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# Generating Functions for Laguerre Type Polynomials of Two Variables $L_n^{\alpha-n}(x,y)$ by Using Group Theoretic method

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**Abstract**

An attempt is made to obtain generating relations of modified Laguerre polynomials. in two-variables by means of Group theoretic method. Laguerre polynomials have special importance in engineering, sciences and constitute good model for many systems in various fields.

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**Keywords:** Two-variable Laguerre polynomials, Recurrence relations, Group Theoretic Method and generating relation

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## 1. INTRODUCTION AND PRELIMINARIES:

The Group theoretical analysis provides us an effective tool for finding generating function of special functions. The above idea was originally generated by Weisner [9] and he [10 and 11] also applied this technique to obtain the generating relation. Miller, McBride, Srivastava and Manocha [4, 3 and 6] respectively reported Group theoretic method for obtaining generating relations in their books. S. Khan, M.A. Pathan and G. Yasmin [2] studied representation of a Lie algebra $G(0,1)$ and three variable generalised Hermite polynomials, $H_n(x,y,z)$. S. Khan and G. Yasmin [2] have also done some works on Laguerre polynomial of two variables by using Group theoretic method. S. Khan and M.A. Pathan [3] studied a paper on Lie-theoretic generating relations of Hermite 2D polynomials. Again they studied Lie theory and two variable generalised Hermite polynomials

Group theoretic method basically gives a connection between special function and the matrix groups. In present paper our aim is to study first order linear differential operator which generates Lie algebra isomorphic to some matrix Lie algebra and apply these operators to determine a local representation $[T(g)f]$ which makes a one to one correspondence between these two Lie algebras., Afterwards by choosing the suitable values of f(x, y) the above representation leads us to generating functions.

Consider the abstract group $G(0,1)$ consists of all $4\times 4$ matrices of the form

$$g=\begin{pmatrix} 1 & ce^t & a & t \\ 0 & e^t & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.1}$$

where the group operation is matrix multiplication.

Now we can introduced coordinates for the elements g in $G(0,1)$ by setting

$$g\equiv(a,b,c,t). \tag{1.2}$$

Thus $G(0,1)$ is a complex 4-dimensional Lie group. Here the coordinates (1.2) are valid over the entire group and the group $G(0,1)$ is simply connected.

The Corresponding Lie algebra of the Lie group $G(0,1)$ is $L[G(0,1)]=G_4$ the space of all $4\times 4$ matrices of the form

$$\alpha=\begin{pmatrix} 0 & x_2 & x_4 & x_3 \\ 0 & x_3 & x_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad x_1,x_2,x_3,x_4\in \mathbb{C} \tag{1.3}$$

with the Lie product $[\alpha,\beta]=\alpha\beta-\beta\alpha$ for $\alpha,\beta\in L[G(0,1)]$.

This is of dimension 4 with the basis $B=\{\mathcal{J}^+,\mathcal{J}^-,\mathcal{J}^3,\varepsilon\}$.

where the tangent matrices corresponding to basis have respectively the forms

$$\mathcal{Z}^{+}=\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad \mathcal{Z}^{-}=\begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$\mathcal{Z}^{3}=\begin{pmatrix}0&0&0&1\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\quad \text{and}\quad \varepsilon=\begin{pmatrix}0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\tag{1.4}$$

They satisfy the commutation relations

$$\begin{gathered}\left[\mathcal{Z}^{3},\mathcal{Z}^{+}\right]=\mathcal{Z}^{+},\left[\mathcal{Z}^{3},\mathcal{Z}^{-}\right]=-\mathcal{Z}^{-},\left[\mathcal{Z}^{+},\mathcal{Z}^{-}\right]=-\varepsilon\\ \left[\varepsilon,\mathcal{Z}^{+}\right]=\left[\varepsilon,\mathcal{Z}^{-}\right]=\left[\varepsilon,\mathcal{Z}^{3}\right]=0\end{gathered}\tag{1.5}$$

where 0 is the $4\times 4$ zero matrix, form a basis for $L\left[G(0,1)\right]$.

