Int. Journal of Math. Analysis, Vol. 4, 2010, no. 12, 565 - 576

Periodicity and Transformation of Difference Equations

Khalil I. T. Al-Dosary

Department of Mathematics, College of Sciences
University of Sharjah, P.O. Box 27272, Sharjah, UAE
dosary@sharjah.ac.ae

Abstract

In this research we develop a method to construct periodic difference equations with desired period. Also, we investigate the transformation of periodicity of difference equations under invertible maps . These techniques are applied to generalize and extend some results that have been obtained recently. One of the interesting results is derivation of some generalized Lyness' equations.

Mathematics Subject Classification: 39A10

Keywords: Lyness' equations, Periodic solution

1 Introduction and Preliminaries

Let $D \subseteq R$, and consider the difference equation

$$x_{n+k} = f(x_n, ..., x_{n+k-1}) \qquad n = 0, 1, 2, ... \tag{1}$$

where f is a continuous function such that $f : D^k \to D$, and $x_{0,...,}x_{k-1} \in D$, and k is a nonnegative integer. The study of periodicity of difference equations has received a great deal of attention $[1, 2]$, $[6]$, $[8 - 11]$ and others. Our objective in this article is to develop a constructive method introduced in [2] and then to furnish a parametric form for the function f for which $Eq.(1)$ is periodic in global sense. This problem is motivated in [9]. Although the theorem establishes an endless applications. It is applied to generalize and extend several results that have been obtained recently, see for example, $[4]$, $[11]$ and references therein. It is worth mentioning that our method is applied to derive some generalized Lyness' equations.

For the sake of completeness, we recall the following definitions and some notations and preliminaries that will be used in the present work.

A sequence $\{x_n\}$ is called periodic, ($\tau - period$) if there is an integer $\tau \in N$ such that $x_{n+\tau} = x_n$ for all $n \in N$, [6] . If a periodic sequence has a smallest period p (>0), this period is called the fundamental period. A difference equation of the form (1) is said to be global periodic of period τ (or for a simple $\tau - period$) if every its solution sequence is $\tau - period$ regardless of the initial terms $x_0, ..., x_{k-1}$.

Definition 1 *A function $I(x_1, ..., x_k)$ is said to be symmetric if it is invariant under any permutation of $(x_1, ..., x_k)$.*

Definition 2 *A function $I(x_1, ..., x_k)$ is said to be isovertible if it is an invertible function in each of its independent variables, i.e.,*

$$I_1(\lambda) = I(\lambda, x_2, ..., x_k),\ I_2(\lambda) = I(x_1, \lambda, x_3, ..., x_k), ..., I_k(\lambda) = I(x_1, ..., x_{k-1}, \lambda)$$

are invertible functions. The inverse of I_j is denoted by $I^{-(j)}$, i.e., $I^{-(j)} = I_j^{-1}$. In other words, $I^{-(j)} : D \to D$, is defined as follows, for any $j \in \{1, ..., k\}$, $\lambda \in D$,

$$I^{-(j)}(x_1, ..., x_{j-1}, \lambda, x_{j+1}, ..., x_k) = x_j \qquad (2)$$

if and only if

$$I(x_1, ..., x_{j-1}, x_j, x_{j+1}, ..., x_k) = \lambda$$

Remark 3 *Observe that if $g, h : D \to D$ are two invertible functions, and $I : D^k \to D$ is symmetric and isovertible, then so the function F defined by $F(x_1, ..., x_k) = g(I(h(x_1), ..., h(x_k)))$.*

This is due to the fact that it is invariant under any permutation of $(\mathrm{x}_1, ..., x_k)$ and it is invertible.

We introduce the following notation that will play an important role in what follows

Definition 4 *Given a nonnegative integer r,*

$$I_r(x_1,...,x_k)=\begin{cases}1 & if\ r=0\\ \sum\limits_{1\le i_1<...<i_r\le k}\prod\limits_{s=1}^{r}x_{i_s} & if\ \ 0<r\le k\\ 0 & if\ r>k\end{cases}$$

For example, for $k=2$ we have $I_1=x_1+x_2$, $I_2=x_1x_2$, for $k=3$ we have $I_1=x_1+x_2+x_3$, $I_2=x_1x_2+x_1x_3+x_2x_3$, $I_3=x_1x_2x_3$, and so on.

Lemma 5 *Let r be a nonnegative integer such that $0<r\le k$ and I_r be as in Definition 4, then*

(a) I_r is symmetric and isovertible.

