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A Complexity Minimization Principle for Dynamical Processes

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Abstract. From an analysis of the concept of complexity for curves, it is demonstrated that geodesic motion of particles is consistent with a minimization of the complexity. The reduction of complexity is also evident in the quantum interactions of strings.

1. Introduction

The principle of least action governs the motion of point particles in Newtonian mechanics and general relativity, and it is relevant for the path integral formulation of quantum mechanics. The equations of motion of a free particle require that it travels in a straight line in Euclidean space. The generalization of dynamics to relativistic scales requires not only the gravitational field equations derived from the Einstein-Hilbert action but also the condition of free particles following geodesics on the curved space-time. With a properly defined measure of complexity of curves, the value of this measure can be shown to be zero for geodesics, and consequently, the geodesic equation may be cast in the form of a constraint of vanishing complexity. The extension of the concept to surfaces leads to a characterization of string worldsheets. Again, the complexity is reduced for the classical motion of the string. From the formulation of the quantum theory of strings through the path integral, it can be shown that the minimization of complexity, within the class of theories describing string interactions, occurs if conformal invariance is maintained.

2. Characterization of Path Complexity

A path of length L can be characterized by the number of occurrences ℓ_i of a given element and the number of elements, n. Each length-element in a particle path, shall be characterized by a distinct set of Frenet-Serrat vectors (cf. §5). The average complexity for nonoverlapping elements, that are counted separately, is defined by the frequencies $f_i = \frac{\ell_i}{L}$ and $C = -\sum_{i=1}^{n} f_i ln\ f_i$, and, for span g, it would be $C_{span\ g} = -\sum_i \frac{\ell_i^{(g)}}{\lfloor \frac{L}{g} \rfloor} ln \frac{\ell_i^{(g)}}{\lfloor \frac{L}{g} \rfloor}$, $1 \le g \le \frac{L}{2}$, where $\ell_i^{(g)}$ is the number of occurrences of the i^{th} type of g-element subsequence [1]. The dynamics of a particle provide a correlation between segments of its path and will affect the measure of complexity at nontrivial span. There is a higher correlation between the segments of a path in geodesic and periodic motion.

This measure of complexity is non-negative and it has a maximum value when $f_i = \frac{1}{L}$. The extent to which a sequence can be described by the average complexity is established in the following theorem.

Theorem 2.1. *There is a unique value of C for each sequence $\{[\ell_i, n, L]\}$.*

Proof. By definition, for each element of the set $\{[\ell_i, n, \ell]\}$, there is one value of the complexity, $C = -\sum_{i=1}^{n} \left(\frac{\ell_i}{L}\right) ln\ \left(\frac{\ell_i}{L}\right)$. Given a value of C, it can be written always in the form $-\sum_{i=1}^{n} \left(\frac{\ell_i}{L}\right) ln\ \left(\frac{\ell_i}{L}\right) = -ln\ \prod_{i=1}^{n} \left(\frac{\ell_i}{L}\right)^{\frac{\ell_i}{L}}$. Suppose that there are two elements $[\ell_i, n, L]$ and $[k_j, m, L]$ such that the values of the complexity are equal. Then

$$\frac{\prod_{i=1}^{n} \ell_i^{\ell_i}}{L^{\sum_i \ell_i}} = \frac{\prod_{j=1}^{m} k_j^{k_j}}{L^{\sum_j k_j}} \tag{2.1}$$

However, $\sum_{i=1}^{n} \ell_i = \sum_j k_j = L$, and this equality implies that $\prod_{i=1}^{n} \ell_i^{\ell_i} = \prod_{j=1}^{m} k_j^{k_j}$.

Lemma 2.2. *If $\{\ell_i\}$, $\{k_j\}$ are sets of non-negative integers satisfying $\prod_{i=1}^{n} \ell_i^{\ell_i} = \prod_{j=1}^{m} k_j^{k_j}$ and $\sum_{i=1}^{n} \ell_i = \sum_{j=1}^{m} k_j$, then $\{\ell_i\} = \{k_j\}$ and $n = m$.*

Proof. By the prime factorization theorem, $\ell_i = \prod_r \ell_r^{q_{ir}}$ and $k_j = \prod_s k_s^{p_{js}}$, where $\ell_r,\ k_s$ are primes, and

$$\begin{aligned} \prod_{i=1}^{n} \ell_i^{\ell_i} &= \prod_{i=1}^{n} \prod_r \ell_r^{\ell_i q_{ir}} = \prod_r \prod_{i=1}^{n} \ell_r^{\ell_i q_{ir}} = \prod_r \ell_r^{\sum_{i=1}^{n} \ell_i q_{ir}} \\ \prod_{j=1}^{m} k_j^{k_j} &= \prod_{j=1}^{m} \prod_s k_s^{k_j p_{js}} = \prod_s \prod_{j=1}^{m} k_s^{k_j p_{js}} = \prod_s k_s^{\sum_{j=1}^{m} k_j p_{js}} \end{aligned} \tag{2.2}$$

The number $\prod_{i=1}^{n} \ell_i^{\ell_i} = \prod_j k_j^{k_j}$ also must have a unique prime factorization, which forces the identification of ℓ_r with k_s and $\sum_{i=1}^{n} \ell_i q_{ir} = \sum_{j=1}^{m} k_j p_{js}$. The following equations must hold

$$(2.3)\qquad \sum_{i=1}^{n}(\prod_{r}\ell_r^{q_{ir}}) - \sum_{j=1}^{m}(\prod_{r}\ell_r^{p_{jr}}) = 0$$

$$\sum_{i=1}^{n}(\prod_{s}\ell_s^{q_{is}})q_{ir} - \sum_{j=1}^{m}(\prod_{s}\ell_s^{p_{js}})p_{jr} = 0$$

Multiplying the first equation by $q_{kr}\delta_{ki}$ and summing over k gives

$$(2.4)\qquad \sum_{i=1}^{n}\left(\prod_{s}\ell_s^{q_{is}}\right)q_{ir} - \sum_{j=1}^{m}\left(\prod_{s}\ell_s^{p_{js}}\right)q_{ir} = 0$$

Subtracting this equation from the second condition in Eq. (2.3) yields

$$(2.5)\qquad \sum_{j=1}^{m}\left(\prod_{s}\ell_s^{p_{js}}\right)(q_{ir} - p_{jr}) = 0$$

and $Lq_{ir} = \sum_{j=1}^{m}\ell_j p_{jr}$. Multiplying Eq. (2.5) by δ_{ij} and summing over i,

$$(2.6)\qquad \sum_{j=1}^{m}\left(\prod_{s}\ell_j^{p_{js}}\right)(q_{jr} - p_{jr}) = 0$$

