Int. Journal of Math. Analysis, Vol. 4, 2010, no. 47, 2311 - 2320

Quasi-Normal Operators

Shqipe Lohaj

Electronic Faculty, University of Prishtina
Prishtinë, 10000, Kosova
shqipe_ lohaj@hotmail.com

Abstract

In this article we will give some properties of quasinormality of operators in Hilbert spaces. Exactly we will show that every invertible N-quasinormal operator is a quasinormal operator and we will show that if T is singular N-quasinormal operator then for $N \neq 1$, $T(T^*T)$ and $(T^*T)T$ are quasinilpotent operators. Also we will study some operator equalities of the form $T(T^*T) = \lambda S$, for the special kind of operators S.

Mathematics Subject Classification: 47B20

Keywords: Quasinormal operators, N-quasinormal operators

1 Introduction

Let us denote by H the complex Hilbert space and with $B(H)$ the space of all bounded linear operators defined in Hilbert space H. Let T be an operator in $B(H)$. The operator T is called normal if it satisfies the following condition: $T^*T = TT^*$. The operator T is called quasi-normal if: $T(T^*T) = (T^*T)T$. We will defined a new class of operators, the class of N-quasinormal operators. We will say that T is a N- quasinormal operator, if $T(T^*T) = N((T^*T)T)$. In this paper we will study some properties of N- quasi-normal operators. Exactly we will show that every invertible N-quasinormal operator is a quasinormal operator and we will show that if T is a singular N-quasinormal operator then for $N \neq 1$, $T(T^*T)$ and $(T^*T)T$ are quasinilpotent operators. Also, there it is shown that if T is N-quasinormal unilateral weighted shift operator then for $N > 1, T$ is quasinilpotent compact operator.Further we will give some results in the case of operator equalities of the form $T(T^*T) = \lambda S$, depending on the choice of the operator S.

2 Quasinormal operators and N-quasinormal operators

Lemma 2.1 *If $T \in B(H)$ is an invertible N-quasinormal operator, then*

1. $T^*(T^*T) = \frac{1}{N}(T^*T)T^*$;
2. $T(T^*T) = N(T^*T)T$;
3. $T^{-1}(T^*T) = \frac{1}{N}(T^*T)T^{-1}$.

Proposition 2.2 *If T is an invertible N-quasinormal operator, then T is a quasinormal operator.*

Proof. Since T is a N-quasinormal operator, then

$$T^*T(TT^*)^{-1} = T^*T(T^{*-1}T^{-1})$$

$$= \frac{1}{N}T^{*-1}(T^*T)T^{-1} = \frac{1}{N}I.$$

From the last relation we conclude that

$$T^*T\Big(\frac{TT^*}{N}\Big)^{-1} = I.$$

Further on,

$$(T^*T)^{-1} = \Big(\frac{TT^*}{N}\Big)^{-1}$$

$$T^{-1}T^{*-1} = NT^{*-1}T^{-1}$$

$$\|T^{-1}T^{*-1}\| = N\|T^{*-1}T^{-1}\|$$

$$\|(T^{*-1})^*T^{*-1}\| = N\|(T^{-1})^*T^{-1}\|$$

$$\|T^{*-1}\|^2 = N\|T^{-1}\|^2.$$

From the last equality we have $N = 1$, which means that T is a quasinormal operator.

Lemma 2.3 *If T is an invertible quasinormal operator, then T is normal operator.*

Theorem 2.4 *If T is N-quasinormal unilateral weighted shift operator with weighted sequence $\{\alpha_i\}$, then $|\alpha_{i+1}| = \frac{1}{\sqrt{N^i}}|\alpha_1|$. For $N > 1, T$ is quasinilpotent compact operator and for $N < 1, \lim\limits_{i\to\infty} |\alpha_{i+1}| = \infty$.*

Proof. From estimations

$$T(T^*T)(e_i) = T(T^*(T(e_i))) = T(T^*(\alpha_i e_{i+1}))$$
$$= T((\alpha_i\alpha_i)e_i) = T(|\alpha_i|^2 e_i) = |\alpha_i|^2\alpha_i e_{i+1}$$

and

$$(T^*T)T(e_i) = (T^*T)(\alpha_i e_{i+1}) = T^*(\alpha_i\alpha_{i+1}e_{i+2}) = \alpha_i|\alpha_{i+1}|^2 e_{i+1}$$

and since $T(T^*T) = N(T^*T)T$, we will have

$$|\alpha_i|^2\alpha_i = N\alpha_i|\alpha_{i+1}|^2.$$

If $\alpha_1 \neq 0$ then

$$|\alpha_i| = \sqrt{N}|\alpha_{i+1}|$$

or

$$|\alpha_{i+1}| = \frac{1}{\sqrt{N}}|\alpha_i| = \frac{1}{\sqrt{N^i}}|\alpha_1|.$$

Now, if $N > 1$ then

$$\lim_{i\to\infty} |\alpha_{i+1}| = 0.$$

Therefore, T is quasinilpotent compact operator (because $\sigma(T) = \{0\}$.)