(b) For every $j=1,...,k,$ and $0<r\le k$, $I_r^{-(j)}$ is given by

$$I_r^{-(j)}(x_1,...,x_j,...,x_k)=\frac{\lambda-I_r(x_1,...,x_{j-1},x_{j+1},...,x_k)}{I_{r-1}(x_1,...,x_{j-1},x_{j+1},...,x_k)}$$

Proof. *Part (a) is straightforward, and Part (b) follows from the following argument,*

$$\lambda=I_r(x_1,...,x_j,...,x_k)=\sum_{1\le i_1<...<i_r\le k}\prod_{s=1}^{r}x_{i_s}$$

$$=\sum_{\substack{1\le i_1<...<i_r\le k\\ i_s\ne j}}\prod_{s=1}^{r}x_{i_s}+x_j\sum_{\substack{1\le i_1<...<i_r\le k\\ i_s\ne j}}\prod_{s=1}^{r-1}x_{i_s}$$

$$=I_r(x_1,...,x_{j-1},x_{j+1},...,x_k)+x_jI_{r-1}(x_1,...,x_{j-1},x_{j+1},...,x_k).$$

Now, the result follows by solving for x_j. ■

There is some relationship between the symmetric and isovertible function $I(x_1,...,x_k)$ and the invariants and then Liapunov function [9].

The paper is organized as follows. In the current Section, we present some definitions, preliminaries and the adopted notations. In the next sections, we introduce our main results. These results are of three issues. The first one is to establish, under restriction on the desired period p, a constructive method that generates one-parameter families of functions for which all solutions are periodic of period p and this will be given in Section 2. The second issue is to construct from a given periodic difference equation of period p, a new difference equation of period lp, where p,l are positive integers and these periods are fundamental periods and this is given in Section 3. In the third issue we investigate a form of periodic difference equation that is transformed from a given periodic difference equation under invertible map which is given in Section 4. In each section we provide applications and some examples to illustrate the applicability of the results.

2 Global Periodicity

Theorem 6 *Let p be a positive integer such that $p > k$. Suppose that $p-k$ divides k with $L = k/(p-k)$. Then all solutions of Equation (1) are periodic of period p if*

$$f(x_n, ..., x_{n+k-1}) = I^{-(n+k)}(x_n, x_{n+l}, ..., x_{n+(L-1)l}, \lambda)$$

for some symmetric and isovertible function $I : D^{L+1} \to D$ and arbitrary constant $\lambda \in D$.

Proof. *From Eq.(1) and the hypothesis, we get*

$$x_{n+k} = f(x_n, ..., x_{n+k-1}) = I^{-(n+k)}(x_n, x_{n+l}, ..., x_{n+(L-1)l}, \lambda)$$

Hence

$$I(x_n, x_{n+l}, ..., x_{n+(L-1)l}, x_{n+k}) = \lambda$$

From invertibility in the $(n+k)$st independent variable and the fact that $l = p-k$, we have

$$I(x_n, x_{n+(p-k)}, ..., x_{n+(L-1)(p-k)}, x_{n+L(p-k)}) = \lambda \qquad (3)$$

This is for all $n \geq 0$, in particular for $n+p-k$, we get

$$I(x_{n+(p-k)}, x_{n+2(p-k)}, ..., x_{n+L(p-k)}, x_{n+(L+1)(p-k)}) = \lambda$$

Now, by symmetry of I, this equation can be written in the form,

$$I(x_{n+(L+1)(p-k),} x_{n+(p-k)}, ..., x_{n+(L-1)(p-k)}, x_{n+L(p-k)}) = \lambda \qquad (4)$$

From Eq.(3) and Eq.(4) and invertibility of I, if we equate the corresponding first terms, we obtain that,

$$x_n = x_{n+(L+1)(p-k)} = x_{n+p} \qquad \text{since } L(p-k) = k$$

and hence the result follows ■

Using Lemma 5, Part b and Theorem 6, we obtain the following result.

Corollary 7 *Suppose that k, p are positive integers such that $p > k \geq 2$ and $L = k/(p-k)$ is a positive integer. Then Eq.(1) is periodic with period p if*

$$f(x_n, x_{n+l}, ..., x_{n+(L-1)l}) = \frac{\lambda - I_r(x_n, x_{n+l}, ..., x_{n+(L-1)l})}{I_{r-1}(x_n, x_{n+l}, ..., x_{n+(L-1)l})}, \qquad r = 1, 2, ..., \frac{p}{p-k}$$

where λ is an arbitrary real number.