The exponential map $\exp\alpha,\alpha\in L\left[G(0,1)\right]$, is of the form

$$\exp\alpha=e^{\alpha}=\sum_{k=0}^{\infty}\frac{\alpha^{k}}{k!}$$

and it is an analytic diffeomorphism mapping all of $L\left[G(0,1)\right]$ onto $G(0,1)$.

In particular

$$\exp(a\mathcal{Z}^{+})=\begin{pmatrix}1&0&0&0\\0&1&a&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\qquad \exp(b\mathcal{Z}^{-})=\begin{pmatrix}1&b&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},$$

$$\exp(c\mathcal{Z}^{3})=\begin{pmatrix}1&0&0&c\\0&e^{c}&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\quad\text{and}\quad \exp(d\varepsilon)=\begin{pmatrix}1&0&d&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\tag{1.6}$$

Let the Laguerre type polynomial of two variables $L_n^{\alpha-n}(x,y)$ is defined as:

$$L_n^{\alpha-n}(x,y)=\sum_{r=0}^{n}\frac{y^{n-r}\Gamma(\alpha+1)(-x)^{r}}{(n-r)!r!\Gamma(\alpha+r-n+1)},\ \text{where}\ \Gamma(z)=\int_0^{\infty}t^{z-1}e^{-t}dt,\operatorname{Re}(z)>0\tag{1.7}$$

and its generating function is given by as

$$\sum_{n=0}^{\infty} L_n^{\alpha-n}(x,y)t^n = \frac{1}{(1-yt)^{\alpha-n+1}}\exp\left(-\frac{xt}{1-yt}\right) \tag{1.8}$$

where $\alpha$ is a non negative integer.

The polynomial $L_n^{\alpha-n}(x,y)$ satisfy the differential equation

$$\left[x\frac{d^2}{dx^2}+\left(\alpha+1-n-\frac{x}{y}\right)\frac{d}{dx}+\frac{n}{y}\right]L_n^{\alpha-n}(x,y)=0 \tag{1.9}$$

## 2. GROUP THEORETIC METHOD:

Now replacing $\frac{d}{dx}$ by $\frac{\partial}{\partial x}$ and n by $z\frac{\partial}{\partial z}$ in equation (1.9) we construct a partial differential equation

$$\left[x\frac{\partial^2}{\partial x^2}-z\frac{\partial^2}{\partial x\partial z}+\left(\alpha+1-\frac{x}{y}\right)\frac{\partial}{\partial x}+\frac{z}{y}\frac{\partial}{\partial z}\right]f(x,y,z)=0 \tag{2.1}$$

Thus by [observation 1, pp.327] of H. M. Srivastava and H. L. Manocha [6], $f(x,y,z)=L_n^{\alpha-n}(x,y)z^n$ is a solution of equation (3.1) since $L_n^{\alpha-n}(x,y)$ is a solution of equation (2.1).

Consider the first order linear differential operators $J^+, J^-, J^3$ and E defined by

$$J^3 = z\frac{\partial}{\partial z}$$

$$J^+ = -xz\frac{\partial}{\partial x}+z^2\frac{\partial}{\partial z}+\frac{xz}{y}-\alpha z \tag{2.2}$$

$$J^- = \frac{y}{z}\frac{\partial}{\partial x}$$

and $E=1$

These operators satisfy the commutation relations

$$\left[J^3,J^+\right]=J^+,\ \left[J^3,J^-\right]=-J^- \text{ and} \left[J^+,J^-\right]=-E. \tag{2.3}$$

$$\left[E,J^+\right]=\left[E,J^-\right]=\left[E,J^3\right]=0$$

This is identical with the commutation relation (1.5) for the generators of $G(0,1)$. Thus by **Theorem 1.10** of Miller W. Jr. [4], the operators $J^+, J^-, J^3$ and E generates the Lie algebra of generalized Lie derivatives.