If $N < 1$, $\lim_{i\to\infty} |\alpha_{i+1}| = \infty$.

From the above Theorem we conclude that for $N < 1$ don't exist bonded unilateral weighted shift operator with weighted sequence $\alpha_i \neq 0$, which is N-quasinormal.

Theorem 2.5 *If T is a singular N-quasinormal operator, then for $N \neq 1, T(T^*T)$ and $(T^*T)T$ are quasinilpotent operators.*

Proof. From the equality

$$T(T^*T) = N(T^*T)T$$

we have

$$\sigma(T(T^*T))\backslash\{0\} = \sigma(N(T^*T)T)\backslash\{0\}$$
$$\sigma(T(T^*T))\backslash\{0\} = N\sigma((T^*T)T)\backslash\{0\}.$$

For $\lambda \neq 0, \lambda \in \sigma(T(T^*T))$, we assume

$$\sigma(T(T^*T)) = N\sigma((T^*T)T).$$

Now, because $\sigma((T^*T)T)\backslash\{0\} = \sigma(T(T^*T))\backslash\{0\}$ (see [4]), from the last equality we have two possibilities:

For $N = 1$ the operator T is quasinormal;

For $N \neq 1$ the equality holds true if and only if

$$\sigma(T(T^*T)) = \sigma((T^*T)T) = \{0\},$$

therefore $T(T^*T)$ and $(T^*T)T$ are quasinilpotent operators.

Lemma 2.6 *If T is N-quasinormal operator, then the equality*

$$(T(T^*T))^n = \frac{1}{N^{\frac{n(n-1)}{2}}} T^n(T^*T)^n,$$

holls true for all $n \in \mathbf{N}$

Proof. Observe that

$$\Big((T^*T)T\Big)^2 = (T^*T)T(T^*T)T = \frac{1}{N^{1+2}} T^2(T^*T)^2$$

$$\Big((T^*T)T\Big)^3 = ((T^*T)T)^2(T^*T)T = \frac{1}{N^{1+2}}\Big(T^2(T^*T)^2\Big)(T^*T)T = \frac{1}{N^{1+2+3}} T^3(T^*T)^3.$$

Bay inductive argument it is obvius that

$$\Big((T^*T)T\Big)^n = \frac{1}{N^{1+2+3+\ldots+n}} T^n(T^*T)^n,$$

$$\Big((T^*T)T\Big)^n = \frac{1}{N^{\frac{n(n+1)}{2}}} T^n(T^*T)^n.$$

Further on,

$$(T(T^*T))^n = (N(T^*T)T)^n$$

from which we have

$$(T(T^*T))^n = N^n((T^*T)T)^n,$$

$$(T(T^*T))^n = N^n \frac{1}{N^{\frac{n(n+1)}{2}}} T^n(T^*T)^n,$$

$$(T(T^*T))^n = \frac{1}{N^{\frac{n(n-1)}{2}}} T^n(T^*T)^n.$$

Proposition 2.7 *If T is a singular N-quasinormal operator then*

$$\lim_{n\to\infty} \|T^n(T^*T)^n\|^{\frac{1}{n}} = 0$$

for $N < 1$.

Proof. From Theorem 2.5 and Lemma 2.6 we have

$$0 = r(T(T^*T)) = \lim_{n\to\infty}\Big(\frac{1}{N^{\frac{n(n-1)}{2}}}\Big)^{\frac{1}{n}} \|T^n(T^*T)^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{N^{\frac{n-1}{2}}} \lim_{n\to\infty} \|T^n(T^*T)^n\|^{\frac{1}{n}}.$$

From the last relation it is obvius that

$$\lim_{n\to\infty} \|T^n(T^*T)^n\|^{\frac{1}{n}} = 0$$

3 Some operator equalities

Lemma 3.1 *Let T be invertible quasinormal operator in $B(H)$ and*

$$T(T^*T) = \lambda I + Q. \tag{1}$$

Then, Q is normal operator.