Either $q_{jr} = p_{jr}$ for all j or

$$(2.7)\qquad \sum_{j=1}^{m}\left(\prod_{s}\ell_j^{p_{js}}\right)q_{jr} = Lq_{ir}\ for\ all\ i \neq j'$$

for some j'. Then

$$(2.8)\qquad \sum_{j=1}^{m}\ell_j q_{jr} = \ell_{j'}q_{j'r} + (L - \ell_{j'})q_{ir} = Lq_{ir}$$

and $\ell_{j'}q_{j'r} = \ell_{j'}q_{ir}$ or $q_{j'r} = q_{ir}$. However, if q_{ir} is identical for all $i \leq m$, it will have this value for $i \geq m$ if m and n are not necessarily equal, and

$$(2.9)\qquad \sum_{i=1}^{n}\prod_{r}\ell_r^{q_{ir}} = n\prod_{r}\ell_r^{q_{ir}} = n\ell_i = L$$

$$\ell_i = \frac{L}{n}$$

From Eq. (2.5),

$$(2.10)\qquad \sum_{j=1}^{m}\frac{L}{n}(q_{ir} - p_{jr}) = \frac{L}{n}\sum_{j=1}^{m}(q_{ir} - p_{jr}) = 0$$

A similar argument, reversing the roles of q_{ir} and p_{jr}, implies that p_{jr} is constant for all j and $\ell_j = \frac{L}{m}$. By Eq. (2.10), q_{ir} must equal p_{jr} and $m = n$. The equality q_{ir} and p_{jr} implies that $\ell_i = k_j$. ■

It follows from the lemma that the elements $[\ell_i, n, L]$ and $[k_j, m, L]$ can be identified and that there is a unique element of $\{[\ell_i, n, L]\}$ corresponding to C for fixed L. There is then a one-to-one correspondence between the two sets $\{[\ell_i, n, L]\}$ and $\{C, L\}$. ∎

The function $-p\, ln\, p$ is non-negative between 0 and 1 and it has a maximum value at $p = \frac{1}{e}$. For each value of the function, there are two probabilities p_1 and p_2. However, except for an exceptional circumstance, these probabilities cannot both be fractions with the same denominator for finite L.

Lemma 2.3. *Given frequences $f_i = \frac{\ell_i}{L}$ and $f_i' = \frac{\ell_i'}{L}$, $0 < f_i,\ f_i' < 1$, the only example of equality of $-f_i ln\ f_i$ and $-f_i' ln\ f_i'$ with $f_i \neq f_i'$ occurs for finite L if $f_i = \frac{1}{2}$ and $f_i' = \frac{1}{4}$ or $f_i = \frac{1}{4}$ and $f_i' = \frac{1}{2}$.*

Proof. Let $-f_i\ ln\ f_i = -f_i'\ ln\ f_i'$. Then

$$\left(\frac{\ell_i}{L}\right)^{\frac{\ell_i}{\ell_i'}} = \frac{\ell_i'}{L} \tag{2.11}$$

Since $\frac{\ell_i}{L}$ and $\frac{\ell_i'}{L}$ are proper fractions, equality can be attained if $\ell_i = k\ell_i'$ for some integer k. From Eq.(2.11),

$$\frac{L}{\ell_i'} = k^{\frac{k}{k-1}} \tag{2.12}$$

When $k \geq 3$, $\frac{k}{k-1}$ is fractional and the condition cannot be valid for integer ℓ_i' and L. If $k = 2$, $\frac{L}{\ell_i'} = 2^2 = 4$ or $f_i' = \frac{\ell_i'}{L} = \frac{1}{4}$. Furthermore, $\ell_i = 2\ell_i'$ and $f_i = 2f_i' = \frac{1}{2}$.

If $\ell_i' \not| \ell_i$ then it is necessary for $\frac{\ell_i}{L}$ to be a product of a power of $\frac{\ell_i'}{L}$ and an integer that is a corresponding power. First, let $\frac{\ell_i}{L} = \left(\frac{\ell_i'}{L}\right)^{\lambda}$, such that $\ell_i' = \ell_i^{\frac{1}{\lambda}} L^{\frac{\lambda-1}{\lambda}}$. It is necessary for $\frac{\ell_i}{\ell_i'}\lambda = 1$ for the equality (2.11) to be valid and

$$\frac{L}{\ell_i'} = \lambda^{\frac{\lambda}{\lambda-1}} \tag{2.13}$$

Therefore, $\lambda = 2$ and $\frac{\ell_i}{L} = \frac{1}{4}$. Then $\frac{\ell_i'}{L} = \frac{1}{2}$.

More generally, let $\frac{\ell_i}{L} = k^{\lambda}\left(\frac{\ell_i'}{L}\right)^{\lambda}$, $k \in \mathbb{Z}^+$. The exponent λ must have a value such that

$$\frac{L}{\ell_i} = \left(\frac{k\lambda}{k'}\right)^{\frac{\lambda}{\lambda-1}}, \qquad k' \in \mathbb{Z}^+ \tag{2.14}$$

By Eq.(2.11), $\frac{L}{\ell_i'} = k^{\frac{k'}{k'-1}}$, and

$$k^{\lambda}\lambda = k'^{\lambda-1}k^{\frac{k'(\lambda-1)}{k'-1}} \tag{2.15}$$

Divisibility conditions resulting from Eqs.(2.14) and (2.15) require $k' < 3$. If $k' = 2$, Eq.(2.15) becomes $k^\lambda \lambda = 2^{\lambda-1} k^{2(\lambda-1)}$, which is satisfied only when $\lambda = 2$. It follows that

$$\frac{\ell_i}{L} = k^2 \left(\frac{\ell_i'}{L}\right)^2 = \left(\frac{2k}{2}\right)^{-2} = k^{-2} = \frac{\ell_i'}{L} \tag{2.16}$$

and equal frequencies yield equal values of the function $-f_i ln\ f_i$. When $k' = 1$,

$$k^{k'} \left(\frac{\ell_i'}{L}\right)^{k'} = k\frac{\ell_i'}{L} = \frac{\ell_i'}{L} \tag{2.17}$$

and $k = 1$. Then the previous inequality $\frac{\ell_i}{L} = \left(\frac{\ell_i'}{L}\right)^2$ is recovered, with the solution $f_i = \frac{1}{4}$ and $f_i' = \frac{1}{2}$. ■

The relation between the frequencies in the anomalous condition cannot hold for each i-type element because the constraints $\sum_i \ell_i = L$ and $\sum_i \ell_i' = L$ prevent $\ell_i = 2\ell_i'$ for each i. However, if the relations are interchanged between primed and unprimed elements, consistency may be achieved, with the sequences being represented by a relabelling of the elements.