In order to illustrate the applicability of Theorem 6 and Corollary 7, we present the following applications.

2.1 Applications

In this subsection we will illustrate the use of our result by choosing different values for k and p in order to obtain parametric $p-periodic$ difference equations of two parameters λ (arbitrary) and r $(= 1, 2, ..., \frac{p}{p-k})$.

2.1.1 Example 1: Case k=2

There are only two possible values, 3 and 4, for p for which $\frac{k}{p-k}$ is positive integer.

Thus, for $p = 3$, we have $l = p - k = 1$ and $L = \frac{k}{p-k} = 2$. So from Corollary 7, equation (1) will be in the form

$$x_{n+2} = f(x_n, x_{n+1}) = \frac{\lambda - I_r(x_{n+1}, x_n)}{I_{r-1}(x_{n+1}, x_n)}, \qquad r = 1, 2, 3$$

where λ is an arbitrary constant, and these difference equations are periodic of period 3. In this case, we have

$$for \quad r = 1: \qquad x_{n+2} = \frac{\lambda - I_1(x_n, x_{n+1})}{I_0(x_n, x_{n+1})} \tag{5}$$

But from the Definition 4 of I_r we have $I_0(x_{n+1}, x_n) = 1$ and $I_1(x_{n+1}, x_n) = x_{n+1} + x_n$. Thus equation (5) will be of the form

$$x_{n+2} = \lambda - x_{n+1} - x_n$$

Similarly, for other values of r.

$$for \quad r = 2: \qquad x_{n+2} = \frac{\lambda - I_2(x_n, x_{n+1})}{I_1(x_n, x_{n+1})} = \frac{\lambda - x_n x_{n+1}}{x_n + x_{n+1}} \tag{6}$$

$$for \quad r = 3: \qquad x_{n+2} = \frac{\lambda - I_3(x_{n+1}, x_n)}{I_2(x_{n+1}, x_n)} = \frac{\lambda}{x_n x_{n+1}} \tag{7}$$

Now, for $p = 4$, we have $l = p - k = 2$ and $L = 1$, and so all solutions of the difference equations

$$x_{n+2} = \frac{\lambda - I_r(x_n)}{I_{r-1}(x_n)}, \qquad r = 1, 2$$

where λ is an arbitrary constant, are periodic of period 4. In this case, we have the difference equations:

$$for \quad r = 1: \qquad x_{n+2} = \frac{\lambda - I_1(x_n)}{I_0(x_n)} = \lambda - x_n \tag{8}$$

$$for \quad r = 2: \qquad x_{n+2} = \frac{\lambda - I_2(x_n)}{I_1(x_n)} = \frac{\lambda}{x_n} \tag{9}$$

2.1.2 Example 2: Case k=3

There are two possible values for p in order to have $\frac{k}{p-k}$ positive integer, these values are 4 and 6.

For $p = 4$; we have $l = p - k = 1$ and $L = \frac{k}{p-k} = 3$, so all solutions of the difference equations

$$x_{n+3} = \frac{\lambda - I_r(x_n, x_{n+1}, x_{n+2})}{I_{r-1}(x_n, x_{n+1}, x_{n+2})} \qquad r = 1, 2, 3, 4$$

are periodic of period 4. In this case, we have the following difference equations:

$$r = 1: \qquad x_{n+3} = \lambda - x_n - x_{n+1} - x_{n+2} \tag{10}$$

$$r = 2: \qquad x_{n+3} = \frac{\lambda - x_n x_{n+1} - x_n x_{n+2} - x_{n+1} x_{n+2}}{x_n + x_{n+1} + x_{n+2}} \tag{11}$$

$$r = 3: \qquad x_{n+3} = \frac{\lambda - x_n x_{n+1} x_{n+2}}{x_n x_{n+1} + x_n x_{n+2} + x_{n+1} x_{n+2}} \tag{12}$$

$$r = 4: \qquad x_{n+3} = \frac{\lambda}{x_n x_{n+1} x_{n+2}} \tag{13}$$

For $p = 6$, we have $l = 3$, and $L = 1$, so all solutions of the difference equations

$$x_{n+3} = \frac{\lambda - I_r(x_n)}{I_{r-1}(x_n)} \qquad r = 1, 2$$

are periodic of period 6. In this case, we have the difference equations:

$$r = 1: \qquad x_{n+3} = \lambda - x_n \tag{14}$$

$$r = 2: \qquad x_{n+3} = \frac{\lambda}{x_n} \tag{15}$$

3 Transformation of Periodicity

In order to widen the area of applicability of Theorem 6 and Corollary 7, we introduce the following Proposition

