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Some Properties in Generalized n-Inner Product Spaces

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Abstract

The main aim of this paper is to prove parallelogram law, Polarization identity in generalized n-inner product spaces over the field $K = R$ of real numbers or the field $K = C$ of complex numbers.

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1 Introduction

The concept of n-inner product and n-inner product spaces has been investigated by A.Misiak [1, 2] and has been developed extensively in different subjects by others, for example [3, 4, 5]. A.Misiak [1] has introduced an n-normed space by the following definition.

Definition 1.1. *Let $n \in N$ and let X be a linear space of dimension $\geq n$ over the field $K = R$ of real numbers or the field $K = C$ of complex numbers and $\|\bullet, \bullet, \dots, \bullet\|$ is a real valued function on $\underbrace{X \times X \times \cdots \times X}_{n} = X^n$ such that*

$nN_1 : \|x_1, x_2, \dots, x_n\| = 0$ if and only if $x_1, x_2, \dots, x_n$ are linearly dependent vectors,

$nN_2 : \|x_1, x_2, \dots, x_n\|$ is invariant under any permutation of $x_1, x_2, \dots, x_n$,

$nN_3 : \|x_1, x_2, \dots, x_{n-1}, \alpha x_n\| = |\alpha|\ \|x_1, x_2, \dots, x_n\|$ for all $\alpha \in K$,

$nN_4 : \|x_1, x_2, \dots, x_{n-1}, y + z\| \leq \|x_1, x_2, \dots, x_{n-1}, y\| + \|x_1, x_2, \dots, x_{n-1}, z\|$ for all $x_1, x_2, \dots, x_{n-1}, y, z \in X$,

then the function $\|\bullet, \bullet, \dots, \bullet\|$ is called an n-norm on X and the pair $(X, \|\bullet, \bullet, \dots, \bullet\|)$ is called linear n-normed space.

A.Misiak [1] has introduced an n-inner product space by the following definition.

Definition 1.2. *Let $n \in N$ and X be a linear space of dimension $\geq n$ over the field $K = R$ of real numbers or the field $K = C$ of complex numbers and $(\bullet, \bullet/\bullet, \bullet, \ldots, \bullet)$ is a K-valued function defined on $\underbrace{X \times X \times \cdots \times X}_{n+1} = X^{n+1}$ $(n \geq 2)$ such that*

$nI_1 : (x_1, x_1/x_2, x_3, \ldots, x_n) \geq 0$ for any $x_1, x_2, \ldots, x_n \in X$ and $(x_1, x_1/x_2, x_3, \ldots, x_n) = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent vectors,

$nI_2 : (a, b/x_1, x_2, \ldots, x_{n-1}) = (\varphi(a), \varphi(b)/\pi(x_1), \pi(x_2), \ldots, \pi(x_{n-1}))$ for any $a, b, x_1, x_2, \ldots, x_{n-1} \in X$ and for any bijection $\pi : \{x_1, x_2, \ldots, x_{n-1}\} \longrightarrow \{x_1, x_2, \ldots, x_{n-1}\}$ and $\varphi : \{a, b\} \longrightarrow \{a, b\}$,

nI_3 : If $n > 1$ then $(x_1, x_1/x_2, x_3, \ldots, x_n) = \overline{(x_2, x_2/x_1, x_3, \ldots, x_n)}$ for any $x_1, x_2, x_3, \ldots, x_n \in X$,

$nI_4 : (\alpha a, b/x_1, x_2, \ldots, x_{n-1}) = \alpha(a, b/x_1, x_2, \ldots, x_{n-1})$ for any $a, b, x_1, x_2, \ldots, x_{n-1} \in X$ and for any scalar $\alpha \in K$,

$nI_5 : (a+a_1, b/x_1, x_2, \ldots, x_{n-1}) = (a, b/x_1, x_2, \ldots, x_{n-1})+(a_1, b/x_1, x_2, \ldots, x_{n-1})$ for any $a, a_1, b, x_1, x_2, \ldots, x_{n-1} \in X$,

then $(\bullet, \bullet/\bullet, \bullet, \ldots, \bullet)$ is called the n-inner product and $(X, (\bullet, \bullet/\bullet, \bullet, \ldots, \bullet))$ is called the n-inner product space.