For the addition of two sequences, there is a relation between the total complexity of the entire sequence, the sum of the total complexities of one sequence and a subsequence of the second sequence and the average complexity of the combined regions [1].

3. Change in the Complexity under Entropic Processes

Path complexity can be anticipated to remain constant, or decrease, in a deterministic transition, which may be characterized by one-to-one or many-to-one mappings between initial and subsequent states. If one-to-many mappings are introduced in an order-disorder process, an increase in the complexity, regarded as a generalized correlation measure of the final sequence, can occur.

If an element of the first set can be mapped to more than one element of the second set, a probability can be assigned to each mapping. Let $i \to j$ have probability p_{ij}. Then the entropy of the entire process is $S = -\sum_{i=1}^n \sum_{j=1}^m p_{ij} ln p_{ij}$. First, consider the probability of ℓ_i elements i being mapped to k_{ij} elements j. If $p_{ij} = 0$, $p(\ell_i \to k_j) = \delta_{k_{ij},0}$, and if $p_{ij} = 1$, $p(\ell_i \to k_{ij}) = \delta_{k_{ij},\ell_i}$. In general,

$$p(\ell_i \to k_{ij}) = p_{ij}^{k_{ij}} (1 - p_{ij})^{\ell_i - k_{ij}} \frac{\ell_i!}{k_{ij}!(\ell_i - k_{ij})!} \tag{3.1}$$

and

$$p(\ell_i \to k_{i1}, k_{i2}, ..., k_{im}) = \frac{\ell_i!}{k_{i1}!k_{i2}!...k_{im}!} p_{i1}^{k_{i1}} p_{i2}^{k_{i2}} ..p_{im}^{k_{im}} \tag{3.2}$$

$$\sum_j p_{ij} = 1 \quad \sum_j k_{ij} = \ell_i$$

The probability of the sequence transformation $[\ell_i, n, L] \to [k_j, m, L]$ occurring is

(3.3)
$$p([\ell_i, n, L] \to [k_j, m, L]) = \prod_{i=1}^{n} p(\{\ell_i\} \to \{k_{ij} | \sum_i k_{ij} = k_j, \ \sum_j k_{ij} = \ell_i\}) = \prod_{i=1}^{n} \sum_{\substack{k_{ij} \\ \sum_i k_{ij} = k_j \\ \sum_j k_{ij} = \ell_i}} \prod_{j=1}^{m} \frac{\ell_i!}{k_{ij}!} p_{ij}^{k_{ij}}$$

To determine the probability of obtaining a sequence of greater complexity, it may be noted that the complexity of a sequence reaches its maximum value when the frequency of each of the elements is approximately equal. Setting $m = n$ by allowing k_j to be equal to zero, if there are no j^{th} elements, the probability of the frequency of the i^{th} element of the final sequence being closer to $\frac{L}{n}$ than the i^{th} element of the initial sequence.

$$p\left(\left|k_i - \frac{L}{n}\right| < \left|\ell_i - \frac{L}{n}\right|\right) = \sum_{t_i = -\left(\ell_i - \frac{L}{n}\right)+1}^{\ell_i - \frac{L}{n} - 1} p\left(k_i = t_i + \frac{L}{n}\right) \tag{3.4}$$

Given that the probability of the map $a \to i$ is p_{ai},

$$p(\ell_a \to k_{ai}) = p_{ai}^{k_{ai}} (1 - p_{ai})^{\ell_i - k_{ai}} \frac{\ell_i!}{k_{ai}!(\ell_i - k_{ai}!)} \tag{3.5}$$

and

$$p(\{\ell_a\} \to k_i) = \sum_{\sum_a \{k_{ai}\} = k_i} \prod_a p(\ell_a \to k_{ai}) = \sum_{\sum_a \{k_{ai}\} = k_i} \prod_a p_{ai}^{k_{ai}} (1 - p_{ai})^{\ell_a - k_{ai}} \frac{\ell_a!}{k_{ai}!(\ell_a - k_{ai}!)} \tag{3.6}$$

Then

(3.7)
$$p\left(\left|k_i - \frac{L}{n}\right| < \left|\ell_i - \frac{L}{n}\right|\right) = \sum_{t_i = -\left(\ell_i - \frac{L}{n}\right)+1}^{\ell_i - \frac{L}{n} - 1} p\left(\{\ell_a\} \to t_i + \frac{L}{n}\right) = \sum_{t_i = -\left(\ell_i - \frac{L}{n}\right)+1}^{\ell_i - \frac{L}{n} - 1} \sum_{\sum_a k_{ai} = t_i + \frac{L}{n}} \prod_a p_{ai}^{k_{ai}} (1 - p_{ai})^{\ell_a - k_{ai}} \frac{\ell_a!}{k_{ai}!(\ell_a - k_{ai}!)}$$

The probability of the final sequence $[k_i, n, L]$ having greater complexity would be

$$(3.8) \qquad \prod_{i=1}^{n} \sum_{\substack{t_i=-\left(\ell'_i-\frac{L}{n}\right) \\ \prod_i \ell'^{\ell'_i}_i < \prod_i \ell^{\ell_i}_i}}^{\ell'_i+\frac{L}{n}} \sum_{\sum_a \{k_{ai}\}=t_i+\frac{L}{n}} \prod_a p_{ai}^{k_{ai}} \frac{\ell_a!}{k_{ai}!}$$

4. A Relation between Path Complexity and Configuration Space Entropy

Statistical mechanical processes are irreversible if the entropy increases. A one-to-one mapping from an initial to a subsequent state will conserve the complexity. Since the complexity must remain constant or decrease in a deterministic process [1], the system cannot be invariant under time reversal if there is a change in the complexity.

Theorem 4.1. *Complexity is a conserved quantity if the process is deterministic and reversible.*

Proof. The complexity either remains constant or decreases in a deterministic process since injective mappings can only merge the elements in the range. If $\{f_{i,init}\}$ and $\{f_{i,fin}\}$ are initial and final frequencies, $\frac{1}{n}\sum_i |f_{i,fin}-\frac{1}{n}| > \frac{1}{n}\sum_i |f_{i,init}-\frac{1}{n}|$. It follows that $-\sum_i f_{i,fin} ln\ f_{i,fin} < \sum_i f_{i,init} ln\ f_{i,init}$. When the complexity decreases, it is not possible to return to the initial state as the complexity of the final sequence cannot equal its original value.