Proposition 8 *Eq.*(1) *is periodic with fundamental period* p *if and only if the difference equation*

$$y_{n+lk} = f(y_n, y_{n+l}, ..., y_{n+(k-1)l}) \tag{16}$$

is periodic with fundamental period lp*.*

Proof. The change of variables

$$x_{n+i} = y_{n+il} \tag{17}$$

for $i = 0, 1, 2, ...$ and $l = 1, 2, 3, ...$reduces Eq.(1) to $Eq.(16)$. The fact, p is period of Eq.(1) if and only if lp is period of Eq.(16), is directly proved from (17).

In order to show that, p is fundamental if and only if lp is fundamental, we firstly assume that p is fundamental for Eq.(1), but lp is not, then we establish the proof by contradiction.

Assume that

$$x_{n+p} = x_n \qquad for\ all\ n$$

Hence

$$y_{n+lp} = y_n \qquad for\ all\ n$$

since, by assumption, lp is not the smallest such number, then there is some α, $0 < \alpha < p$ such that

$$y_n = y_{n+l(p-\alpha)} \qquad for\ all\ n$$

so

$$x_n = x_{n+p-\alpha} \qquad for\ all\ n$$

This contradicts the assumption. Hence lp must be fundamental for Eq,(16). The proof of the other direction is similar and will be omitted. ■

3.1 Applications

The following applications generalize some results published recently.

3.1.1 Example 1

The difference equation

$$x_{n+l} = \frac{1}{x_n} \qquad l = 1, 2, ...$$

is periodic with period $2l$, recalling that the difference equation

$$x_{n+1} = \frac{1}{x_n}$$

is periodic with period 2.

3.1.2 Example 2

The difference equation

$$x_{n+2l} = \frac{\max\{x_{n+l}, 1\}}{x_n} \qquad l = 1, 2, ...$$

is periodic with period $5l$, recalling that the difference equation

$$x_{n+2} = \frac{\max\{x_{n+1}, 1\}}{x_n}$$

is periodic with period 5, see [4].

3.1.3 Example 3

A generalized Lyness' equation

$$x_{n+2l} = \frac{x_{n+l}+1}{x_n} \qquad l = 1, 2, ...$$

is periodic with period $5l$, recalling that, Lyness' equation

$$x_{n+2} = \frac{x_{n+1}+1}{x_n} \tag{18}$$

is periodic with period 5, see [6]. In the next section we present another generalization for this well known equation.

4 Transformation of Periodic Equations

Theorem 9 *Let $T : D \to D$ be an invertible transformation. If Eq.(1) is periodic with fundamental period p, then the difference equation*

$$y_{n+k} = T(f(T^{-1}(y_n), ..., T^{-1}(y_{n+k-1}))) \tag{19}$$

where $y_n = T(x_n)$, is periodic with fundamental period p.

Proof. *By the given transformation, we have*

$$y_{n+p} = T(x_{n+p}) = T(x_n) = y_n \qquad for\ n \in N$$

which implies the result. ■

Now, if we combine Theorem 6 and Theorem 9 together, we obtain the following theorem of more general statement

Theorem 10 *Let $p \in N$, such that $\frac{k}{p-k} = L \in N$. Let $I : D^{L+1} \to D$ be a symmetric, isovertible and continuous function, and $x_n = T(y_n)$ be an invertible transformation, then Eq.(1) is periodic with period p if*

$$f(x_n, ..., x_{n+k-1}) = T(I^{-(n+k)}(T^{-1}(x_n), ..., T^{-1}(x_{n+(L-1)l}), \lambda))$$

where λ is an arbitrary real number independent of n and $l = p - k$.

4.1 Applications

The applications of these above theorems are endless. However, in the following, we choose some particular periodic equations that have been derived in previous sections in order to illustrate the applicability of the theorems using the particular invertible transformation,

$$y_n = \frac{ax_n + b}{cx_n + 1}, \qquad a \neq bc \tag{20}$$

This transformation will blow up the form of the equations inserting parameters a, b and c in the form of the equation, see the following examples;

4.1.1 Example 1

If we apply the transformation (20) in the Eq.(5) and simplify, we obtain that the difference equation

$$y_{n+2} = \frac{A_1 + A_2(y_n + y_{n+1}) + A_3 y_n y_{n+1}}{A_2 + A_3(y_n - y_{n+1}) + A_4 y_n y_{n+1}}$$

where

$$A_1 = a^2(\lambda a + 3b)$$
$$A_2 = -a(\lambda ac + 2bc + a)$$
$$A_3 = c(\lambda ac + 2a + bc)$$
$$A_4 = c^2(\lambda c + 3)$$

and $a \neq bc$, λ is an arbitrary constant, is periodic of period 3.

