Int. Journal of Math. Analysis, Vol. 4, 2010, no. 44, 2167 - 2198

Regional Controllability of Linear and Semi Linear Hyperbolic Systems

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Abstract

This work concerns regional controllability of linear and semi linear hyperbolic systems as in ([9, 10, 11]) but the results are established with more flexible conditions. Also we consider the case where the subregion of interest is a boundary part of the system evolution domain.Then we give illustrations of different results through many examples and simulations.

Keywords: Distributed hyperbolic systems - Regional Controllability - boundary target - Fixed point

Introduction

Appropriate mathematical model of distributed parameter systems yields most often nonlinear partial differential equations. Clearly, such model involves using very sophisticated mathematical methods, which requires to describe the process more accurately and to implement more effective control strategies. Early lumping, which means approximation of a nonlinear systems by linear ordinary differential equation of possibly high order with exaggerated simplifications, may completely mask the distributed nature of the system and therefore is not always satisfactory. Hence, the need to explore non-linear models.

Hyperbolic systems may be efficient models for many real problems in various fields. Indeed, our main means of perception of the world, sight and hearing runs through light waves and sound. They are also used in the fusion of metals, medicine (surgery and ophthalmology), in communications technology and also in laser printers, CD readers. However, there is a fundamental difference between the hyperbolic systems and the parabolic ones. Indeed, for the heat equation, a small initial disturbance is felt immediately throughout the whole domain (we say that heat spreads with infinite speed). However, for the wave equation, the effect of a disturbance at t is felt at the moment $t + \varepsilon$. Moreover, the heat equation has a regularizing effect at the initial conditions, contrary to the wave equation where the solution is not more regularly than the initial state. For the controllability issues, one normally considers a control system in the time interval $[0, T]$ and asks in particular how to reach the state space, (Exact controllability) or a dense set in the state space (approximate controllability).

An extension which is very important in practical applications is that of regional controllability concept. The term regional has been used to refer to control problems in which the target of interest is not fully specified as a state, but refers only to a smaller region ω of the system domain Ω. More precisely, regional controllability consists in finding a control that steers a system at time T, to a desired state only on ω which may be interior or on the boundary of Ω .

This concept has been developed for parabolic systems by Zerrik (1993), El Jai et al. (1995) where one establishes interesting results, in particular characterization of control with minimum energy that steers a system to a desired state on such a subregion, and examples of systems that are regionally controllable but not controllable on the whole domain. Then extended to linear and semi linear hyperbolic systems see ([9, 10, 11]).

The present work concerns the regional controllability of linear and semi linear hyperbolic systems as in ([9, 10, 11]) but the results are established with more flexible conditions. Then we study the case where the target region is located on the boundary of Ω. Illustrations of different results through many examples and simulations are also given.

Divided into two parts, the first one concerns the regional controllability of linear hyperbolic systems, the target region is considered to be interior or on the boundary of the system evolution domain. So we consider the regional optimal control problems and we concentrate on the determination of a control achieving regional controllability with minimum energy using an extension of the Hilbert Uniqueness Method. Then we develop a numerical approach that leads to explicit formulas of the optimal control. Thereafter we give illustrations through examples and simulations.

In the second part we deal with regional controllability for distributed semi linear hyperbolic systems using fixed point techniques. So we begin with some preliminaries and recalls. Thereafter, we concentrate on the determination of a control that achieves regional controllability when the system is asymptotically linear. Next the analytical case is considered using the generalized inverse techniques. In all cases the control achieving regional target is characterized via fixed point theorems and depends on the final time T, the subregion in question (ω or Γ) and the actuator location. We give numerical approach which leads to explicit formulas for such a control with illustrations through numerical examples and simulations.

