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A Note on Families of Stable Matrices

Handan Akyar

Anadolu University, Science Faculty
Department of Mathematics
26470 Eskisehir, Turkey
hakyar@anadolu.edu.tr

Abstract

In this paper, some properties of families of stable matrices are studied. It is shown that various families of stable matrices are contractible, and as a consequence these families are simply-connected.

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1 Introduction

Many topological properties of the whole family of stable polynomials and matrices have been studied in a number of publications (see e.g., [1,3–7,11,12]). The present paper continues these investigations with the aim of strengthening these results in the sense that instead of simple-connectivity we show stronger property of contractibility of the various families of stable matrices from this point of view.

We now recall some topological notions needed below. A metric (or topological) space which can be reduced to one of its points by a continuous deformation is said to be contractible. In other words let X be a metric space, $Y \subset X$ and $x_0 \in Y$. Then Y is called contractible to the point $x_0 \in Y$, if there exists a continuous function $F : Y \times [0, 1] \to Y$ such that

$$F(x, 0) = x, \; F(x, 1) = x_0$$

for all $x \in Y$ ([8], [10]). If Y is contractible to a point $x_0 \in Y$, then it is contractible to any other point of Y.

The set Y is called path-wise connected if for all $a, b \in Y$ there exists a continuous function $f : [0, 1] \to Y$ such that $f(0) = a, \; f(1) = b$. Such a function is called a curve in Y connecting a and b. A curve is called closed if $f(0) = f(1)$.

Let Y be a path-wise connected subset in the metric space X. Then Y is called simply-connected if there exists $x_0 \in Y$ such that any closed curve $f : [0,1] \to Y$ with $f(0) = f(1) = x_0$ can be deformed to the constant curve at x_0 in the following sense: There exists a continuous function $F : [0,1] \times [0,1] \to Y$ such that $F(t,0) = f(t),\ F(t,1) = F(0,s) = F(1,s) = x_0$ for all $t, s \in [0,1]$. (This property holds then for any other point of Y also)

It follows from these definitions that if Y is contractible then Y is path-wise connected and simply-connected. The converse is not true in general. For example, ball's surface is path-wise connected and simply-connected but it is not contractible.

Suppose that $\mathcal{D}$ is a simply-connected open set in the complex plane $\mathbb{C}$. An $n \times n$ matrix A (n-th degree polynomial $p(s)$) is called $\mathcal{D}$-stable if all its eigenvalues (roots) lie in the region $\mathcal{D}$. If $\mathcal{D}$ is the open left half plane then a $\mathcal{D}$-stable matrix (polynomial) is called Hurwitz stable; if $\mathcal{D}$ is the open unit disc then a $\mathcal{D}$-stable matrix (polynomial) is called Schur stable.

An $n \times n$ real matrix A is called negative definite matrix if A is symmetric and $x^T Ax < 0$ for all nonzero vectors $x \in \mathbb{R}^n$, where x^T denotes the transpose. This is denoted $A < 0$.

An $n \times n$ matrix A is called Hurwitz diagonally stable if there exists a positive diagonal matrix P such that $A^T P + PA$ is negative definite. If there exists a positive diagonal matrix P such that $A^T PA - P$ is negative definite then A is called Schur diagonally stable.

If for all positive diagonal $n \times n$ matrices D the matrix AD is Hurwitz stable then A is said to be Hurwitz D-stable. The matrix A is said to be Schur D-stable if AD is Schur stable for all real diagonal matrices D whose elements are all less than or equal to unity in absolute value, i.e., $|D| \leq I$. Here, I denotes the identity matrix.

In [6] the contractibility of the set of Schur stable monic polynomials was established. This result extended to families of $\mathcal{D}$-stable polynomials and it was shown that families of $\mathcal{D}$-stable polynomials are not contractible, but they can be divided into two disjoint subsets each of which is contractible (see [1]). In [5] it was proved that set of all Hurwitz stable matrices of order n is the product of two convex open cones and itself forms a simply-connected open cone with a vertex at the origin. In [7] it was shown that the family of Schur D-stable matrices is open, unbounded, path-wise connected, nonconvex set in the matrix space.

In this paper we show that the families of complex Hurwitz and Schur stable matrices are contractible. Moreover if $\mathcal{D}$ is convex open set in $\mathbb{C}$ then the family of $n \times n$ complex $\mathcal{D}$-stable matrices is contractible. We show that the families of $n \times n$ Hurwitz diagonally stable, Schur diagonally stable and Schur

D-stable matrices are also contractible. On the other hand, the contractibility property of family of Hurwitz D-stable matrices is established for $n = 2, 3$. In the general case we could not prove this.

2 Contractibility of Families of Stable Matrices

In this section, it is shown that the family of $n \times n$ stable matrices is contractible. In addition, contractibility of the family of diagonally stable and D-stable matrices are also investigated.

Proposition 2.1.

