \emptyset$. Indeed, for $x < 0$, $F^{-1}(x) = \{y \in \mathbb{R} | x \in [|y|, \infty)\} = \emptyset$. Because $|x + y| \leq |x| + |y|$, we have $[|x + y|, \infty) \supseteq [|x|, \infty) + [|y|, \infty)$ and from $a \in [\lambda |x|, \infty)$, we deduce $a = \lambda\, u$, with $u \geq |x|$, that is $a \geq \lambda |x|$. We have $F(x) - F(x) \subseteq F(0)$, too.

Example 2.4. $F : \mathbb{R} \rightrightarrows \mathbb{R}$, $F(x) = [x, \infty)$, for which $0 \in F(0)$, $F(\lambda\, x) \supseteq \lambda F(x)$ and $F(x + y) \supseteq F(x) + F(y)$. For $x \in \mathbb{R}$ we have $F^{-1}(x) = \{y \in \mathbb{R} : x \in [y, \infty)\} = (-\infty, x]$. So we have $F^{-1}(x) + F(x) = (-\infty, x] \cup [x, \infty) = \mathbb{R}$, so $x \in F^{-1}(x) + F(x)$. We have $F(x) - F(x) \subseteq F(0)$, too, for $x \geq 0$.

Example 2.5. $F : \mathbb{R} \rightrightarrows \mathbb{R}$, $F(x) = \{-x, x\}$, for which $F^{-1}(x) = \{-x.x\}$ and we have $0 \in F(0) = \{0\}$, $F(\lambda\, x) \supseteq \lambda F(x)$ and $F(x + y) = \{-x - y, x + y\} \not\supseteq \{-y, y\} + \{x, -x\}$ so it does not the $\mathcal{F}$-property, but we also have $F^{-1}(x) = F(x)$ and $x \in F(x)$, for any $x \in \mathbb{R}$.

For the sublinear multifunctions we have from Example 2.1 ([1]): $(\forall)(x, y) \in Gph\, F$, $reg\, F(\bar{x}|\bar{y}) \leq reg\, F(0|0) = \inf\{k \in (0, \infty) | y \in \mathbb{B}_Y \Rightarrow F^{-1}(y) \cap k\mathbb{B}_X \neq \emptyset\}$, where $\mathbb{B}_X$ is the unit ball from the Banach space X and $F : X \rightrightarrows Y$. In the example 2.4. we have $reg\, F(0|0) = 1$ so F is metrically regular on $(0|0)$.

Theorem 2.5. *If $F : X \rightrightarrows X$, where X is complete metric space, is a set-valued operator and $V \subset X \times X$ is closed so that:*
(i) $d'(x, F^{-1}(y)) \leq k\, d'(y, F(x))$, for any $(x, y) \in V$
(ii) $k \in (0, 1)$
(iii) $F^{-1}(y) \subseteq F(y)$, $\forall\, y \in pr_2V$,
(iv) $F^{-1}(x) \subseteq F(x)$, $\forall\, x \in pr_1V$,
then we have $y \in F(F(y))$, $\forall\, y \in pr_2V$.

Proof. From $F^{-1}(y) \subseteq F(y)$, we have $d'(x, F(y)) \leq d'(x, F^{-1}(y))$, and from the condition (i), we have $d'(x, F^{-1}(y)) \leq k \cdot d'(y, F(x))$. From $F^{-1}(x) \subseteq F(x)$, we have $k \cdot d'(y, F(x)) \leq k \cdot d'(y, F^{-1}(x)) \leq k^2 \cdot d'(x, F(y))$, so
$(1 - k^2)d'(x, F(y)) \leq 0$, that is $x \in F(y)$. In an analogical way, $y \in F(x)$. From the condition (iii), we deduce $x \in F(F^{-1}(x)) \subseteq F(F(x))$, so the operator $F \circ F$, defined by $(F \circ F)(y) = \bigcup_{y \in F(x)} \{F(y) : x \in X\}$ has any element from V as a fixed point. □

Theorem 2.6. *If $F : X \rightrightarrows X$, where X is complete metric space is a set-valued operator and $V \subset X \times X$ is closed so that:*
(i) F has the $\mathcal{F}$-property
(ii) $F^{-1}(x) \subseteq F(0)$, for any $x \in pr_1V$.
Then we have $x \in F(x)$, $(\forall) x \in pr_1V$.

Proof. From $F^{-1}(x) \subseteq F(0)$ and $x \in F(x) + F^{-1}(x)$, we have: $x \in F(x) + F^{-1}(x) \subseteq F(x) + F(0) \subseteq F(x)$, so $x \in F(x)$ for any $x \in pr_1V$. □

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