Part I. Regional controllability of linear systems

1 Definitions and properties

Let Ω be an open bounded domain in $I\!R^n (n = 1, 2, 3)$, with a regular boundary $\partial\Omega$. For $T > 0$ we denote by $Q = \Omega \times]0, T[$, $\Sigma = \partial\Omega \times]0, T[$ and consider the system described by the equation

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} + Ay = Bu & Q \\ y(x, 0) = y_0(x),\ \dfrac{\partial y}{\partial t}(x, 0) = y_1(x) & \Omega \end{cases} \tag{1-1}$$

where A is a second order elliptic linear operator such that $\begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$ has a compact resolvent and generates a strongly continuous semi-group $S(t)_{t\geq 0}$ on a subspace of the Hilbert state space $\mathcal{F}$, $U = L^2(0, T; I\!R^p)$ is the controls space, $(y_0, y_1) \in \mathcal{F}$, $(y_u, \dfrac{\partial y_u}{\partial t})$ denotes the solution of the equation (1-1) and assume that $(y_u(T), \dfrac{\partial y_u}{\partial t}(T)) \in \mathcal{F}$.
Let ω be an open set of Ω and

$$\begin{array}{cccc} \chi_\omega & : & L^2(\Omega) \times L^2(\Omega) & \longrightarrow & L^2(\omega) \times L^2(\omega) \\ & & (z_1, z_2) & \longrightarrow & (z_1, z_2)|_\omega \end{array}$$

while χ^*_ω is the adjoint operator defined from $L^2(\omega) \times L^2(\omega) \longrightarrow L^2(\Omega) \times L^2(\Omega)$ given by

$$\chi^*_\omega(z_1, z_2)(x) = \begin{cases} (z_1, z_2)(x) & x \in \omega \\ 0 & x \in \Omega \backslash \omega \end{cases}$$

For $\Gamma \subseteq \partial\Omega$ we denote by $\chi_\Gamma : \quad L^2(\partial\Omega) \times L^2(\partial\Omega) \longrightarrow L^2(\Gamma) \times L^2(\Gamma)$ the restriction operator to Γ while its adjoint is

$$\chi^*_\Gamma(z_1, z_2)(\xi) = \begin{cases} (z_1, z_2)(\xi) & \xi \in \Gamma \\ 0 & \xi \in \partial\Omega \setminus \Gamma \end{cases}$$

$\gamma_0 : \mathcal{F} \longrightarrow H^{\frac{3}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ denotes the trace operator of order zero. It is linear, continuous and surjective, while γ^*_0 is its adjoint.

We recall that an actuator is conventionally defined by a couple (D, f), where D is the support of the actuator and f is the spatial distribution of the action on the support D. In the case of pointwise actuator (internal or boundary), D is reduced to the location $b \in \Omega$ and $f = \delta(. - b)$ where δ_b is the dirac mass concentrated in b. For more details of strategic and regional strategic actuators, we refer to (El Jai-Pritchard 1988, El Jai et al. 1995,).

Definition 1.1
The system (1-1) is said to be ω-exactly (resp. ω-approximately) regionally controllable if for all $(p^d, v^d) \in L^2(\omega) \times L^2(\omega)$ (resp. for all $\varepsilon > 0$) there exists $u \in U$ such that
$\chi_\omega(y_u(T), \frac{\partial y_u}{\partial t}(T)) = (p^d, v^d)$ (*resp.* $\|y_u(T) - p^d\|_{L^2(\omega)} + \|\frac{\partial y_u}{\partial t}(T) - v^d\|_{L^2(\omega)} \leq \varepsilon$).

Definition 1.2
The system (1-1) is said to be Γ-exactly (resp. Γ-approximately) regionally controllable if for all $(z_1^d, z_2^d) \in H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$ (resp. for all $\varepsilon > 0$) there exists $u \in U$ such that
$\gamma_0(y_u(T), \frac{\partial y_u}{\partial t}(T)) = (z_1^d, z_2^d)$ *on* Γ (*resp.* $\|\gamma_0(y_u(T), \frac{\partial y_u}{\partial t}(T)) - (z_1^d, z_2^d)\|_{H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)} \leq \varepsilon$).