Again these operators also satisfy the following properties

$$J^+\left[z^nL_n^{\alpha-n}(x,y)\right]=(n+1)L_{n+1}^{\alpha-n}(x,y)z^{n+1}$$

$$J^-\left[z^nL_n^{\alpha-n}(x,y)\right]=-\alpha L_{n-1}^{\alpha-n}(x,y)z^{n-1} \tag{2.4}$$

$$J^3\left[z^n L_n^{\alpha-n}(x,y)\right]=n\,L_n^{\alpha-n}(x,y)\;z^n$$

$$E\left[z^n L_n^{\alpha-n}(x,y)\right]=L_n^{\alpha-n}(x,y)\;z^n$$

In terms of the *J* operators, we introduced the Casimir operator [p.32] of Miller W. Jr. [4],

$$C \;=\; J^+J^- - E\,J^3$$

$$=\;-y\left\{x\frac{\partial^2}{\partial x^2}-z\frac{\partial^2}{\partial x\partial z}+\left(\alpha+1-\frac{x}{y}\right)\frac{\partial}{\partial x}+\frac{z}{y}\frac{\partial}{\partial z}\right\}$$

Hence,

$$C \;=\; -y\left\{x\frac{\partial^2}{\partial x^2}-z\frac{\partial^2}{\partial x\partial z}+\left(\alpha+1-\frac{x}{y}\right)\frac{\partial}{\partial x}+\frac{z}{y}\frac{\partial}{\partial z}\right\} \tag{2.5}$$

We can easily verified from the commutation relation (2.3), that *C* commutes with $J^+, J^-, J^3$ and E.

$$\left[C,J^3\right]=\left[C,J^+\right]=\left[C,J^-\right]=\left[C,E\right]=0 \tag{2.6}$$

by using equation (2.5) ,we can write equation (2.1) as

$$C\;f(x,y;z)=\left[-y\left\{x\frac{\partial^2}{\partial x^2}-z\frac{\partial^2}{\partial x\partial z}+\left(\alpha+1-\frac{x}{y}\right)\frac{\partial}{\partial x}+\frac{z}{y}\frac{\partial}{\partial z}\right\}\right]f(x,y;z)$$

$$=0 \tag{2.7}$$

Now on proceeding to compute the multiplier representation $[T\;(g)\,f\,](x,y,z),\; g\in G(0,1)$ induced by the *J*- operators.

In order to determine the multiplier representation T of $G(0,1)$, we first compute the action of $\exp\left(a\,J^+\right)f$ , $\exp\left(b\,J^-\right)f$ , $\exp\left(c\,J^3\right)f$ and $\exp\left(d\,E\right)f$.

To obtain $\exp\left(a\,J^+\right)f$ by using [Theorem -7] of H. M. Srivastava and H. L. Manocha [6], we need following facts

$$\frac{d}{da}x(a)=-x(a)\,z(a) \tag{2.8}$$

$$\frac{d}{da}y(a)=0 \tag{2.9}$$

$$\frac{d}{da}z(a)=\left\{z(a)\right\}^2 \tag{2.10}$$

$$\frac{d}{da}v(a)=v(a)\left\{\frac{x(a)}{y(a)}-\alpha\right\}z(a) \tag{2.11}$$

and satisfying the condition $x(0)=x$ , $y(0)=y$, $z(0)=z$ and $v(0)=1$.

as, $\left[\exp\left(a\,J^+\right)f\,\right](x,y;z)=v(t)\;f\left(x(a),y(a);z(a)\right)$,

where $x(a)$, $y(a)$ and $z(a)$ are the solutions of the aforesaid equation.
On solving the above differential equation and by using initial conditions, we get

$$\left[T\left(\exp\left(a\mathbf{z}^{+}\right)\right)f\right](x,y;z) = (1-az)^{\alpha}\exp\left(\frac{axz}{y}\right)\times f\left(x(1-az),y;\frac{z}{1-az}\right) \quad (2.12)$$

where $|az|<1$.
Similarly

$$\left[T\left(\exp\left(b\mathbf{z}^{-}\right)\right)f\right](x,y;z) = f\left(x+\frac{by}{z},y;z\right) \quad (2.13)$$

$$\left[T\left(\exp\left(c\mathbf{z}^{3}\right)\right)f\right](x,y;z) = f\left(x,y;z\exp(c)\right) \quad (2.14)$$

and $\left[T\left(\exp\left(d\varepsilon\right)\right)f\right](x,y;z) = \exp(d)\,f(x,y;z)$ (2.15)