4.1.2 Example 2

If we apply the transformation (20) in the Eq.(8) and simplify, we obtain that the difference equation

$$y_{n+2} = \frac{A_1 + A_2 y_n}{-A_2 + A_3 y_n}$$

where

$$A_1 = a(\lambda a + 2b)$$
$$A_2 = -(\lambda ac + a + bc)$$
$$A_3 = -c(\lambda c + 2)$$

and $a \neq bc$, λ is an arbitrary constant, is periodic with period 4.

4.1.3 Example 3

If we apply the transformation (20) in Lyness'equation (18) and simplify, we obtain that y_n satisfies the difference equation;

$$y_{n+2} = \frac{a(1-c)y_{n+1} + by_n + a^2 - ab - b^2}{c(1-c)y_{n+1} + y_n + ac - bc - b} \tag{21}$$

where a , b and c are any numbers such that $a \neq bc$.

This parametric difference equation is periodic with period 5. In particular, for $c = 0$, Eq(21) reduces to

$$y_{n+2} = \frac{ay_{n+1} + by_n + a^2 - ab - b^2}{y_n - b} \qquad a \neq 0$$

which is a generalized Lyness' equation [6] .

5 Conclusion

We conclude the paper by mentioning that Th.(10) can be applied with other different forms of transformations T, for instance, $T(x) = \log_a x$, $a > 0$; $T(x) = x^m$, m is an odd integer; and $T(x) = a^x$, $a > 0$, ...etc. or any combination of these injective maps. The cases that have been applied in this paper are nothing but a particular examples whereas the applications of the theorems are endless.

Acknowledgment

The author acknowledges the University of Sharjah for support.

References

[1] R. P. Agarwal, J. Popenda, On Periodic Solutions of First Order Linear Difference Equations, Math. Comput. Modelling 22(1)(1995), 11-19.

[2] K. I. Al-Dosary, Global Periodicity: An Inverse Problem, Applied Mathematics Letters 18(2005), 1041-1045.

[3] K. Cooke, L. Ladeira, Applying Carvalho's Method to Find Periodic Solutions of Difference Equations, J.Difference Eqs. and Applications, 2(1996), 105-115.

[4] Ed Barbeau, B. Gelbord and S. Tanny, Periodic of Solutions of the Generalized Lyness Recursion, J. Difference Eqs. and Applications,1(1995), 291-306.

[5] E.J. Janowski,V. L. Kocic and G. Ladas, and S.W. Schults, Global Behavior of Solutions of $x_{n+1} = \max\{x_n, A\}/x_{n-1,}$ Proc. of the first ICDEA (Eds.: S. Elaydi, J.Greaf, G. Ladas, and A. Petterson), (1995),273-282.

[6] V. L. Kocic and G. Ladas, Global behavior of Nonlinear Difference Equations of Higher Order With Applications, Kluwer Academic Publications, London, 1995.

[7] M.Kulenovic, Invariant and Related Liapunov Functions for Difference Equations, Appl. Math. Lett. 13(7)(200),1-8.

[8] M.Kulenovic, and G. Ladas. Open Problems and Conjectures: On Period Two Solutions of $x_{n+1} = ((\alpha + \beta x_n + \gamma x_{n-1})/(A + Bx_n + cx_{n-1}))$, J. Difference Eqs. and Applications, 5(2000),641-646.

[9] G. Ladas, Open Problems and Conjectures, J.Difference Eqs. and Applications,1(1995), 95-97.

[10] G. Papaschinopoulos, C. Schinas and G. Stefanidou, Boundedness, Periodicity and Stability of the Difference Equations $x_{n+1} = A_n + x_{n-1}$. Intr. J of Dyn. Syst. and Diff. Equ. Vol. 1, No. 2, (2007),109-116.

[11] Y. Zheng and L. Longta, The Periodicity of the Rational Sequence $x_{n+1} = f(x_n)/x_{n-1}$, J. Changsha Univ. Elec. Power, 11(1996),113-117.

Received: October, 2009