This n-inner product induces an n-norm by $\|x_1, x_2, \ldots, x_n\| = \sqrt{(x_1, x_1/x_2, \ldots, x_{n-1})}$. If $n = 1$ then the definition 1.1 reduces to the ordinary inner product.

2 Preliminaries

K.Trenceveski and R.Malceski [5] has introduced the following definition of generalized n-inner product space.

Definition 2.1. *Let $n \in N$ and X be a linear space of dimension $\geq n$ over the field $K = R$ of real numbers or the field $K = C$ of complex numbers and $\langle \bullet, \bullet, \ldots, \bullet/\bullet, \bullet, \ldots, \bullet \rangle$ is a K-valued function defined on $\underbrace{X \times X \times \cdots \times X}_{2n} = X^{2n}$ such that*

$gnI_1 : \langle a_1, a_2, \ldots, a_n/a_1, a_2, \ldots, a_n \rangle > 0$ if $a_1, a_2, \ldots, a_n$ are linearly independent vectors,

$gnI_2 : \langle a_1, a_2, \ldots, a_n/b_1, b_2, \ldots, b_n \rangle = \overline{\langle b_1, b_2, \ldots, b_n/a_1, a_2, \ldots, a_n \rangle}$ for any $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in X$,

$gnI_3 : \langle \lambda a_1, a_2, \ldots, a_n/b_1, b_2, \ldots, b_n \rangle = \lambda \langle a_1, a_2, \ldots, a_n/b_1, b_2, \ldots, b_n \rangle$ for any scalar $\lambda \in K$ and any $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in X$,

gnI_4 : $\langle a_1, a_2, \ldots, a_n/b_1, b_2, \ldots, b_n\rangle = -\langle a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)}/b_1, b_2, \ldots, b_n\rangle$ for any odd permutation σ in the set $\{1, 2, \ldots, n\}$ and any $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in X$,

gnI_5 : $\langle a_1 + c, a_2, \ldots, a_n/b_1, b_2, \ldots, b_n\rangle = \langle a_1, a_2, \ldots, a_n/b_1, b_2, \ldots, b_n\rangle + \langle c, a_2, \ldots, a_n/b_1, b_2, \ldots, b_n\rangle$ for any $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n, c \in X$,

gnI_6 : If $\langle a_1, b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n/b_1, b_2, \ldots, b_n\rangle = 0$ for each $i \in \{1, 2, \ldots, n\}$ then $\langle a_1, a_2, \ldots, a_n/b_1, b_2, \ldots, b_n\rangle = 0$ for arbitrary vectors $a_1, a_2, \ldots, a_n$, then the function $\langle \bullet, \bullet, \ldots, \bullet/\bullet, \bullet, \ldots, \bullet\rangle$ is called the generalized n-inner product and the pair $(X, \langle \bullet, \bullet, \ldots, \bullet/\bullet, \bullet, \ldots, \bullet\rangle)$ is called the generalized n-inner product space (or generalized n-pre Hilbert space).

In the special case, if we consider only such pair of sets $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ which differ for at most one vector, for example $a_1 = a$, $b_1 = b$ and $a_2 = b_2 = x_1$, $a_3 = b_3 = x_2, \ldots,$ $a_n = b_n = x_{n-1}$ then by putting $(a, b/x_1, x_2, \ldots, x_{n-1}) = \langle a, x_1, x_2, \ldots, x_{n-1}/b, x_1, x_2, \ldots, x_{n-1}\rangle$ we obtain an n-inner product.