These are all defined for $|a|,|b|,|c|\, and\, |d|$ sufficiently small.
Now,

$$\left[T\left(\exp\left(a\mathbf{z}^{+}\right)\exp\left(b\mathbf{z}^{-}\right)\exp\left(c\mathbf{z}^{3}\right)\exp\left(d\varepsilon\right)\right)f\right](x,y;z)$$

$$=\left[T\left(\exp\left(a\mathbf{z}^{+}\right)\right)T\left(\exp\left(b\mathbf{z}^{-}\right)\right)T\left(\exp\left(c\mathbf{z}^{3}\right)\right)T\left(\exp\left(d\varepsilon\right)\right)f\right](x,y;z)$$

$$=(1-az)^{-\alpha}\exp\left(\frac{axz}{y}+d\right)f\left(\left(x+\frac{by}{z}\right)(1-az),\ y\ ;\ z\exp(c)\right)$$

Let $g=\exp(a\mathbf{z}^{+})\exp(b\mathbf{z}^{-})\exp(c\mathbf{z}^{3})\exp(d\varepsilon)$, $g\in G_4$
By using (1.6) we can write i.e

$$\exp(a\mathbf{z}^{+})\exp(b\mathbf{z}^{-})\exp(c\mathbf{z}^{3})\exp(d\varepsilon)=\begin{pmatrix}1 & be^{c} & d & c\\ 0 & e^{c} & a & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \quad (2.16)$$

$$[T(g)f](x,y;z)=(1-az)^{-\alpha}\exp\left(\frac{axz}{y}+d\right)f\left(\left(x+\frac{by}{z}\right)(1-az),\ y\ ;\ z\exp(c)\right) \quad (2.17)$$

Where

$$g=\begin{pmatrix}1 & be^{c} & d & c\\ 0 & e^{c} & a & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\in G_4 \quad (2.18)$$

Setting $f(x,y;z)=z^{n}L_{n}^{\alpha-n}(x,y)$ which is a common Eigen function of $C$ and $J^{3}$.

### *3. GENERATING RELATIONS:*

To accomplish our task for obtaining generating functions of $L_n^{\alpha-n}(x,y)$, we find a function $f(x,y;z)$ which satisfies the differential equation (1.9).

Consider $f(x,y;z)$ is a common eigenfunction of $C$ and $J^3$, and also a solution of the simultaneous equation

$$C\, f(x,y;z) = 0 \tag{3.1}$$

$$J^3 f(x,y;z) = n\, f(x,y;z) \tag{3.2}$$

Equation (3.1) and (3.2) can also be written as

$$-y\left\{x\frac{\partial^2}{\partial x^2} - z\frac{\partial^2}{\partial x \partial z} + \left(\alpha + 1 - \frac{x}{y}\right)\frac{\partial}{\partial x} + \frac{z}{y}\frac{\partial}{\partial z}\right\} f(x,y;z) = 0 \tag{3.3}$$

$$\left(z\frac{\partial}{\partial z} - n\right) f(x,y;z) = 0 \tag{3.4}$$

respectively.

By using (3.3) and (3.4), we arrive at conclusion that

$$f(x,y;z) = L_n^{\alpha-n}(x,y)\, z^n$$

On applying the (observation 3, p.324, [8]), we have

$$[T(g)f](x,y;z) = (1-az)^{-\alpha} \exp\left(\frac{axz}{y} + d\right) L_n^{\alpha-n}\left(\left(x + \frac{by}{z}\right)(1-az),\ y\right)(z\exp(c))^n$$

$$= (1-az)^{-\alpha} \exp\left(\frac{axz}{y} + nc + d\right) L_n^{\alpha-n}\left(\left(x + \frac{by}{z}\right)(1-az),\ y\right) z^n,\ |az| \prec 1 \tag{3.5}$$