A finite complex system can be characterized by its sequential and geometric properties, which may be labelled by a set of scalar functions in space-time $\{\phi_r\}$. For a reversible process, the system would be defined by time-independent scalar functions. The scalar functions can be selected to be the frequencies of occurrence of the elements in a system, since a change in the frequencies $\{\frac{\ell_i}{L}\}$ leads to a change in the complexity by Theorem 1. Invariance under time translations yields a balance equation [5] $\partial_t C = 0$. Therefore, complexity is a conserved quantity of the system during reversible processes.

■

The specification of a sequence leads to a decrease in the uncertainty. It can be conjectured then that there is a balance relation between the complexity and a configurational entropy, since both quantities have a fixed value in a reversible process and the entropy increases while the complexity decreases under an irreversible change.

Consider a sequence $[\ell_i, n, L]$, and suppose that $n \geq L$ such that it is possible to select a sequence with every element different. Then the total complexity would be $C_{tot} = -\sum_i \ell_i ln \frac{\ell_i}{L}$. The choice of the sequence implies that the information is the distinction between the different sets of letters, whereas each set of identical letters is selected from amongst a large category. The space of potential configurations for the i^{th} set is determined by ℓ_i different letters since other choices would correspond to isomorphic sets leading to an overcounting of states. For each block of identical letters in the initial sequence, the maximum-entropy transitions, defined equal probabilities for the final state elements, would be used. The entropy is constrained by the size of set of sites with identical elements in the initial sequence, since it will bound the number of final state elements in the image of the this set.

The number of elements in this category would be $\prod_{i=1}^{n} \ell_i^{\ell_i}$ and, given equal probabilities for the mappings,

$$S_{conf} = -\sum_{i,i' \in (i),j} p_{i'j} ln\ p_{i'j} = -\sum_{i,j} \frac{\ell_i^2}{\ell_i} ln\ \frac{1}{\ell_i} = ln \prod_{i=1}^{n} \ell_i^{\ell_i} \tag{4.1}$$

The configurational entropy is generally different from the entropy defined by specified probabilities for each mapping, and equality is achieved only if all of the probabilities are equal.

The sum of the total complexity and the configuration space entropy then is

$$\begin{aligned} C_{tot} + S_{conf} &= -\sum_i \ell_i ln\ \ell_i + \sum_i \ell_i ln\ L + \sum_i \ell_i ln\ \ell_i \\ &= L\ ln\ L = ln\ L^L \end{aligned} \tag{4.2}$$

It follows that the sum is a conserved quantity equal to the logarithm of the total number of states of the system.

5. Complexity of Curves

The foregoing analysis can be generalized to curved paths, on defining their intrinsic complexity. The concept of complexity can be extended from a set of linear path elements to curved paths. Each curve can be described by a function $f(t)$, and the derivatives shall be used to define the complexity. For a straight line, $f(t) = at + b$, the derivative is constant. The line, therefore, would be equivalent to a sequence of identical length elements and C should equal zero. The function defining a parabola $f(t) = t^2$ would have derivatives $f'(t) = 2t$, $f''(t) = 2$, and one of the derivative sequences would consist of differing letters, while the others would comprise of identical letters. The complexity therefore should be increased.

If there exists a frame of reference such that every point in a portion of the curve, relative to that frame, is identical, again leading to a reduction of the complexity. An example is the circular arc, where the magnitude of the tangent vector is fixed, and the angle changes at a constant rate. Since $\frac{d|\vec{t}|}{dt}$ is zero, $C_r^{(n)} = 0$. However, $arg\ \vec{t}$ does change, $C_\theta^{(n)}$ is not zero. By analogy with the definition of complexity for sequences consisting of different elements, the quantity $-\sum_i \frac{\delta\ell_i}{L} ln \frac{\delta\ell_i}{L}$ may be considered, with $\delta\ell_i$ representing the length of an infinitesimal segment and L being the length of the arc. Given that the infinitesimal segments all have the same length, $\delta\ell$, the sum is equal to

$-ln\frac{\delta\ell}{L}$. However, the divergence must be removed with a regularization factor, yielding the intrinsic complexity,

$$
\begin{aligned}
(5.1) \qquad C_{int}^{(1)} &= -\sum_{\substack{i \\ lim_{\delta\ell\to 0}\ \frac{\delta\ell}{\ell_i}\neq 0}} \frac{\ell_i}{L} ln \frac{\ell_i}{\frac{L}{\ell_{i'min}}\delta\ell} - \sum_{\substack{i' \\ lim_{\delta\ell\to 0}\frac{\delta\ell}{\ell_i}=0}} \frac{\frac{\ell_{i'}}{\ell_{i'min}}}{\frac{L}{\ell_{i'min}}} ln\ \frac{\frac{\ell_{i'}}{\ell_{i'min}}}{\frac{L}{\ell_{i'min}}} \\
&= -\int_{\mu=L_\delta} \frac{f_1(\ell)d\ell}{L} ln\ \frac{f_1(\ell)\ell_{imin}}{L} - \sum_{i'} \frac{\ell_{i'}}{L} ln\ \frac{\ell_{i'}}{L}
\end{aligned}
$$

where $f_1(\ell)$ equals the number of times that the tangent vectors can be identified, the index i' labels all of the segments of non-zero measure and $\ell_{i'min}$ is the minimum length of the segments of non-zero measure. The index set i' must be finite as L is finite, and $\sum_{i'} \ell_{i'} = L - L_\delta$. It is not necessary to multiply the first factor by $\frac{1}{\delta\ell}$, since the vanishing of the factor as $\delta\ell \to 0$ is removed by the summation $\sum_{\substack{i \\ lim_{\delta\ell\to 0}\ \frac{\delta\ell}{\ell_i}\neq 0}}$, which is then written as an integral. When a curve contains no set of identical points of non-zero measure, the length scale ℓ_{imin} could be set equal to $\delta\ell$, giving rise to a divergence in the complexity as $\delta\ell \to 0$. This infinity can be removed by setting ℓ_{imin} to a fixed value such as 1. For the circular arc, setting $C_{int\theta}^{(1)} = -\int\ \frac{d\ell}{L} ln\ \frac{1}{L} = ln\ L$.

