Int. Journal of Math. Analysis, Vol. 4, 2010, no. 9, 413 - 418

Pseudo Almost Automorphic Solutions of Nonautonomous Semilinear Differential Equations in Banach spaces

Saud M. Alsulami

Department of Mathematics
King Abdulaziz University
P.O. Box 138381
Jeddah 21323, Saudi Arabia
alsulami@kau.edu.sa

Abstract

We consider the existence and uniqueness of the Pseudo almost automorphic solutions to the nonautonomous semilinear differential equation:

$$u'(t) = A(t)u(t) + f(t, u(t)),\ t \in \mathbb{R}$$

where $A(t), t \in \mathbb{R}$, generates an exponentially stable evolution family $\{U(t, s)\}$ and $f : \mathbb{R} \times X \to X$ satisfies a Lipschitz condition with respect to the second argument.

Mathematics Subject Classification: 43A60, 34G20, 47D

Keywords: Pseudo Almost automorphic; Nonautonomous equations

1 Introduction

In this paper we study the existence and uniqueness of Pseudo almost automorphic solutions to the following nonautonomous semilinear differential equation:

$$u'(t) = A(t)u(t) + f(t, u(t)),\ t \in \mathbb{R} \tag{1}$$

where $U(t, s)$ generated by $A(t)$, is exponentially stable, and $f \in PAA(X)$and $f : \mathbb{R} \times X \to X$ satisfies a Lipschitz condition with respect to the second argument.

S. Bochner introduced the concept of almost automorphic functions in his papers ([2]-[4]) in relation to some aspects of differential geometry. This concept became a generalization of almost periodicity which is one of the most attractive topics in the qualitative theory of differential equations because of their significance and applications in physics, mathematical biology, control theory, and other related fields. A natural generalization of the concept of almost automorphic functions is the concept of Pseudo almost automorphic functions which has widely been used in investigation of the existence of almost automorphic solutions of various kinds of evolution equations by many others. For more information of the latter concept, we refer the reader to [5] and [6] . After this paper has been accepted, we know about the appearance of [10] where our problem and more general cases have been considered.

2 Preliminaries

Definition 2.1 *(i) A continuous function $f : \mathbb{R} \to X$ is said to be almost automorphic (in Bochner's sense) if for every sequence of real numbers $\{s'_n\}$, there exists a subsequence $\{s_n\}$ such that*

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t - s_n) = f(t)$$

for each $t \in \mathbb{R}$. $AA(\mathbb{R}, X)$ stands for the set of all such functions.
(ii) A continuous function $f : \mathbb{R} \times X \to X$ is said to be almost automorphic if $f(t, x)$ is almost automorphic in $t \in \mathbb{R}$ uniformly for all x in any bounded subsets of X . $AA(\mathbb{R} \times X, X)$ is the set of all such functions.
(iii) A continuous function $f : \mathbb{R} \to X$ (resp. $f : \mathbb{R} \times X \to X$) is said to be pseudo almost automorphic if it can be decomposed as $f = g + \phi$ where $g \in AA(\mathbb{R}, X)$ (resp. $AA(\mathbb{R} \times X, X)$) and ϕ is bounded continuous function with vanishing mean value (resp. ϕ is bounded continuous function with

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|\phi(\sigma, x)\| d\sigma = 0$$

uniformly for all x in any bounded subsets of X) . Denote by $PAA(\mathbb{R}, X)$ (resp. $PAA(\mathbb{R} \times X, X)$) the set of all such functions.

Note that $(PAA(\mathbb{R}, X), \|\cdot\|_0)$ turns out to be a Banach space, where $\|\cdot\|_0$ is the supremum norm, see [Theorem 2.2, [9]].

Throughout this paper, X stands for a Banach space with norm $\|\cdot\|$. We denote by $C(\mathbb{R}, X)$ the Banach space of all continuous functions from $\mathbb{R}$ to X. Similarly, $BC(\mathbb{R}, X)$ is the Banach space of all bounded continuous functions from $\mathbb{R}$ to X. Note that $(BC(\mathbb{R}, X), \|\cdot\|_\infty)$ is a Banach space with the sup norm $\|\cdot\|_\infty$.

Note that $(AA(X), \|\cdot\|_\infty)$ turns out to be a Banach space.