Remarks 1.3 .

1. *Consider the linear and continuous extension operator*
$R : H^{\frac{3}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \longrightarrow \mathcal{F} = H^2(\Omega) \times H^1(\Omega)$ *such that* $\gamma_0 R(g_1, g_2) = (g_1, g_2), \forall (g_1, g_2) \in H^{\frac{3}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.
Let $(z_1^d, z_2^d) \in H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$ *and* $(\bar{z}_1^d, \bar{z}_2^d)$ *its extension to* $H^{\frac{3}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ *and assume that* Γ *is a part of the boundary of* Ω *such that* $\Gamma = \partial\Omega \cap \partial\omega$, *then we have the result*

   If the system (1-1) is $\omega-$exactly (resp. $\omega-$approximately) regionally controllable, then it is $\Gamma-$exactly (resp. $\Gamma-$approximately) regionally controllable.

2. *The above definitions do not allow for pointwise or boundary controls, since for such systems B is not bounded. However the extension can be carried out if we take a regular controls such that* $(y_u(T), \frac{\partial y_u}{\partial t}(T)) \in \mathcal{F}$.

3. *A system which is exactly (resp. approximately) controllable is ω-exactly (resp. ω-approximately) regionally controllable.*

4. *A system which is ω_2-exactly (resp. ω_2-approximately) controllable is ω_1-exactly (resp. ω_1-approximately) controllable for any $\omega_1 \subseteq \omega_2$.*

5. *Let* $\mathcal{J}(u) = \int_0^T \|u(t)\|^2 dt$ *be the transfer cost. Then for any $\omega \subseteq \Omega$, the regional transfer cost in ω is smaller than the transfer cost in all Ω.*

   Consider the system (1-1), the desired state (p^d, v^d) and the sets

$$\mathcal{W}_\Omega = \{u \in L^2[0,T] \;/\; (y_u(T), \frac{\partial y_u}{\partial t}(T)) = (p^d, v^d) \quad on \quad \Omega\}$$

$$\mathcal{W}_\omega = \{u \in L^2[0,T] \;/\; (y_u(T), \frac{\partial y_u}{\partial t}(T)) = (p^d, v^d) \quad on \, \omega\}$$

   We have $\mathcal{W}_\Omega \subseteq \mathcal{W}_\omega$ *then* $\min_{u \in \mathcal{W}_\omega} \mathcal{J}(u) \leq \min_{u \in \mathcal{W}_\Omega} \mathcal{J}(u)$.

6. *Due to finite speed of the propagation of hyperbolic systems, the time plays one of the most important role in the controllability of these systems. Here we show that the minimum time of the regional controllability of the wave equation is less than the one of the controllability in the whole domain* Ω*. Consider the system (1-1), the desired state* (p^d, v^d) *and the sets*

$$\mathcal{T}_\Omega = \left\{ T > 0 \text{ such that}: (y_u(T), \frac{\partial y_u}{\partial t}(T)) = (p^d, v^d) \text{ on } \Omega \right\}$$

$$\mathcal{T}_\omega = \left\{ T > 0 \text{ such that }: (y_u(T), \frac{\partial y_u}{\partial t}(T)) = (p^d, v^d) \text{ on } \omega \right\}$$

then $\mathcal{T}_\Omega \subset \mathcal{T}_\omega$ *and* $\inf_{T\in\mathcal{T}_\omega} \leq \inf_{T\in\mathcal{T}_\Omega}$*. This is illustrated by the following example:*