Example 2.2. *Let X be a linear space with inner product $\langle \bullet/\bullet\rangle$ then*

$$\langle a_1, a_2, \ldots, a_n/b_1, b_2, \ldots, b_n\rangle = \begin{vmatrix} \langle a_1, b_1\rangle & \langle a_1, b_2\rangle & \ldots & \langle a_1, b_n\rangle \\ \langle a_2, b_1\rangle & \langle a_2, b_2\rangle & \ldots & \langle a_2, b_n\rangle \\ \vdots & \ddots & \ddots & \vdots \\ \langle a_n, b_1\rangle & \langle a_n, b_2\rangle & \ldots & \langle a_n, b_n\rangle \end{vmatrix}$$

defined the generalized n-inner product on X.

K.Trenceveski and R.Malceski [5] proved Cauchy-Schwarz inequality in generalized n-inner product on X as

$$\langle a_1, a_2, \ldots, a_n/b_1, b_2, \ldots, b_n\rangle^2 \leq \langle a_1, a_2, \ldots, a_n/a_1, a_2, \ldots, a_n\rangle\langle b_1, b_2, \ldots, b_n/b_1, b_2, \ldots, b_n\rangle$$

$$\text{i.e .,} \quad |\langle a_1, a_2, \ldots, a_n/b_1, b_2, \ldots, b_n\rangle| \leq \|a_1, a_2, \ldots, a_n\| \, \|b_1, b_2, \ldots, b_n\|$$

The generalized n-inner product on X induces an n-norm by

$$\|x_1, x_2, \ldots, x_n\| = \sqrt{\langle x_1, x_2, \ldots, x_n/x_1, x_2, \ldots, x_n\rangle}$$

and it is the same n-norm induced by the definition 1.2. Let $(X, \langle \bullet, \bullet, \ldots, \bullet/\bullet, \bullet, \ldots, \bullet\rangle)$ be the generalized n-inner product space and $\|\bullet, \bullet, \ldots, \bullet\|$ be the induced n-norm. We observe that if $x_k \longrightarrow x$ then $\|x_k, z_2, \ldots, z_n\| \longrightarrow \|x, z_2, \ldots, z_n\|$ for every $z_2, \ldots, z_n \in X$. This result tells us that n-norm $\|\bullet, \bullet, \ldots, \bullet\|$ is continuous in first variable and by property nN_2 of n-norms $\|\bullet, \bullet, \ldots, \bullet\|$ is continuous in each variable.

3 Continuity of generalized n-inner product

If $x_{1k} \longrightarrow y_1,\ x_{2k} \longrightarrow y_2,\ \ldots\ ,\ x_{nk} \longrightarrow y_n$ then by property gnI_5 and Cauchy-Schwarz inequality for the generalized n-inner product, we have