Theorem 2.2 *([8]) Assume that $A(t), t \in \mathbb{R}$ is a bounded linear operator on a Banach space X and $t \to A(t)$ is continuous in the uniform operator topology, then for $-\infty < s \leq t < \infty$, $U(t,s)$ generated by $A(t)$, is a bounded linear operator satisfying the following:*

(i) $\|U(t,s)\| \leq \exp(\int_s^t \|A(\tau)\| d\tau)$.

(ii) $U(t,t) = I, \;\; U(t,s) = U(t,r)U(r,s),$ *for* $-\infty < s \leq r \leq t < \infty$.

(iii) $(t,s) \to U(t,s)$ *is continuous in the uniform operator topology for* $-\infty < s \leq t < \infty$.

(iv) $\partial U(t,s)/\partial t = A(t)U(t,s)$ *for* $-\infty < s \leq t < \infty$.

(v) $\partial U(t,s)/\partial s = -U(t,s)A(s)$ *for* $-\infty < s \leq t < \infty$.

In this paper we assume that $\{A(t)\}_{t\in\mathbb{R}}$ satisfies the 'Acquistapace-Terreni' conditions introduced in ([1]), that is,

(H_1) there exist constants $\lambda_0 \geq 0,\; \theta \in (\frac{\pi}{2}, \pi),\; L,\; K \geq 0$, and $\alpha,\; \beta \in (0,1]$ with $\alpha + \beta > 1$ such that

$$\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0),\; \|R(\lambda, A(t) - \lambda_0)\| \leq \frac{K}{1 + |\lambda|}$$

and

$$\|(A(t) - \lambda_0)R(\lambda, A(t) - \lambda_0)[R(\lambda_0, A(t)) - R(\lambda_0, A(s))]\| \leq L|t-s|^\alpha |\lambda|^{-\beta}$$

for $t, s \in \mathbb{R},\; \lambda \in \Sigma_\theta := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| \leq \theta\}$.

Remark 2.3 *If the condition (H_1) holds, then there exists a unique evolution family $\{U(t,s)\}_{-\infty < s \leq t < \infty}$ on X, which satisfies the homogenous equation $u'(t) = A(t)u(t),\; t \in \mathbb{R}$.*

3 Main Results

Definition 3.1 *A mild solution to (1) is a continuous function $u(t) : \mathbb{R} \to X$ satisfying*

$$u(t) = U(t,a)u(a) + \int_a^t U(t,s)f(s,u(s))ds \tag{2}$$

for all $t \geq a$ and all $a \in \mathbb{R}$.

In the proof of the following theorem we follow the same reasoning as in the proof of Theorem 3.3 in [9] with the proper modification .

Theorem 3.2 *suppose that the evolution family $U(t,s)$ generated by $A(t)$ is exponentially stable, that is, there are constants $K, \omega > 0$ such that $\|U(t,s)\| \leq Ke^{-\omega(t-s)}$ for all $t \geq s$. and f satisfies*

$$\|f(t,u) - f(t,v)\| \leq L\|u - v\|$$

$t \in \mathbb{R}$ and $u, v \in X$ for $L < \frac{\omega}{K}$. Then the equation (1) has a unique pseudo almost automorphic mild solution.

Proof.
Consider the nonlinear operator $\mathcal{F}$ given by

$$(\mathcal{F}x)(t) = \int_{-\infty}^t U(t,s)f(s,u(s))ds.$$

If $u(s) \in PAA(\mathbb{R}, X)$, then Lemma 3.1 in [9] gives that $f(s,u(s)) \in PAA(\mathbb{R}, X)$. i.e., there exist $g \in AA(\mathbb{R}, X)$ and a bounded continuous function ϕ with vanishing mean value, such that $f = g + \phi$. Therefore $(\mathcal{F}x)(t)$ can be expressed as $(\mathcal{F}x)(t) = G(t) + \Phi(t)$, where

$$G(t) = \int_{-\infty}^t U(t,s)g(s,u(s))ds.$$

$$\Phi(t) = \int_{-\infty}^t U(t,s)\phi(s,u(s))ds.$$

From the proof of N'Guerekata [[7], Theorem 3.2], it follows that $t \to G(t)$ is almost automorphic.