1. *The family $\mathcal{S}$ of complex $n \times n$ Schur stable matrices is contractible to the zero matrix,*

2. *The family $\mathcal{H}$ of complex $n \times n$ Hurwitz stable matrices is contractible to the matrix $-I$.*

Proof. 1. The function

$$F : \mathcal{S} \times [0,1] \to \mathcal{S}, \; F(A,t) = (1-t)A$$

is continuous. If a matrix A is Schur stable, then for all $t \in [0,1]$, $(1-t)A$ is also Schur stable. By the definition of F

$$F(A,0) = A, \; F(A,1) = 0.$$

2. The function

$$F : \mathcal{H} \times [0,1] \to \mathcal{H}, \; F(A,t) = (1-t)A - tI$$

is continuous. For $t \in (0,1)$ denote $\alpha = \dfrac{t}{1-t}$.

If A is a Hurwitz stable matrix, then $A - \alpha I$ is also Hurwitz stable for all $\alpha > 0$. Additionally,

$$F(A,0) = A, \; F(A,1) = -I.$$

□

Proposition 2.2. *Let $\mathcal{D} \subset \mathbb{C}$ be a convex set and $s_* \in \mathcal{D}$. Then the family $\mathcal{M}$ of complex $n \times n$ $\mathcal{D}$-stable matrices is contractible to the matrix $s_* I$.*

Proof. The function

$$F : \mathcal{M} \times [0,1] \to \mathcal{M}, \; F(A,t) \;=\; (1-t)A + ts_* I$$

is continuous. If A is $\mathcal{D}$-stable then the matrix $(1-t)A+ts_*I$ is also $\mathcal{D}$-stable. Indeed, if λ is an eigenvalue of the matrix $(1-t)A+ts_*I$ ($t \in [0,1]$), then there exists a nonzero complex vector x such that

$$((1-t)A + ts_* I)x = \lambda x$$

or

$$Ax = \frac{(\lambda - ts_*)}{1-t} x$$

is satisfied. Since A is a $\mathcal{D}$-stable matrix, then $\dfrac{(\lambda - ts_*)}{1-t} \in \mathcal{D}$. Therefore, we get

$$\lambda \in (1-t)\mathcal{D} + ts_*, \; \lambda \;=\; (1-t)d + ts_*, \; d \in \mathcal{D}.$$

Since $\mathcal{D}$ is convex set, it follows that $\lambda \in \mathcal{D}$. Therefore, $(1-t)A + ts_*I$ is also a $\mathcal{D}$-stable matrix. In addition,

$$F(A,0) = A, \; F(A,1) = s_* I.$$

□

Proposition 2.2 implies the following corollary.

Corollary 2.3. *Let $\mathcal{D} \subset \mathbb{C}$ be a convex set, $\mathcal{D} \cap \mathbb{R} \neq \emptyset$ and assume that $s^* \in \mathcal{D} \cap \mathbb{R}$. Then the family of real $n \times n$ $\mathcal{D}$-stable matrices is contractible to the matrix $s^* I$.*

Proposition 2.4. *The family $\mathcal{D}_c$ of $n \times n$ Hurwitz diagonally stable matrices is contractible to the matrix $-I$.*

Proof. Obviously, the function

$$F : \mathcal{D}_c \times [0,1] \to \mathcal{D}_c, \; F(A,t) \;=\; (1-t)A - tI$$

is continuous. If $\dfrac{t}{1-t} = \alpha$ then $\alpha > 0$ for all $t \in (0,1)$. If A is a Hurwitz diagonally stable matrix, then $A - \alpha I$ is also Hurwitz diagonally stable. Indeed, if A is a Hurwitz diagonally stable matrix, then there exists a positive diagonal matrix P such that $A^T P + PA < 0$. Then

$$(A - \alpha I)^T P + P(A - \alpha I) = A^T P + PA - 2\alpha P < 0$$

is satisfied. As a result, $A - \alpha I$ is Hurwitz diagonally stable. Additionally,

$$F(A,0) = A, \; F(A,1) = -I.$$

□

Proposition 2.5. *If A is a Schur diagonally stable matrix, then αA is also Schur diagonally stable for all real α with $|\alpha| < 1$.*

Proof. Let A be a Schur diagonally stable matrix. Then there exists a positive diagonal matrix P such that $A^T PA - P < 0$ or equivalently for all nonzero vectors $x \in \mathbb{R}^n$ the inequality

$$x^T(A^T PA)x - x^T Px < 0 \tag{1}$$

holds.

If $\alpha = 0$ then $\alpha A = 0$ and the matrix 0 is Schur diagonally stable.