$$
\begin{aligned}
&|\langle x_{1k}, x_{2k}, \ldots, x_{nk}/z_1, z_2, \ldots, z_n\rangle - \langle y_1, y_2, \ldots, y_n/z_1, z_2, \ldots, z_n\rangle| \\
&= |\langle x_{1k}, x_{2k}, \ldots, x_{nk}/z_1, z_2, \ldots, z_n\rangle - \langle y_1, x_{2k}, \ldots, x_{nk}/z_1, z_2, \ldots, z_n\rangle \\
&\quad + \langle y_1, x_{2k}, \ldots, x_{nk}/z_1, z_2, \ldots, z_n\rangle - \langle y_1, y_2, x_{3k}, \ldots, x_{nk}/z_1, z_2, \ldots, z_n\rangle \\
&\quad + \langle y_1, y_2, x_{3k}, \ldots, x_{nk}/z_1, z_2, \ldots, z_n\rangle - \ldots - \langle y_1, y_2, y_3, \ldots, y_n/z_1, z_2, \ldots, z_n\rangle \\
&\quad + \langle y_1, y_2, y_3, \ldots, y_n/z_1, z_2, \ldots, z_n\rangle - \langle y_1, y_2, y_3, \ldots, y_n/z_1, z_2, \ldots, z_n\rangle| \\
&= |\langle x_{1k}-y_1, x_{2k}, \ldots, x_{nk}/z_1, z_2, \ldots, z_n\rangle + \langle y_1, x_{2k}-y_2, x_{3k}, \ldots, x_{nk}/z_1, z_2, \ldots, z_n\rangle \\
&\qquad + \cdots + \langle y_1, y_2, \ldots, x_{nk} - y_n/z_1, z_2, \ldots, z_n\rangle| \\
&\leq |\langle x_{1k} - y_1, x_{2k}, \ldots, x_{nk}/z_1, z_2, \ldots, z_n\rangle| + |\langle y_1, x_{2k} - y_2, x_{3k}, \ldots, x_{nk}/z_1, z_2, \ldots, z_n\rangle| \\
&\qquad + \ldots + |\langle y_1, y_2, \ldots, x_{nk} - y_n/z_1, z_2, \ldots, z_n\rangle| \\
&\leq \|x_{1k} - y_1, x_{2k}, \ldots, x_{nk}\|\,\|z_1, z_2, \ldots, z_n\| + \|y_1, x_{2k} - y_2, x_{3k}, \ldots, x_{nk}\|\,\|z_1, z_2, \ldots, z_n\| \\
&\qquad + \ldots + \|y_1, y_2, \ldots, x_{nk} - y_n\|\,\|z_1, z_2, \ldots, z_n\| \longrightarrow 0 \text{ as } k \longrightarrow \infty
\end{aligned}
$$

Since we know that n-norms $\|\bullet, \bullet, \ldots, \bullet\|$ is continuous in each variable. Hence

$$\langle x_{1k}, x_{2k}, \ldots, x_{nk}/z_1, z_2, \ldots, z_n\rangle \longrightarrow \langle y_1, y_2, \ldots, y_n/z_1, z_2, \ldots, z_n\rangle.$$

This shows that $\langle \bullet, \bullet, \ldots, \bullet/\bullet, \bullet, \ldots, \bullet\rangle$ is continuous in first n-variable and hence by the property gnI_2 and gnI_4 we get $\langle \bullet, \bullet, \ldots, \bullet/\bullet, \bullet, \ldots, \bullet\rangle$ is continuous in each variable.

Now, we give the parallelogram law and Polarization identity in generalized n-inner product space.

Proposition 3.1. *Parallelogram law in generalized n-inner product space*

$$\|x+y, x_2, \ldots, x_n\|^2 + \|x-y, x_2, \ldots, x_n\|^2 = 2\|x, x_2, \ldots, x_n\|^2 + 2\|y, x_2, \ldots, x_n\|^2$$

Proof. $\|x + y, x_2, \ldots, x_n\|^2 + \|x - y, x_2, \ldots, x_n\|^2 =$

$$
\begin{aligned}
&\langle x + y, x_2, \ldots, x_n/x + y, x_2, \ldots, x_n\rangle + \langle x - y, x_2, \ldots, x_n/x - y, x_2, \ldots, x_n\rangle \\
&= \langle x, x_2, \ldots, x_n/x + y, x_2, \ldots, x_n\rangle + \langle y, x_2, \ldots, x_n/x + y, x_2, \ldots, x_n\rangle \\
&\quad + \langle x, x_2, \ldots, x_n/x - y, x_2, \ldots, x_n\rangle - \langle y, x_2, \ldots, x_n/x - y, x_2, \ldots, x_n\rangle \\
&= \overline{\langle x + y, x_2, \ldots, x_n/x, x_2, \ldots, x_n\rangle} + \overline{\langle x + y, x_2, \ldots, x_n/y, x_2, \ldots, x_n\rangle} \\
&\quad + \overline{\langle x - y, x_2, \ldots, x_n/x, x_2, \ldots, x_n\rangle} - \overline{\langle x - y, x_2, \ldots, x_n/y, x_2, \ldots, x_n\rangle}
\end{aligned}
$$