A function satisfying an n^{th} order differential equation $\frac{d^n f}{dt^n} + c_{n-1}\frac{d^{(n-1)}f}{dt^{(n-1)}} + ... + c_1\frac{df}{dt} + c_0 f = 0$. Derivatives of n^{th} and higher order can be determined by the lower-order derivatives, and the complexity is reduced. The weighted sum of the contributions of the derivative complexity would be $\sum_{k=1}^{\infty}\ \frac{1}{k!}C_{int}^{(k)}$. Typically, $C_{int}^{(k)}$ would achieve the maximum value $ln\ L$ for $k = 1, ..., n-1$. When $k \geq n$, this value would be divided by two since the value of the derivative is determined by the derivatives of order less than or equal to $n-1$, implying that there is no new information in the additional data and the complexity of the curve then would be $\sum_{k=1}^{n-1} \frac{1}{k!}C_{int}^{(k)} + \sum_{k=n}^{\infty} \frac{1}{k!}C_{int}^{(k)} = \sum_{k=1}^{n-1} \frac{1}{k!} ln\ L + \sum_{k=n}^{\infty} \frac{1}{k!}\left[-\int^{\frac{L}{2}}\ \frac{d\ell}{L} ln\ \frac{1}{L} - \frac{\frac{L}{2}}{L} ln\ \frac{\frac{L}{2}}{L}\right]$.

A general curve, not described by an n^{th}-order differential equation, would have complexity $\sum_{k=1}^{\infty} \frac{1}{k!} ln\ L = (e-1) ln\ L$ if $C_{int}^{(k)} = ln\ L$ for all k.

Theorem 5.1. *The complexity measure has the properties*

(1) The maximum value of C_{int} occurs for nonrepeating curves.
(2) For two curves $\gamma_1,\ \gamma_2$, $C_{int}(\gamma_1 \times \gamma_2) = C_{int}(\gamma_1) + C_{int}(\gamma_2)$.
(3) Translations, rotations and dilations of the curve do not alter C_{int}.

Proof. Every infinitesimal segment is different in a nonrepeating curve and the complexity is $\sum_{k=1}^{\infty}\ \frac{1}{k!} ln\ \frac{L}{\ell_{i'\ min}^{(k)}}$. In general,

(5.2)

$$
\begin{aligned}
C_{int}^{(k)} &= -\int_{\mu=L_\delta} \frac{f_k(\ell)d\ell}{L} ln\ \frac{f_k(\ell)}{L} - \sum_{i'} \frac{\ell_{i'}^{(k)}}{L} ln\ \frac{\ell_{i'}^{(k)}}{L} \\
&\leq \frac{L_\delta}{L} ln\ \frac{L}{\ell_{i'\ min}^{(k)}} + \sum_{\substack{i' \\ \sum_{i'} \ell_{i'}^{(k)} = L - L_\delta}} ln\ \left(\frac{L}{\ell_{i'}^{(k)}}\right)^{\frac{\ell_{i'}^{(k)}}{L}} \\
&= \frac{L_\delta + (L - L_\delta)}{L} ln\ \frac{L}{\ell_{i'\ min}^{(k)}} - \sum_{\substack{i' \\ \sum_{i'} \ell_{i'}^{(k)} = L - L_\delta}} \frac{\ell_{i'}^{(k)}}{L} ln\ \frac{\ell_{i'}^{(k)}}{\ell_{i'\ min}^{(k)}} \leq ln\ \frac{L}{\ell_{i'\ min}^{(k)}}
\end{aligned}
$$

To compare the complexity of a nonrepeating curve and a curve with set of identical points of non-zero measure, it is sufficient to use a scaling such that $\ell_{i_{min}}^{(k)} = 1$.

(5.3)
$$
C_{int} = \sum_{k=1}^{\infty} \frac{1}{k!} C_{int}^{(k)} \leq \sum_{k=1}^{\infty} \frac{1}{k!} ln\ \frac{L}{\ell_{i'\ min}^{(k)}}
$$

with equality obtained only when the curve is nonrepeating.

Property (2) is valid since a surface of area $L_1 L_2$ defined by the product of two independent nonrepeating curves, $\gamma_1,\ \gamma_2$ with lengths $L_1,\ L_2$ would have intrinsic complexity

(5.4)
$$
\begin{aligned}
C_{int}(\gamma_1 \times \gamma_2) &= ln\ \frac{A}{A_{\sigma' min}} = ln\ \frac{L_1}{\ell_{i' min}^1} + ln\ \frac{L_2}{\ell_{j' min}^2} \\
&= C_{int}(\gamma_1) + C_{int}(\gamma_2)
\end{aligned}
$$

where the index σ' denotes the pair (i', j'). It would follow that

$$(5.5)\quad C_{int}^{(k)}(\gamma_1 \times \gamma_2) = ln\ \frac{A}{A_{\sigma'\ min}^{(k)}} - \sum_{\sigma'} \frac{A_{\sigma'}^{(k)}}{A} ln\ \frac{A_{\sigma'}^{(k)}}{A_{\sigma'\ min}^{(k)}}$$
$$- \sum_{i'} \frac{\ell_{i'}^{(k)}}{L_1}\frac{L_{2\delta}}{L_2} ln\ \frac{\ell_{i'}^{(1,k)}}{\ell_{i'\ \ min}^{(1,k)}} - \sum_{j'} \frac{L_{1\delta}}{L_1}\frac{\ell_{j'}^{(2,k)}}{L_2} ln\ \frac{\ell_{j'}^{2,k)}}{\ell_{j'\ \ min}^{(2,k)}}$$
$$= ln\ \frac{L_1}{\ell_{i'\ \ min}^{(1,k)}} + ln\ \frac{L_2}{\ell_{j'\ \ min}^{(2,k)}}$$
$$- \sum_{i',j'} \frac{\ell_{i'}^{(1,k)}\ell_{j'}^{(2,k)}}{L_1 L_2}\left[ln\ \frac{\ell_{i'}^{(1,k)}}{\ell_{i'\ \ min}^{(1,k)}} + ln\ \frac{\ell_{j'}^{(2,k)}}{\ell_{j'\ \ min}^{(2,k)}}\right]$$
$$- \sum_{i'} \frac{\ell_{i'}^{(k)}}{L_1}\frac{L_{2\delta}}{L_2} ln\ \frac{\ell_{i'}^{(1,k)}}{\ell_{i'\ \ min}^{(1,k)}} - \sum_{j'} \frac{L_{1\delta}}{L_1}\frac{\ell_{j'}^{(2,k)}}{L_2} ln\ \frac{\ell_{j'}^{(2,k)}}{\ell_{j'\ \ min}^{(2,k)}}$$

as the factor of $\delta\ell$ has been removed in the sums containing the lengths, and

$$(5.6)\quad C_{int}^{(k)}(\gamma_1) + C_{int}^{(k)}(\gamma_2) = ln\ \frac{L_1}{\ell_{i'\ \ min}^{(1,k)}} - \sum_{i'} \frac{\ell_{i'}^{(1,k)}}{L_1} ln\ \frac{\ell_{i'}^{(1,k)}}{\ell_{i'\ \ min}^{(1,k)}}$$
$$+ ln\ \frac{L_2}{\ell_{j'\ \ min}^{(2,k)}} - \sum_{j'} \frac{\ell_{j'}^{(2,k)}}{L_2} ln\ \frac{\ell_{j'}^{(2,k)}}{\ell_{j'\ \ min}^{(2,k)}}$$