Let $\alpha \neq 0$. Set $P_* = \frac{1}{\alpha^2}P$. It is obvious that

$$\begin{aligned} \alpha A^T P_* \alpha A - P_* &= \alpha^2 A^T P_* A - P_* = \alpha^2 A^T \frac{1}{\alpha^2} PA - \frac{1}{\alpha^2} P \\ &= A^T PA - \frac{1}{\alpha^2} P \end{aligned}$$

holds. Since $\frac{1}{\alpha^2} > 1$ for all $|\alpha| < 1$, using (1) we obtain

$$x^T(A^T PA)x - x^T \frac{1}{\alpha^2} Px = x^T(A^T PA)x - \frac{1}{\alpha^2} x^T Px < 0$$

for all nonzero vectors x. Hence

$$\alpha A^T P_* \alpha A - P_* < 0$$

holds. Therefore, the matrix αA is Schur diagonally stable. □

Proposition 2.6. *The family $\mathcal{D}_d$ of $n \times n$ Schur diagonally stable matrices is contractible to the zero matrix.*

Proof. The function

$$F : \mathcal{D}_d \times [0, 1] \to \mathcal{D}_d, \ \ F(A, t) \ = \ (1 - t)A$$

is continuous. If A is Schur diagonally stable, then by Proposition 2.5 $(1-t)A$ is also Schur diagonally stable for all $t \in [0, 1]$. Additionally,

$$F(A, 0) = A, \ \ F(A, 1) = 0.$$

□

Proposition 2.7. *The family $\mathbb{D}_d$ of $n \times n$ Schur D-stable matrices is contractible to the zero matrix.*

Proof. The function

$$F : \mathbb{D}_d \times [0,1] \to \mathbb{D}_d,\ F(A,t) \ = \ (1-t)A$$

is continuous. If A is a Schur D-stable matrix, then AD is Schur stable for all $|D| \leq I$. Moreover, $(1-t)AD$ is also a Schur stable matrix for all $t \in [0,1]$. Thus $(1-t)A$ is Schur D-stable for all $t \in [0,1]$. Additionally,

$$F(A,0) = A,\ F(A,1) = 0.$$

□

Proposition 2.8. *The family $\mathbb{D}_c^2$ of real 2×2 Hurwitz D-stable matrices is contractible to the matrix $-I$.*

Proof. The function

$$F : \mathbb{D}_c^2 \times [0,1] \to \mathbb{D}_c^2,\ F(A,t) \ = \ (1-t)A - tI$$

is continuous. If we set $\dfrac{t}{1-t} = \alpha$, then $\alpha > 0$ for all $t \in (0,1)$. If the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is Hurwitz D-stable, then $A - \alpha I$ is also Hurwitz D-stable matrix. Indeed, if A is Hurwitz D-stable, then

$$AD = \begin{pmatrix} ad_1 & bd_2 \\ cd_1 & dd_2 \end{pmatrix}$$

is Hurwitz stable for all diagonal matrices

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix},\ d_1, d_2 > 0.$$

The characteristic polynomial of AD is

$$p_{AD}(s) = s^2 - (ad_1 + dd_2)s + d_1 d_2 (ad - bc).$$

By the Hurwitz stability criterion of second order polynomials [2] we obtain

$$ad_1 + dd_2 < 0 \text{ and } d_1 d_2 (ad - bc) > 0. \tag{2}$$

The characteristic polynomial $p_{(A-\alpha I)D}(s)$ of $(A - \alpha I)D$ is given by

$$\begin{aligned} p_{(A-\alpha I)D}(s) \ &= \ s^2 + (\alpha(d_1 + d_2) - (ad_1 + dd_2))\, s + \\ &\quad + \left(ad - bc + \alpha^2 - \alpha(a+d)\right) d_1 d_2. \end{aligned}$$

From inequality (2) we obtain that all coefficients of the polynomial $p_{(A-\alpha I)D}(s)$ are positive. Therefore, this polynomial is Hurwitz stable and $(A-\alpha I)D$ is a Hurwitz stable matrix. Hence, $A-\alpha I$ is a Hurwitz D-stable. Additionally,

$$F(A,0)=A,\ F(A,1)=-I.$$

□

Proposition 2.9. *The family $\mathbb{D}_c^3$ of real 3×3 Hurwitz D-stable matrices is contractible to the matrix $-I$.*

Proof. The function

$$F:\mathbb{D}_c^3\times[0,1]\to\mathbb{D}_c^3,\ F(A,t) \quad = \quad (1-t)A-tI$$

is continuous. If $\dfrac{t}{1-t}=\alpha$ then $\alpha>0$ for all $t\in(0,1)$. If A is Hurwitz D-stable then $A-\alpha I$ is also Hurwitz D-stable. It is known that a real 3×3 matrix A is Hurwitz D-stable if and only if $A-D$ is Hurwitz D-stable for all diagonal matrices D with nonnegative diagonal entries ([9], page 40). Taking $D=\alpha I$, we conclude that $A-\alpha I$ is also a Hurwitz D-stable matrix. Additionally,

$$F(A,0)=A,\ F(A,1)=-I.$$

□

It is not known whether a family of real $n\times n$ Hurwitz D-stable matrices is contractible or not in the case of $n>3$.

Corollary 2.10. *The followings hold.*

1. *If $\mathcal{D}$ is a convex open set in the complex plane then the whole family of $n\times n$ complex $\mathcal{D}$-stable matrices is simply-connected.*

2. *If $\mathcal{D}$ is a convex open set in the complex plane and $\mathcal{D}\cap\mathbb{R}\neq\emptyset$ then the whole family of $n\times n$ real, $\mathcal{D}$-stable matrices is simply-connected.*

3. *The family of real Hurwitz (Schur) diagonally stable matrices is simply-connected.*

4. *The family of real Schur D-stable matrices is simply-connected.*

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