$$
\begin{aligned}
&= \overline{\langle x, x_2, \ldots, x_n / x, x_2, \ldots, x_n\rangle} + \overline{\langle y, x_2, \ldots, x_n / x, x_2, \ldots, x_n\rangle} \\
&\quad + \overline{\langle x, x_2, \ldots, x_n / y, x_2, \ldots, x_n\rangle} + \overline{\langle y, x_2, \ldots, x_n / y, x_2, \ldots, x_n\rangle} \\
&\quad + \overline{\langle x, x_2, \ldots, x_n / x, x_2, \ldots, x_n\rangle} - \overline{\langle y, x_2, \ldots, x_n / x, x_2, \ldots, x_n\rangle} \\
&\quad - \overline{\langle x, x_2, \ldots, x_n / y, x_2, \ldots, x_n\rangle} + \overline{\langle y, x_2, \ldots, x_n / y, x_2, \ldots, x_n\rangle} \\
&= \langle x, x_2, \ldots, x_n / x, x_2, \ldots, x_n\rangle + \langle x, x_2, \ldots, x_n / y, x_2, \ldots, x_n\rangle \\
&\quad + \langle y, x_2, \ldots, x_n / x, x_2, \ldots, x_n\rangle + \langle y, x_2, \ldots, x_n / y, x_2, \ldots, x_n\rangle \\
&\quad + \langle x, x_2, \ldots, x_n / x, x_2, \ldots, x_n\rangle - \langle x, x_2, \ldots, x_n / y, x_2, \ldots, x_n\rangle \\
&\quad - \langle y, x_2, \ldots, x_n / x, x_2, \ldots, x_n\rangle + \langle y, x_2, \ldots, x_n / y, x_2, \ldots, x_n\rangle \\
&= 2\langle x, x_2, \ldots, x_n / x, x_2, \ldots, x_n\rangle + 2\langle y, x_2, \ldots, x_n / y, x_2, \ldots, x_n\rangle \\
&= 2\|x, x_2, \ldots, x_n\|^2 + 2\|y, x_2, \ldots, x_n\|^2
\end{aligned}
$$

□

Proposition 3.2. *Polarization Identity in the generalized n-inner product space*

$$
\begin{aligned}
4\langle x, x_2, \ldots, x_n / y, x_2, \ldots, x_n\rangle &= \|x+y, x_2, \ldots, x_n\|^2 - \|x-y, x_2, \ldots, x_n\|^2 \\
&\quad + i\ \|x+iy, x_2, \ldots, x_n\|^2 - i\ \|x-iy, x_2, \ldots, x_n\|^2 \\
&= \sum_{k=1}^{4} i^4\ \|x + i^k y, x_2, x_3, \ldots, x_n\| \qquad (\textit{where } i^2 = -1)
\end{aligned}
$$

Proof. $\|x+y, x_2, \ldots, x_n\|^2 = \langle x+y, x_2, \ldots, x_n / x+y, x_2, \ldots, x_n\rangle$

$$
\begin{aligned}
&= \langle x, x_2, \ldots, x_n / x+y, x_2, \ldots, x_n\rangle + \langle y, x_2, \ldots, x_n / x+y, x_2, \ldots, x_n\rangle \\
&= \overline{\langle x+y, x_2, \ldots, x_n / x, x_2, \ldots, x_n\rangle} + \overline{\langle x+y, x_2, \ldots, x_n / y, x_2, \ldots, x_n\rangle} \\
&= \overline{\langle x, x_2, \ldots, x_n / x, x_2, \ldots, x_n\rangle} + \overline{\langle y, x_2, \ldots, x_n / x, x_2, \ldots, x_n\rangle} \\
&\quad + \overline{\langle x, x_2, \ldots, x_n / y, x_2, \ldots, x_n\rangle} + \overline{\langle y, x_2, \ldots, x_n / y, x_2, \ldots, x_n\rangle}
\end{aligned}
$$