Since

$$(5.7)\quad \sum_{i',j'} \frac{\ell_{i'}^{(1,k)}\ell_{j'}^{(2,k)}}{L_1 L_2} ln\ \frac{\ell_{i'}^{(1,k)}}{\ell_{i'\ \ min}^{(1,k)}} = \sum_{i'} \frac{\ell_{i'}^{(1,k)}}{L_1}\frac{L_2 - L_{2\delta}^{(k)}}{L_2} ln\ \frac{\ell_{i'}^{(1,k)}}{\ell_{i'\ \ min}^{(1,k)}}$$
$$\sum_{i',j'} \frac{\ell_{i'}^{(1,k)}\ell_{j'}^{(2,k)}}{L_1 L_2} ln\ \frac{\ell_{j'}^{(2,k)}}{\ell_{j'\ \ min}^{(2,k)}} = \sum_{j'} \frac{L_1 - L_{1\delta}^{(k)}}{L_1}\frac{\ell_{j'}^{(2,k)}}{L_2} ln\ \frac{\ell_{j'}^{(2,k)}}{\ell_{j'\ \ min}^{(2,k)}}$$

as $\sum_{i'} \ell_{i'}^{(1,k)} = L_1 - L_{1\delta}^{(k)}$ and $\sum_{j'} \ell_{j'}^{(2,k)} = L_2 - L_{2\delta}^{(k)}$. It follows that $C_{int}^{(k)}(\gamma_1 \times \gamma_2) = C^(k)_{int}(\gamma_1) + C_{int}^{(k)}(\gamma_2)$.

Translations and rotations do not change the lengths of the curve or the measures of any of the sets of identical points. Under a dilation, all lengths are scaled by a factor of λ, implying that

(5.8)

$$C_{int}(\lambda) = \sum_{k=1}^{\infty} \frac{1}{k!}\left[-\int_{\mu=\lambda L_\delta} \frac{f_k^\lambda(\lambda\ell)d\lambda\ell}{\lambda L} ln\ \frac{f_k^\lambda(\lambda\ell)\lambda\ell_{i'\ min}^{(k)}}{\lambda L} - \sum_{i'} \frac{\lambda\ell_{i'}^{(k)}}{\lambda L} ln\ \frac{\lambda\ell_{i'}^{(k)}}{\lambda L}\right]$$
$$= \sum_{k=1} \frac{1}{k!}\left[-\int_{\mu=L_\delta} \frac{f_k(\ell)d\ell}{L} ln\ \frac{f_k(\ell)\ell_{i'\ min}^{(k)}}{L} - \sum_{i'} \frac{\ell_{i'}^{(k)}}{L} ln\ \frac{\ell_{i'}^{(k)}}{L}\right] = C_{int}$$

where $\ell_{i'}^{(k)}$ represents the measure of the i'^{th} set of points with identical k^{th} derivatives and $f_k^{\lambda}(\lambda\ell) = f_k(\ell)$. ■

It has been demonstrated that there is a unique measure $S = -\lambda \sum_i p_i ln\ p_i$ [6] for finite probability schemes satisfying the properties

(1) For given n and for $\sum_{i=1}^{n} p_i = 1$, the function $S(p_1, ..., p_n)$ has its maximum value at $p_i = \frac{1}{n}$.
(2) $S(AB) = S(A) + S_A(B)$ for two probability schemes, with $S_A(B)$ being the conditional entropy
(3) $S(p_1, ..., p_n, 0) = S(p_1, ..., p_n)$

Similarly, the measure which satisfies the three properties for complexity is unique up to a proportionality constant, since the relation $\frac{L}{L'} = \frac{C_{int}(L)}{C(L')}$ for nonrepeating curves holds.

An extrinsic complexity also can be introduced. A similar quantity has been defined previously through $log\ \bar{N}$ where $\bar{N}$ is the average number of times a straight line intersects the curve [10]. Each critical point of a curve with nonvanishing second derivative can be labelled by an element of a sequence, and the elements can be distinguished by the functional form of the curve about the extrema up to translations and rotations. The extrinsic complexity then would be $C_{ext} = -\sum_i \frac{n_i}{N} ln\ \frac{n_i}{N}$, where N is the number of minima and maxima of the curve. The extrinsic complexity is also equal to the average complexity with the lag given by the distance from one extremum to the next.

If a curve is embedded in three-dimensional space, a triad of unit vector fields, the tangent $\mathbf{T}$, the normal $\mathbf{N}$ and $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ can be defined. From the Frenet formulae [11],

$$\begin{aligned} \vec{T}' &= \kappa \vec{N} \\ \vec{N}' &= -\kappa \vec{T} + \tau \vec{B} \\ \vec{B}' &= -\tau \vec{N} \end{aligned} \tag{5.9}$$

where κ is the curvature and τ is the torsion. when the curvature and the torsion are zero, the curve has zero complexity. whereas if they are constant, the only contribution to the intrinsic complexity is $\vec{C}_{int}^{(1)}$.

Moreover, defining the covariant derivative of a vector along a curve as the projection of the derivative along the tangent vector, the connection equations of a frame field $\mathbf{E}_1,\ \mathbf{E}_2,\ \mathbf{E}_3$ are

$$\nabla_{\mathbf{V}} \mathbf{E}_i = \sum_j \omega_{ij}(\mathbf{V}) \mathbf{E}_j \tag{5.10}$$

for some vector field $\mathbf{V}$. When $\mathbf{V}$ is selected to be the tangent vector field, and the frame field is fitted to the curve, with $\mathbf{E}_1 = \mathbf{T}$, $\mathbf{E}_2 = \mathbf{N}$, $\mathbf{E}_3 = \mathbf{B}$. The connection equations give the Frenet formulae with ω_{ij} antisymmetric and $\omega_{12}(\mathbf{T}) = \kappa$ and $\omega_{23}(\mathbf{T}) = \tau$.