Consider the system described by the wave equation excited by a boundary control

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} - \triangle y = 0 & Q \\ y(x,0) = y_0, \dfrac{\partial y}{\partial t}(x,0) = y_1 & \Omega \\ y = u & \Sigma(x^0) = \Gamma(x^0)\times]0,T[\\ y = 0 & \Sigma\backslash\Sigma(x^0) \end{cases} \tag{1-2}$$

where $x^0 \in I\!R^n$*,* $\Sigma(x^0) = \{x \in \partial\Omega \quad \text{such that} \quad \nu(x)(x - x^0) > 0\} \times]0,T[$*,* $\nu(x)$ *is the unit normal vector on* x *directed toward the exterior of* $\partial\Omega$ *and* $u \in L^2(\Sigma(x^0))$*. We denote by*

$$R_\Omega(x^0) = \max_{x\in\bar\Omega}(\sum_{k=1}^n (x_k - x_k^0)^2)^{\frac{1}{2}} \quad \text{and} \quad R_\omega(x^0) = \max_{x\in\bar\omega}(\sum_{k=1}^n (x_k - x_k^0)^2)^{\frac{1}{2}}$$

It is known (Lions 1988) that for $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *the system (1-2) is exactly controllable in the whole domain* Ω *for all* T *such that* $T > T_\Omega(x^0) = 2R_\Omega(x^0)$*. But the system (1-2) is* ω*-exactly regionally controllable for all* T *such that* $T > T_\omega(x^0)$*, where* $T_\omega(x^0) = R_\Omega(x^0) + R_\omega(x^0)$*. For more details see ([9]).*

2 Regional internal control problem

In this section we characterize the control with minimum energy that steers an hyperbolic system evolving on Ω to a desired state on a subregion $\omega \subset \Omega$.
Consider the following system excited by an internal zone actuator

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} - \triangle y = \chi_D f u & Q \\ y(x,0) = y_0(x), \dfrac{\partial y}{\partial t}(x,0) = y_1(x) & \Omega \\ y(\xi,t) = 0 & \Sigma \end{cases} \tag{2-1}$$

where $(y_0, y_1) \in \mathcal{F} = H_0^1(\Omega) \times L^2(\Omega)$ and $f \in L^2(\Omega)$.

and the problem

$$\begin{cases} \min\limits_{u \in L^2(0,T)} \mathcal{J}(u) = \|u\|^2_{L^2(0,T)} \\ y_u(T) = p^d \qquad \textit{and} \qquad \dfrac{\partial y_u}{\partial t}(T) = v^d \quad \text{on} \quad \omega \end{cases} \tag{2-2}$$

where $(y_u, \dfrac{\partial y_u}{\partial t})$ the solution of (2-1), $(p^d, v^d) \in L^2(\omega) \times L^2(\omega)$ is a desired state at time T. This problem will be solved by an approach which is an extension of the Hilbert Uniqueness Method initiated by (Lions 1988). The steps are the following

For $(\phi_1, -\phi_0) \in \bar{G}$, the system

$$\begin{cases} \dfrac{\partial^2 \phi}{\partial t^2} - \triangle \phi = 0 & Q \\ \phi(x,T) = \phi_0(x), \quad \dfrac{\partial \phi}{\partial t}(x,T) = \phi_1(x) & \Omega \\ \phi(\xi, t) = 0 & \Sigma \end{cases} \tag{2-3}$$

has a unique solution $\phi \in C(0,T,H^1_0(\Omega)) \cap C^1(0,T,L^2(\Omega))$.

In $\bar{G}$ we define the following semi norm

$$\|(\phi_1, -\phi_0)\|_{\bar{G}} = \left(\int_0^T < \phi, f >^2_{L^2(D)} dt \right)^{\frac{1}{2}} \tag{2-4}$$

and we consider the system

$$\begin{cases} \dfrac{\partial^2 \psi}{\partial t^2} - \triangle \psi = - < \phi, f >_{L^2(D)} \chi_D f & Q \\ \psi(x,0) = y_0(x), \dfrac{\partial \psi}{\partial t}(x,0) = y_1(x) & \Omega \\ \psi(\xi, t) = 0 & \Sigma \end{cases} \tag{2-5}$$

(2-5) has a unique solution such that $(\psi(T), \dfrac{\partial \psi}{\partial t}(T)) \in H^1_0(\Omega) \times L^2(\Omega)$.