Proof. Since T is invertible quasinormal operator then $T^*T = TT^*$. From the equality (1) we have

$$QQ^* = (T(T^*T) - \lambda I)((T^*T)T^* - \overline{\lambda} I)$$
$$= T(T^*T)^2T^* - \overline{\lambda}T(T^*T) - \lambda(T^*T)T^* + |\lambda|^2 I$$
$$Q^*Q = ((T^*T)T^* - \overline{\lambda} I)(T(T^*T) - \lambda I)$$
$$= (T^*T)T^*T(T^*T) - \lambda(T^*T)T^* - \overline{\lambda}T(T^*T) + |\lambda|^2 I.$$

Therefore Q is normal operator in $B(H)$.

Proposition 3.2 *If $T \in B(H)$, $T^n = U$ where U is a unitary operator and $\|T\| = r(T), \|T^{-1}\| = r(T^{-1})$ then, $T = \lambda U_1, \lambda \in \mathbf{C}$ where U_1 is a unitary operator.*

Proof. Since U is a unitary operator $\sigma(U) \subseteq \{\lambda \in \mathbf{C} : |\lambda| = 1\}$. Then,

$$\sigma(T) \subseteq \{\lambda \in \mathbf{C} : |\lambda^n| = 1\} = \{\lambda \in \mathbf{C} : |\lambda|^n = 1\} = \{\lambda \in \mathbf{C} : |\lambda| = 1\}.$$

By the conditions of the proposition we get

$$\|T\| \cdot \|T^{-1}\| = r(T) \cdot r(T^{-1}) = 1.$$

Therefore (see [6]) $T = \lambda U_1$ for some $\lambda \in \mathbf{C}$ and some U_1 unitary operator in $B(H)$.

Theorem 3.3 *Let A, B and Q be operators in $B(H)$ with the following properties:*

1. *A is invertible operator;*
2. *$B(A^*A) = (A^*A)B$;*
3. *$B(A^*A) = \lambda I + Q$.*

Then,

1. *Q is normal operator, if only if B is normal;*

2. Q is hyponormal operator, if only if B is hyponormal;

3. Q is quasinormal operator, if only if B is quasinormal;

Proof. From the conditions of the theorem, we have

$$QQ^* = B(A^*A)^2B^* - \overline{\lambda}B(A^*A) - \lambda(A^*A)B^* + |\lambda|^2 I$$
$$Q^*Q = (A^*A)B^*B(A^*A) - \lambda(A^*A)B^* - \overline{\lambda}B(A^*A) + |\lambda|^2 I$$

1. Since Q is normal operator, we have

$$Q^*Q = QQ^* \Leftrightarrow (A^*A)B^*B(A^*A) = B(A^*A)^2B^* \Leftrightarrow$$

$$\Leftrightarrow (A^*A)^2B^*B = (A^*A)^2BB^* \Leftrightarrow (A^*A)^2[B^*B - BB^*] = 0 \Leftrightarrow B^*B = BB^*.$$

   It implies that B is normal operator.

2. Since Q is hyponormal operator, we have

$$Q^*Q - QQ^* = (A^*A)B^*B(A^*A) - B(A^*A)^2B^* \geq 0$$
$$\Leftrightarrow \Big(((A^*A)B^*B(A^*A) - B(A^*A)^2B^*)x, x\Big) \geq 0$$
$$\Leftrightarrow \Big(((A^*A)B^*B(A^*A) - (A^*A)^2BB^*)x, x\Big) \geq 0$$
$$\Leftrightarrow \Big((A^*A)(B^*B(A^*A) - (A^*A)BB^*)x, x\Big) \geq 0$$
$$\Leftrightarrow \Big((B^*B(A^*A) - (A^*A)BB^*)x, (A^*A)x\Big) \geq 0$$
$$\Leftrightarrow \Big((B^*B - BB^*)(A^*A)x, (A^*A)x\Big) \geq 0$$

$$\Leftrightarrow B^*B - BB^* \geq 0$$

   $\Leftrightarrow$ B is hypornormal operator.