Next, we show that $\Phi(t)$ is a bounded continuous function with vanishing mean value. Since $\Phi(t)$ is bounded on $\mathbb{R}$, we have $M := \sup_{t\in\mathbb{R}} \|\Phi(t)\| < +\infty$. For any $T \in [-t, t]$, we get

$$\int_{-T}^{T} \int_{-\infty}^{-T} \|U(t,s)\Phi(s)\| dsdt \leq \frac{MK}{\omega^2}$$

On the other hand,

$$\int_{-T}^{T}\int_{-T}^{t}\|U(t,s)\Phi(s)\|dsdt \le \frac{K}{\omega}\int_{-T}^{T}\|\Phi(s)\|ds$$

By the above two equations, we get

$$\lim_{T\to\infty}\int_{-T}^{T}\int_{-\infty}^{t}\|U(t,s)\Phi(s)\|dsdt = 0$$

Thus, $\Phi(t)$ is a bounded continuous function with vanishing mean value. Therefore, $(\mathcal{F}x)(t) = G(t) + \Phi(t)$ is a pseudo almost automorphic function on $\mathbb{R}$. That is , $\mathcal{F}(PAA(\mathbb{R}, X)) \subseteq PAA(\mathbb{R}, X)$. Moreover, for every $u, v \in PAA(\mathbb{R}, X)$.

$$\|\mathcal{F}u - \mathcal{F}v\| \le \sup_{t\in\mathbb{R}}\int_{-\infty}^{t}\|U(t,s)\|[f(s,u(s)) - f(s,v(s))]ds$$

$$\le \frac{LK}{\omega}\|u - v\|$$

Thus $\mathcal{F}$ is a contraction on $PAA(\mathbb{R}, X)$. Therefore, by the contraction mapping theorem $\mathcal{F}$ has a unique fixed point $u(t) \in PAA(\mathbb{R}, X)$ since $PAA(\mathbb{R}, X)$ is complete. The fixed point satisfies the integral equation

$$u(t) = \int_{-\infty}^{t} U(t,s)f(s,u(s))ds,$$

for all $t \in \mathbb{R}$. Fixing $a \in \mathbb{R}$, we have

$$u(a) = \int_{-\infty}^{a} U(a,s)f(s,u(s))ds,$$

Since $U(t,s) = U(t,a)U(a,s)$, for $-\infty < s \le a \le t < \infty$, it follows that $u(t)$ satisfies (2). Hence $u(t)$ is a mild solution to (1).

On the other hand, let $v(t)$ be a pseudo almost automorphic mild solution to (1).

Then $v(t)$ satisfies the equation (2), with u replaced by v. Letting $a \to -\infty$ yields

$$v(t) = \int_{-\infty}^{t} U(t,s)f(s,v(s))ds, t \in \mathbb{R}$$

Since $v(t)$ is bounded on $\mathbb{R}$ and $U(t,s)$ is exponentially stable. Hence $u(t) \equiv v(t)$ on $\mathbb{R}$. i.e., $u(t)$ is the unique mild solution to (1).

ACKNOWLEDGMENTS. This work has been done under the project no. 428/199. The author is grateful to the Deanship of Scientific Research at King Abdulaziz University for their financial support. The author also likes to thank Haewon Lee for suggesting the problem.

References

[1] P. Acquistapace, B. Terreni, A unified approach to abstract linear parabolic equations, Rend. Sem. Mat. Univ. Padova 78 (1987) 47-107.

[2] S. Bochner, Uniform convergence of monotone sequences of functions, *Proc. Natl. Acad. Sci. USA* **47** (1961) 582-585.

[3] S. Bochner, A new approach to Almost periodicity, *Proc. Natl. Acad. Sci. USA* **48** (1962) 2039-2043.

[4] S. Bochner, Continuous mappings of almost automorphic and almost periodic functions, *Proc. Natl. Acad. Sci. USA* **52** (1964) 907-910.

[5] T. Diagania, Existence of p-almost automorphic mild solution to some abstract differential equations, *Int. J. Evol. Equ.* no. **1** (2005), 57 - 67.

[6] Jin Liang , Jun Zhang and Ti-Jun Xiao, Composition of Pseudo almost automorphic functions, *preprint.*

[7] G.M. N'Guérékata, Existence and uniqueness of almost automorphic mild solutions to some semilinear abstract differential equations, *Semigroup Forum* **69** (2004), 80-86.

[8] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Applied Mathematical Sciences, Vol. 44. Springer-Verlag, New York, 1983.

[9] Ti-Jun Xiao, Jin Liang and Jun Zhang, Pseudo almost automorphic solutions to semilinear differential equations in Banach spaces, *preprint.*

[10] Ti-Jun Xiao, Xing-Xing Zhu, Jin Liang, Pseudo-almost automorphic mild solutions to nonautonomous differential equations and applications, *preprint.*

Received: September, 2008