Let M be the affine operator defined by $M(\phi_1, -\phi_0) = \mathcal{P}(\psi(T), \dfrac{\partial \psi}{\partial t}(T))$ where $\mathcal{P} = \chi^*_\omega \chi_\omega$ but $(\psi(T), \dfrac{\partial \psi}{\partial t}(T)) = (\psi_0(T), \dfrac{\partial \psi_0}{\partial t}(T)) + (\psi_1(T), \dfrac{\partial \psi_1}{\partial t}(T))$, where ψ_0 and ψ_1 are solutions of the systems

$$\begin{cases} \dfrac{\partial^2 \psi_0}{\partial t^2} - \triangle \psi_0 = 0 & Q \\ \psi_0(x,0) = y_0(x), \quad \dfrac{\partial \psi_0}{\partial t}(x,0) = y_1(x) & \Omega \\ \psi_0(\xi, t) = 0 & \Sigma \end{cases} \tag{2-6}$$

and

$$\begin{cases} \dfrac{\partial^2 \psi_1}{\partial t^2} - \triangle \psi_1 \quad = - < \phi, f >_{L^2(D)} \chi_D f & Q \\ \psi_1(x,0) = 0, \quad \dfrac{\partial \psi_1}{\partial t}(x,0) = 0 & \Omega \\ \psi_1(\xi, t) = 0 & \Sigma \end{cases} \tag{2-7}$$

We consider the operator

$$\Lambda(\phi_1, -\phi_0) = \mathcal{P}(\psi_1(T), \frac{\partial \psi_1}{\partial t}(T)) \tag{2-8}$$

Λ is a symmetric and bounded operator. Then the regional controllability problem turns up to solve the equation

$$\Lambda(\phi_1, -\phi_0) = -\mathcal{P}(\psi_0(T), \frac{\partial \psi_0}{\partial t}(T)) + \chi_\omega^*(p^d, v^d) \tag{2-9}$$

In [9], to solve (2-9) one needs the whole controllability condition, but here this condition is relaxed and we have the main following result.

Theorem 2.1
If (2-1) is $\omega-$approximately regionally controllable, then the equation (2-9) has a unique solution ϕ_0, ϕ_1 and the control $u^*(t) = -<\phi, f>_{L^2(D)}$ drives the system (2-1) to (p^d, v^d) on ω at time T, where ϕ is solution of the system (2-3). Moreover, this control is the solution of the problem (2-2).

Proof
The system (2-1) can be written in the first order form $Z' = \bar{\mathcal{A}}Z + \bar{B}u$ with

$$Z = \begin{pmatrix} y \\ y' \end{pmatrix} \quad \bar{\mathcal{A}} = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix} \quad \text{and } \bar{B}u = \begin{pmatrix} 0 \\ \chi_D f u \end{pmatrix}$$

The operator $\bar{\mathcal{A}}$ isn't self-adjoint, but since the Laplacian Δ east

$$\bar{\mathcal{A}}^* = -\bar{\mathcal{A}} = \begin{pmatrix} 0 & -I \\ -\Delta & 0 \end{pmatrix} \text{(see[3])}$$

Also the system (2-3) is equivalent to

$$\begin{cases} \begin{pmatrix} \phi \\ \dfrac{\partial \phi}{\partial t} \end{pmatrix}' - \bar{\mathcal{A}}^* \begin{pmatrix} \phi \\ \dfrac{\partial \phi}{\partial t} \end{pmatrix} = 0 \\ \begin{pmatrix} \phi(T) \\ \dfrac{\partial \phi}{\partial t}(T) \end{pmatrix} = \begin{pmatrix} \phi_0(x) \\ \phi_1(x) \end{pmatrix} \end{cases} \tag{2-10}$$