While the tangent vector is used in all dimensions, it reduces to $\left(1, \frac{df}{dx}\right)$ in two dimensions, providing the connection between the covariant derivatives of the tangent vector and the derivatives of the function.

A logarithmic Mahler measure [3] can be defined for any polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_0$ by

$$
\begin{aligned}
M(f) &= |a_n| \prod_{i=1}^{n} max\,(1, |\alpha_i|) \\
m(f) &= log M(f) \\
\{\alpha_i\}\ &are\ zeros\ of\ f(x)
\end{aligned}
\tag{5.11}
$$

The measure $m(f)$ of the characteristic polynomial of a matrix A is equal to the topological entropy of a transformation T_A, where $h_{top}(T_A) = sup_U \underset{n\to\infty}{lim} \frac{1}{n} log\ N(U \vee T^{-1}(U) \vee ... \vee T^{-(n-1)}(U))$, with U being an open cover of M, on which the matrix A acts [12].

The number of vertices in the spine or subpolyhedron P such that $M - P$ is homeomorphic to $\partial M \times [0,1]$ of a three-manifold is related to a measure of its extrinsic complexity [9]. For example, a class of three-manifolds has been found with Heegard genus $g+1$ such that the number of vertices of the spine is g [4]. The Euler characteristic of the boundary, $\chi(\partial M)$, equals $2-2g$, where g is the genus of the two-dimensional surface ∂M. A cross-section of this surface reveals a union of two curves, each with g maxima and $g-1$ minima. Presuming the functional form of each of the extrema is different, the extrinsic complexity defined previously would be $C_{ext} = ln\ (2g-1)$.

6. Geodesics in Curved Space

Since geodesics are straight lines in flat space, they have minimal complexity. In curved space, as the tangent vector $\tilde{\mathbf{T}}$ is transported by parallel propagation along the geodesic, $\nabla_{\tilde{\mathbf{T}}}\tilde{\mathbf{T}} = f\tilde{\mathbf{T}}$, where $\nabla_{\tilde{\mathbf{T}}}$ is the projection of the covariant derivative $\nabla_\mu V^\rho = \partial_\mu V^\rho + \Gamma^\rho{}_{\mu\nu}V^\nu$ onto $\tilde{\mathbf{T}}$. The geodesic equation is then

$$
\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = f(\lambda)\frac{dx^\mu}{d\lambda}
\tag{6.1}
$$

where $\Gamma^\mu{}_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}(g_{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\mu\nu,\rho})$ is the connection, λ parameterizes the curve and $\tilde{T}^\mu = \frac{dx^\mu}{d\lambda}$. Another parameter $t = h(\lambda)$ can be chosen such that the standard form of the geodesic equation is obtained. From Eq. (6.1),

$$
\begin{aligned}
h'^2\frac{d^2x^\mu}{dt^2} + h''\frac{dx^\mu}{dt} + h'^2\Gamma^\mu{}_{\nu\rho}\frac{dx^\mu}{dt}\frac{dx^\rho}{dt} &= h'f(\lambda)\frac{dx^\mu}{dt} \\
h'^2\left[\frac{d^2x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt}\right] &= (h'f(\lambda) - h'')\frac{dx^\mu}{dt}
\end{aligned}
\tag{6.2}
$$

and if $h'' = h' f(\lambda)$, $x(t)$ satisfies the equation

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\nu}{dt} = 0 \tag{6.3}$$

with t being an affine parameter, or equivalently $\nabla_{\mathbf{T}}\mathbf{T} = 0$, $T^\mu = \frac{dx^\mu}{dt}$. This condition is therefore the equivalent of the vanishing of the second derivative in flat space. Further projected covariant derivatives of the tangent vector field, again, would equal zero. The covariant constancy of the tangent vector field of a geodesic in curved space would imply the equivalence of the geodesic and a sequence with identical letters. The geodesic therefore has zero complexity.

In classical theories in flat space, the introduction of a force leads to motion with greater complexity because the curvature is determined by the acceleration. A constant force produces a curve of less complexity than a time-dependent force. Many theories have second-order differential equations for the fields, giving rise to at most fourth-order equations for the position of the particle. An equilibruium state is obtained only when the forces are balanced, and with no net force, the system belongs to a state of least complexity.

After the metric of a space-time is determined from the Einstein field equations, the motion of free particles follows a geodesic. The principle of minimal complexity is therefore included in the formulation of a relativistic theory. It will be shown in §7 that in string theory, this principle has an essential role in the formalism.

The particle dynamics are not a consequence of the four-dimensional gravitational action. Rather, it is conventional to derived this result from the property of the geodesic as an extremum of the length functional

$$\int_{x_0}^{x_1} \left[g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right]^{\frac{1}{2}} dt \tag{6.4}$$

which is then interpreted as the action of a point-particle multiplied by $\frac{1}{m}$, where m is the mass.

This description of the physics in curved space-times could be derived from one action, by including the geodesic condition with a Lagrange multiplier.

$$I_1 = \int d^4x\ \sqrt{-g}(R - 2\Lambda) + \lambda\ V_\mu \left[\frac{d^2x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt}\right] \tag{6.5}$$

for an arbitrary vector field V^μ. Alternatively, since the point-particle action is a functional of the metric, the action can be chosen to be

$$I_2 = \int d^4x\ \sqrt{-g}(R - 2\Lambda) + \lambda \int \left[g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right]^{\frac{1}{2}} dt \tag{6.6}$$

Variation with respect to $g_{\mu\nu}$ gives

$$\frac{\delta I_2}{\delta g_{\mu\nu}} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} + \lambda\frac{\delta}{\delta g_{\mu\nu}}\int \left[g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right]^{\frac{1}{2}} dt \tag{6.7}$$

where $\frac{\bar{\delta} g_{\mu\nu}}{\bar{\delta} x^\sigma}\frac{\bar{\delta} x^\sigma}{\bar{\delta} g_{\rho\lambda}} = \delta_\rho^{(\mu}\delta_\lambda^{\nu)}$. Similarly,

$$\frac{\delta I_2}{\delta x^\sigma} = \frac{\delta}{\delta x^\sigma}\int d^4x\sqrt{-g}(R-2\Lambda) + \lambda g_{\sigma\mu}\left[\frac{d^2x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt}\right] \tag{6.8}$$

Setting a further variation with respect to λ to zero, $\frac{\delta^2 I_2}{\delta x^\sigma \delta\lambda} = 0$, gives the geodesic equation. Both the gravitational and geodesic equations hold separately.