3. The proof is similar to that in 1.

Remark 3.4 *If the operators A, B and Q are as in the Theorem 3.3. and B is normal (hyponormal, quasinormal), we have that $B(A^*A)$ is normal (hyponormal, quasinormal), also.*

Corollary 3.5 *If B is hyponormal operator, then,*

$$\frac{r(B)}{\|A^{-1}\|^2} \leq r(B(A^*A)) \leq \min\{|\lambda| + r(Q), r(B)\|A\|^2\}.$$

Proof.

$$\|B(A^*A)\|^2 = \|(A^*A)B^*B(A^*A)\| = \|B^*B(A^*A)^2\| \leq$$
$$\leq (|\lambda| + \|Q\|)^2 = (|\lambda| + r(Q))^2 \tag{2}$$

On the other hand

$$\|B(A^*A)\| \leq \|B\|\|A^*A\| = r(B)\|A\|^2. \tag{3}$$

Further,

$$\|B(A^*A)\| \geq \|B(A^*A)\frac{(A^*A)^{-1}}{\|(A^*A)^{-1}\|}\| = \frac{\|B\|}{\|A^{-1}\|^2} = \frac{r(B)}{\|A^{-1}\|^2}. \tag{4}$$

From (2),(3) and (4) we have

$$\frac{r(B)}{\|A^{-1}\|^2} \leq \|B(A^*A)\| \leq \min\{|\lambda| + r(Q), r(B)\|A\|^2\}$$

Since $B(A^*A) = \lambda I + Q$ is hyponormal operator it implies that

$$\|B(A^*A)\| = r(B(A^*A)),$$

and hence

$$\frac{r(B)}{\|A^{-1}\|^2} \leq r(B(A^*A)) \leq \min\{|\lambda| + r(Q), r(B)\|A\|^2\}$$

Lemma 3.6 *If*

$$T(T^*T)^n = \lambda U, \lambda \in C, n \in \mathbf{N} \tag{5}$$

then there exists $\lambda_1 \in \mathbf{R}^+$, *such that* $T = \frac{\lambda}{\lambda_1}U$.

Proof. From

$$T(T^*T)^n = \lambda U$$

and

$$(T^*T)^n T^* = \overline{\lambda} U^*$$

we get

$$(T^*T)^n T^*T(T^*T)^n = |\lambda|^2 I,$$

therefore

$$(T^*T)^{2n+1} = |\lambda|^2 I.$$

Now since $T^*T \geq 0$ it implies that

$$\sigma(T^*T) = \{|\lambda|^{\frac{2}{2n+1}}\}$$

and

$$\sigma((T^*T)^{-1}) = \Big\{\frac{1}{|\lambda|^{\frac{2}{2n+1}}}\Big\}.$$

Further

$$r(T^*T) = \|T^*T\| = |\lambda|^{\frac{2}{2n+1}}$$

and

$$r((T^*T)^{-1}) = \|(T^*T)^{-1}\| = \frac{1}{|\lambda|^{\frac{2}{2n+1}}}.$$

From the above equalities we will have

$$\|T^*T\|\cdot \|(T^*T)^{-1}\| = 1.$$

Therefore, $T^*T = \lambda_1 U_1$ for some $\lambda_1 \in \mathbf{R}^+$ and some unitary operator U_1.

Since $T^*T \geq 0, \lambda_1 \geq 0$ which implies that $T^*T = \lambda_1 I$. Replacing the last one in equality (5) we have $T(\lambda_1^n I) = \lambda U$ and therefore $T = \frac{\lambda}{\lambda_1^n}U$.

Lemma 3.7 *If $T \in B(H)$ and $(T^*T)^n = \lambda I$ then $T = \lambda_1 U$, where $\lambda_1 \in \mathbf{C}$, U is unitary operator, satisfying $U^n = \frac{\lambda}{\lambda_1^n} I$, for some $\lambda_1 \in \mathbf{C}$.*

Proof. Let $(T^*T)^n = \lambda I$, then

$$(T^*T)^{2n} = (T^*T)^n(T^*T)^n = (T^*T)^n((T^*T)^n)^* = |\lambda| I.$$

Since $T^*T \geq 0$ so is $(T^*T)^{2n}$ therefore

$$\sigma((T^*T)^{2n}) = \{|\lambda|\}.$$

Now,

$$\sigma(T^*T) = \{\lambda \in \mathbf{C} : \lambda^{2n} = |\lambda|\} = \{|\lambda|^{\frac{1}{2n}}\}.$$