$$
\begin{aligned}
\|x+y, x_2, \ldots, x_n\|^2 &= \langle x, x_2, \ldots, x_n / x, x_2, \ldots, x_n\rangle + \langle x, x_2, \ldots, x_n / y, x_2, \ldots, x_n\rangle \\
&\quad + \langle y, x_2, \ldots, x_n / x, x_2, \ldots, x_n\rangle + \langle y, x_2, \ldots, x_n / y, x_2, \ldots, x_n\rangle \qquad (3.1)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\|x-y, x_2, \ldots, x_n\|^2 &= \langle x, x_2, \ldots, x_n / x, x_2, \ldots, x_n\rangle - \langle x, x_2, \ldots, x_n / y, x_2, \ldots, x_n\rangle \\
&\quad - \langle y, x_2, \ldots, x_n / x, x_2, \ldots, x_n\rangle + \langle y, x_2, \ldots, x_n / y, x_2, \ldots, x_n\rangle \qquad (3.2)
\end{aligned}
$$

On using (3.1) and (3.2), we get

$$
\|x+y, x_2, \ldots, x_n\|^2 - \|x-y, x_2, \ldots, x_n\|^2 = 2\langle x, x_2, \ldots, x_n / y, x_2, \ldots, x_n\rangle
$$

$$+\ 2\langle y, x_2, \ldots, x_n / x, x_2, \ldots, x_n\rangle \qquad (3.3)$$

Put $y = iy$ in equation (3.3) on both sides, we get

$$\begin{aligned}
&\|x+iy, x_2, \ldots, x_n\|^2 - \|x-iy, x_2, \ldots, x_n\|^2 \\
&= 2\ \langle x, x_2, \ldots, x_n / iy, x_2, \ldots, x_n\rangle\ +\ 2\ \langle iy, x_2, \ldots, x_n / x, x_2, \ldots, x_n\rangle \\
&= 2\ \overline{\langle iy, x_2, \ldots, x_n / x, x_2, \ldots, x_n\rangle}\ +\ 2i\ \langle y, x_2, \ldots, x_n / x, x_2, \ldots, x_n\rangle \\
&= 2\ \bar{i}\ \overline{\langle y, x_2, \ldots, x_n / x, x_2, \ldots, x_n\rangle}\ +\ 2i\ \langle y, x_2, \ldots, x_n / x, x_2, \ldots, x_n\rangle \\
&= -2i\ \langle x, x_2, \ldots, x_n / y, x_2, \ldots, x_n\rangle\ +2i\ \langle y, x_2, \ldots, x_n / x, x_2, \ldots, x_n\rangle
\end{aligned}$$

multiplying the above equation by i on both sides, we get

$$\begin{aligned}
&i\ \|x+iy, x_2, \ldots, x_n\|^2 - i\ \|x-iy, x_2, \ldots, x_n\|^2 \\
&= -\ 2i^2\ \langle x, x_2, \ldots, x_n / y, x_2, \ldots, x_n\rangle\ +\ 2i^2\ \langle y, x_2, \ldots, x_n / x, x_2, \ldots, x_n\rangle \\
&= 2\langle x, x_2, \ldots, x_n / y, x_2, \ldots, x_n\rangle\ -\ 2\langle y, x_2, \ldots, x_n / x, x_2, \ldots, x_n\rangle \qquad (3.4)
\end{aligned}$$

adding (3.3) and (3.4) we get

$$\begin{aligned}
&\|x+y, x_2, \ldots, x_n\|^2 - \|x-y, x_2, \ldots, x_n\|^2\ +\ i\ \|x+iy, x_2, \ldots, x_n\|^2 \\
&-\ i\ \|x-iy, x_2, \ldots, x_n\|^2\ =\ 4\langle x, x_2, \ldots, x_n / y, x_2, \ldots, x_n\rangle \\
&\Longrightarrow \quad 4\ \langle x, x_2, \ldots, x_n / y, x_2, \ldots, x_n\rangle\ =\ \sum_{k=1}^{4} i^4 \|x + i^k y, x_2, x_3, \ldots, x_n\|
\end{aligned}$$

Hence proved. □

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