For

$$H: \quad L^2(0, T, I\!R^p) \longrightarrow L^2(\Omega) \times L^2(\Omega)$$

$$u \longrightarrow (y_u(T), \frac{\partial y_u}{\partial t}(T))$$

After [9] the system (2-1) is $\omega-$approximately regionally controllable if and only if
$\overline{Im \chi_\omega H} = L^2(\omega) \times L^2(\omega) \iff \ker H^* \chi_\omega^* = \{0\}$ with $H^* \chi_\omega^* = \overline{B}^* \overline{S}^*(T - .)\chi_\omega^*$

We show that the semi norm (2-4) is a norm.
$\|(\phi_1, -\phi_0)\|_{\bar{G}} = 0$ gives $< \phi, f >_{L^2(D)} = 0$ on $[0, T]$, therefore

$$< \begin{pmatrix} \chi_D f \\ 0 \end{pmatrix}, \begin{pmatrix} \phi \\ \dfrac{\partial \phi}{\partial t} \end{pmatrix} >_{L^2(\Omega)\times L^2(\Omega)} = 0 \iff \bar{B}^* \begin{pmatrix} \phi \\ \dfrac{\partial \phi}{\partial t} \end{pmatrix} = 0 \quad (see[3])$$

or

$$\begin{pmatrix} \phi(t) \\ \dfrac{\partial \phi}{\partial t}(t) \end{pmatrix} = \overline{S}^*(T-t) \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix}$$

is solution of the system (2-10) where $(\bar{S}^*(-t))_{t \geq 0}$ is the semi group generated by $-\bar{\mathcal{A}}^*$.
Thus

$$< \phi, \chi_D f >_{L^2(\Omega)} = 0 \iff \bar{B}^* \bar{S}^*(T-t) \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} = 0.$$

Consequently $\begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} \in \ker H^* \chi_\omega^*$. As the system (2-1) is $\omega-$approximately regionally controllable, it follows that $\phi_1 = \phi_0 = 0$ and (2-4) is a norm.
Let $\widehat{G}$ be the completion of $\bar{G}$ by the norm (2-4) and $\widehat{G}^*$ be its dual. We show that Λ is an isomorphism from $\widehat{G}$ into $\widehat{G}^*$. Indeed

$< \Lambda(\phi_1, -\phi_0), (\phi_1, -\phi_0) > = < \psi_1(T), \phi_1 > - < \dfrac{\partial \psi_1}{\partial t}(T), \phi_0 >$ and a formal use of Green formula (all this can be made completely rigorous without difficulty) gives, after multiplying (2-3) by ψ_1 and integrating by parts

$$\begin{aligned} -\int_0^T < \phi, f >^2_{L^2(D)} dt &= < \frac{\partial \psi_1}{\partial t}(T), \phi(T) > + < \psi_1(0), \frac{\partial \phi}{\partial t}(0) > \\ &- < \frac{\partial \psi_1}{\partial t}(0), \phi(0) > - < \psi_1(T), \frac{\partial \phi}{\partial t}(T) > \end{aligned}$$

So we have $< \Lambda(\phi_1, -\phi_0), (\phi_1, -\phi_0) > = \|(\phi_1, -\phi_0)\|^2_{\widehat{G}}$.

Hence (2-9) has only one solution (ϕ_1, ϕ_0) and $u^*(t) = - < \phi, f >_{L^2(D)}$ steers the system (2-1) to the desired state (p^d, v^d) on ω at time T.
Now assume that there exist u and v solutions of (2-2), then

$$\begin{aligned} \mathcal{J}'(u^*)(u-v) &= \int_0^T u^*(t)(u(t)-v(t))dt \\ &= -\int_0^T < \phi, f >_{L^2(D)} (u(t)-v(t))dt \end{aligned}$$