Since the minimization of complexity also leads to the geodesic equation, the constraint $C = 0$ can be included with a Lagrange multiplier.

$$I_3 = \int d^4x\ \sqrt{-g}(R-2\Lambda) + \lambda C \tag{6.9}$$

Replacement of the constraint with an action again leads to I_2.

7. Theory of Quantum String Interactions

String theory is described by a σ-model which is invariant under Weyl rescalings

$$I_{string} = \int d^2\xi\sqrt{h}h_{\alpha\beta}\partial^\alpha X^\mu\partial^\nu X^\nu g_{\mu\nu} \tag{7.1}$$

At the quantum level, Weyl invariance, which is equivalent to vanishing of a β-function, is maintained only if modifications of the gravitational field equations are valid. Let Ω be the Weyl rescaling factor. Then $\left.\frac{\delta I_{string}}{\delta\Omega}\right|_{class} = 0$ from the σ-model action, $\left.\frac{\delta I_{string}}{\delta\Omega}\right|_{quantum} = 0$ from the modified gravitational field equations and $\left.\frac{\delta I_{string}}{\delta\Omega}\right|_{class+quantum} = 0$ would follow from variation with respect to both $g^{\mu\nu}$ and λ of an action of the form

$$I = \int d^Dx\sqrt{-g}(R+...) + \lambda\int d^2\xi\sqrt{h}h_{\alpha\beta}\partial^\alpha X^\mu\partial^\beta X^\nu g_{\mu\nu} \tag{7.2}$$

More, as quantum effects can be derived from the partition function, the vanishing of the β-function at higher orders in σ-model perturbation theory may be deduced from

$$Z\ =\ \int D[g]\int D[h]e^{-I_{string}[X,h,g]} \tag{7.3}$$

without the addition of a gravitational action. The gravitational terms may derived as a quantum correction, with the field equations derived from the effective action (7.2).

The string equations imply that surfaces of minimal area are swept out by the string. In flat space, $\dot{X}^2 - X'^2 = 0$ or equivalently, $\dot{X}\pm X' = 0$. Since X' is determined by $\dot{X}$, the complexity would be reduced in that direction. Minimal complexity would be obtained with a plane surface. [8].

In curved space, the equations of motion of the string [2] are

$$
\begin{aligned}
(7.4)\qquad & \partial^\alpha[\sqrt{h}g_{\mu\nu}\partial_\alpha X^\nu] = \frac{1}{2}\sqrt{h}\partial_\mu g_{\nu\rho}\partial_\alpha X^\nu \partial^\alpha X^\rho \\
& T_{\alpha\beta} \equiv g_{\mu\nu}[\partial_\alpha X^\mu \partial_\beta X^\nu - \frac{1}{2}h_{\alpha\beta}\partial_\gamma X^\mu \partial^\gamma X^\nu] = 0
\end{aligned}
$$

In the conformal gauge [2],

$$
\begin{aligned}
(7.5)\qquad & \ddot{x}^\mu - x''^\mu + \Gamma^\mu{}_{\rho\sigma}(\dot{x}^\rho \dot{x}^\sigma - x'^\rho x'^\sigma) = 0 \\
& g_{\mu\nu}(\dot{x}^\mu \dot{x}^\nu + x'^\mu x'^\nu) = 0
\end{aligned}
$$

The first equation implies that

$$
\begin{aligned}
(7.6)\qquad & x''^\mu + \Gamma^\mu{}_{\rho\sigma} x'^\rho x'^\sigma = \ddot{x}^\mu + \Gamma^\mu{}_{\rho\sigma}\dot{x}^\rho \dot{x}^\sigma \\
& \nabla_{\mathbf{S}}\mathbf{S} = \nabla_{\mathbf{T}}\mathbf{T}
\end{aligned}
$$

where **S**, **T** are tangent vector fields in the σ, τ directions. The covariant derivatives along the tangent vector in the σ direction are determined by the covariant derivatives in the τ coordinate. If the surface is the product of two nonrepeating curves $\gamma_1,\ \gamma_2$,

$$
\begin{aligned}
(7.7)\quad C(\gamma_1 \times \gamma_2) &= \sum_{k=1}^{\infty} \frac{1}{k!} ln\ L_1 + \sum_{k=1}^{\infty}\ \frac{1}{k!}\left[-\int^{L_2 - \frac{L_1}{2}}\ \frac{d\ell}{L_2} ln\ \frac{1}{L_2} - \frac{\frac{L_1}{2}}{L_2} ln\ \frac{\frac{L_1}{2}}{L_2}\right] \\
&< \sum_{k=1}^{\infty}\ \frac{1}{k!} ln\ L_1 + \frac{1}{k!} ln\ L_2 = C(\gamma_1) + C(\gamma_2),
\end{aligned}
$$

and the complexity is reduced.

In Euclidean space, the interactions of strings are represented by higher-genus surfaces, which can be uniformized by the upper half-plane if $g \geq 2$ and the Lorentzian section can be conformally mapped to a light-cone diagram with cuts [8].

While a probabilistic interpretation is applied to the worldsheets in the path integral, the continuous path of the string implies that infinitesimal segments do not undergo purely entropic processes at the quantum level. Indeed the entropy is restricted only to the path space and, for the bosonic string, would be given $S = \frac{\int\ D[X^\mu]D[h]\ I(X^\mu,h)e^{-I(X^\mu,h)}}{Vol(Diff\ \Sigma)Vol(Weyl\ \Sigma)}$ when $e^{-I(X^\mu,h)}$ is interpreted as the weighting factor of the map from the two-dimensional space-time to the worldsheet in the target manifold. Moreover, the interaction of the strings, which are perturbations of the classical solution, always must maintain conformal invariance of the theory. It follows that, for string interactions, the nature of the process is consistent with the minimization of complexity as a result of the conformal invariance of the theory at the quantum level and the conformal equivalence of the string worldsheet to the light-cone diagram.

8. Conclusion

Dynamical processes are characterized not only by a minimization of the action but also by the complexity of the path. Point particles travel along geodesics in space-time unless their motion is governed by an external force. Adapting the measure of complexity for sequences of length elements to curves by considering the variation of the derivatives with respect to an affine parameter, it follows that the vanishing of the covariant derivative of a tangent vector field in the same direction implies that the geodesic has zero complexity. The reduction of complexity also occurs in the propagation of strings both in flat and curved space-times. The worldsheets can be mapped to a plane diagram with cuts, which represent surfaces of minimal complexity for the quantum interactions between the strings.

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