This satisfies the relation

$$C\left[\, T(g)\, f\,\right](x,y;z) = 0 \tag{3.6}$$

If $n$ and $\alpha - n$ are non negative integer then (3.6) can be written in the following form

$$[T(g)f](x,y;z) = \sum_{k=-\infty}^{\infty} P_k(g, x, y)\, z^{k+n} \tag{3.7}$$

If $[T(g)f](x,y;z)$ is regular at $x = 0$ and by using (observation 2, p.324, [8]) then we get,

$$P_k(g,x,y) = A_{kn}(g) L_{n+k}^{\alpha-n-k}(x,y) \tag{3.8}$$

Therefore,

$$[T(g)f](x,y;z) = \sum_{k=-\infty}^{\infty} A_{kn}(g) L_{n+k}^{\alpha-n-k}(x,y)\, z^{n+k}$$

$$= \sum_{k=0}^{\infty} A_{kn}(g) L_k^{\alpha-k}(x,y)\, z^k \tag{3.9}$$

From (3.5) and (3.9) we get,

$$(1-az)^{-\alpha}\exp\left(\frac{axz}{y}+nc+d\right)L_n^{\alpha-n}\left(\left(x+\frac{by}{z}\right)(1-az),\ y\right)z^n$$

$$=\sum_{k=-\infty}^{\infty}A_{kn}(g)L_{n+k}^{\alpha-n-k}(x,y)\,z^{n+k} \tag{3.10}$$

On solving the above equation we get,

$$A_{kn}(g)=(-1)^k\,\frac{\Gamma(1+n+k)}{\Gamma(1+n)}\,a^n\,\{\Gamma(1+k)\}^{-1}\;{}_2F_1\left[\begin{matrix}\alpha+n+1\,,\ -n;\\ 1+k;\end{matrix}-ab\right] \tag{3.11}$$

By substituting the value of $A_{kn}(g)$ in equation (3.10) we get the required generating function as:

$$\exp\left(\frac{axz}{y}\right)(1-az)^{-\alpha}L_n^{\alpha-n}\left(\left(x+\frac{by}{z}\right)(1-az),y\right)=\sum_{k=-\infty}^{\infty}\frac{\Gamma(1+n+k)}{\Gamma(1+n)}L_{n+k}^{\alpha-k-n}(x,y)\{\Gamma(1+k)\}^{-1}$$

$${}_2F_1\left[\begin{matrix}\alpha+n+1\,,\ -n;\\ 1+k;\end{matrix}-ab\right](-az)^k$$

where $0\prec|z|\prec|a|^{-1}$. (3.12)

**SPECIAL CASES:**

For $b=0$ and $z=-1$, it becomes

$$\exp\left(-\frac{ax}{y}\right)(1+a)^{-\alpha}L_n^{\alpha-n}(x(1+a),y)=\sum_{k=0}^{\infty}\frac{(1+n)_k}{k!}L_{n+k}^{\alpha-k-n}(x,y)(a)^k \tag{3.13}$$

while for $a=0$ and $z=-1$, we have

$$L_n^{\alpha-n}((x-by),y)=\sum_{k=0}^{\infty}\frac{b^k}{k!}L_{n+k}^{\alpha-k-n}(x,y) \tag{3.14}$$

if n is a positive integer say $n=m$, the generating function following from (3.5) is

$$\exp\left(\frac{axz}{y}\right)(1-az)^{-\alpha}L_m^{\alpha-m}\left(\left(x+\frac{by}{z}\right)(1-az),y\right)=\sum_{k=0}^{\infty}\frac{k!}{m!}L_k^{\alpha-k}(x,y)\{\Gamma(1+k-m)\}^{-1}$$

$${}_2F_1\left[\begin{matrix}\alpha+n+1\,,\ -n;\\ 1+k-m;\end{matrix}-ab\right](-az)^{k-m} \tag{3.15}$$

**5. CONCLUSION:**

In this work, the Group theoretic method has been successfully applied to Modified Laguerre polynomial in two variables $L_n^{\alpha-n}(x,y)$ and this method is easy and straight forward for obtaining the generating relations**.** The reason of interest for this family of Laguerre polynomial is due to their intrinsic mathematical importance and to the fact that these polynomials are shown to be natural solutions of a particular set of partial differential equations which often appear in the treatment of radiation physics problems such as the electromagnetic wave propagation and quantum beam life-time in storage ring.

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