On the other hand $\sigma((T^*T)^{-1}) = \{|\lambda|^{-\frac{1}{2n}}\}$. and this

$$\sigma(T^*T)\sigma((T^*T)^{-1}) = 1.$$

Further, because T^*T and $(T^*T)^{-1}$ are self-adjoint elements,

$$r(T^*T) = \|T^*T\| = |\lambda|^{2n}$$

and

$$r((T^*T)^{-1}) = \|((T^*T)^{-1})\| = |\lambda|^{-\frac{1}{2n}}.$$

Now it is obvious that

$$\|T^*T\| \cdot \|((T^*T)^{-1})\| = 1$$

and therefore $T^*T = \lambda_1 U$, for some $\lambda_1 \in \mathbf{C}$. Since $T^*T \geq 0, T^*T = \lambda_1 I$.

Theorem 3.8 *Let $T \in B(H)$, such that*

$$T(T^*T) = \lambda S. \tag{6}$$

Then

1. *If S is a unilateral weighted shift operator, then $T = \alpha S, \alpha \in \mathbf{R}^+$.*
2. *If $S = P$, where P is an orthogonal projection, then $T = \alpha P$.*
3. *If $S \in F(H)$ ($F(H)$ is the class of Fredholm operators) and S is a partial isometry, then $T = S(\lambda U + K)^{-1}$ for any compact operator K.*

Proof.

1. From $Se_i = e_{i+1}, S^*S = I$ and

$$\begin{aligned}(T(T^*T))^* \cdot T(T^*T) &= \lambda S^* \overline{\lambda} S = |\lambda|^2 I \\ (T^*T)T^*T(T^*T) &= |\lambda|^2 I \\ (T^*T)^3 &= |\lambda|^2 I \\ (T^*T)^3 &= \beta I \end{aligned} \tag{7}$$

for any $\beta \in \mathbf{R}^+$. Replacing (7) in (6) we get

$$T(\beta I) = \lambda S$$

or

$$T = \frac{\lambda}{\beta} S = \alpha S.$$

2. If $S = P$, then

$$T(T^*T) = \lambda P. \tag{8}$$

From the above equality we will have

$$\sigma(T(T^*T)) = \{0, \lambda\}.$$

On the another hand

$$(T^*T)^3 = |\lambda|^2 P$$

it means that

$$\sigma((T^*T) = \{0, \sqrt[3]{|\lambda|^2}\},$$

and hence

$$T^*T = \sqrt[3]{|\lambda|^2} I. \tag{9}$$

Replacing (9) in (6), we have the desired result.

3. Since S is a partial isometry, $S^*S = P$ where P is a projector. Because $S \in F(H), \overline{R(S)} = R(S)$ and $indS = dimKerS^* - dimKerS < \infty$. Similarly as in 2. we have

$$(T^*T)^3 = |\lambda|^2 P, \sigma(T^*T) = \{0, \sqrt[3]{|\lambda|^2}\}.$$

Since $indS < \infty$ it follows that $S^*S \in F(H)$ and $ind(S^*S) = 0$ and hence $dimKerP < \infty$. Now from $(T^*T)^3 = |\lambda|^2 P$ there exists a compact operator K such that $(T^*T)^3 = \alpha I + K$, for any complex number α and $\alpha I + K$ is invertible. Now it follows that there exists an unitary operator U and a compact operator K_1 such that $T^*T = \beta U + K_1$ for some $\beta \in \mathbf{R}^+$. From $T(T^*T) = S$ we have $T(\beta U + K_1) = S$ or $T = S(\beta U + K_1)^{-1}$ (see [6]).

References

[1] S.C. Arora, Ramesh Kumar, M-Paranormal operators, Publications de I'institut Mathematique, Nouvelle serie 29 (43) (1981), 5-13.

[2] S.C. Arora, J.K. Thukral, M*-Paranormal operators, Glas. Math. Ser. III 42 (1987), No.1.

[3] N. Chennappan, S. Karthikeyan, *-Paranormal Operators, Indian J. Pure Appl. Math., 31(6), (2000), 591-600.

[4] P.R. Halmos, A Hilbert space problem book, Van Nostrand, Princeton 1967.

[5] D.J. Harrington, J. Whitley, Seminormal composition operators, J.Operator Theory, 11 (1984) 125-135.

[6] Sh.Lohaj, M.Lohaj, Essentially hyponnormal operators with essential spectrum contained in a circle, Le Matemache, Vol. LXIV (2009)-Fasc. I, pp. 93-96.

Received: June, 2010