Applying Green formula to $\{\phi, y_u - y_v\}$, we obtain

$$\begin{aligned} \int_0^T < \phi, f >_{L^2(D)} (u(t)-v(t))dt &= < \frac{\partial y_u}{\partial t}(T) - \frac{\partial y_v}{\partial t}(T), \phi(T) > - < \frac{\partial y_u}{\partial t}(0) - \frac{\partial y_v}{\partial t}(0), \phi(0) > \\ &+ < y_u(0) - y_v(0), \frac{\partial \phi}{\partial t}(0) > - < y_u(T) - y_v(T), \frac{\partial \phi}{\partial t}(T) > \end{aligned}$$

From boundary and initial conditions, we have

$$< (y_u(T), \frac{\partial y_u}{\partial t}(T)) - (y_v(T), \frac{\partial y_v}{\partial t}(T)), (\phi_1, -\phi_0) > \quad = \quad -\int_0^T < \phi, f >_{L^2(D)} (u(t) - v(t))$$

Hence as u and v steer the system (2-1) to (p^d, v^d) on ω we have $\mathcal{J}'(u^*)(u - v) = 0$. The uniqueness of u^* comes from the strict convexity of $\mathcal{J}$ and this establishes its optimality. ■

Remark 2.2
The problem (2-2) can be solved with the similar techniques when the system (2-1) is excited by an internal pointwise actuator.

3 Regional boundary control problem

In this section we explore an approach which leads to the calculation of the control which drives the system (2-1) to a given regional boundary state via the internal regional controllability techniques developed in the second section.
Assume that the system (2-1) is $\Gamma-$exactly controllable and consider the regional boundary control problem

$$\begin{cases} \min\limits_{u \in U_{ad}^{\Gamma}} \mathcal{J}(u) = \|u\|^2_{L^2(0,T)} \\ \\ U_{ad}^{\Gamma} = \{u \in U \text{ such that } \chi_{\Gamma}\gamma_0(y_u(T), \frac{\partial y_u}{\partial t}(T)) = (z_1^d, z_2^d)\} \end{cases} \tag{3-1}$$

where $(z_1^d, z_2^d) \in H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$ is the desired boundary state.
The problem consists in transferring the initial state (y_0, y_1) of the system (2-1) to (z_1^d, z_2^d) given on Γ at time T.

Let $(\overline{z}_1^d, \overline{z}_2^d)$ be any extension of the desired boundary state (z_1^d, z_2^d) in $H^{\frac{3}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ and consider the systems

$$\begin{cases} \Delta z_1 = 0 & \Omega \\ \\ z_1 = \overline{z}_1^d & \partial\Omega \end{cases} \tag{3-2}$$

and

$$\begin{cases} \Delta z_2 = 0 & \Omega \\ \\ z_2 = \overline{z}_2^d & \partial\Omega \end{cases} \tag{3-3}$$

where Δ is the Laplacien operator.
The system (3-2) (resp. (3-3)) has a unique solution $z_1 \in H^2(\Omega)$ (resp. $z_2 \in H^1(\Omega)$).
Let $(\omega_r)_{r>0}$ be a non increasing sequences of sets of Ω which converges to Γ in Kuratowski sense and assume that the system (2-1) is ω_r-approximately regionally controllable.

For $\omega_r \subset \Omega$, consider the internal controllability problem

$$\begin{cases} \min\limits_{u \in U_{ad}^{\Gamma}} \mathcal{J}(u) = \|u\|^2_{L^2(0,T)} \\ \\ U_{ad}^{\Gamma} = \{u \in U \text{ such that } (y_u(T), \dfrac{\partial y_u}{\partial t}(T)) = (z_1, z_2) \text{ on } \omega_r\} \end{cases} \tag{3-4}$$

where z_1 (resp. z_2) is solution of (3-2), (3-3).
There exists a sequence of controls $(u_r)_r$ solution of the internal regional controllability problem (3-4) on ω_r using similar techniques to those used for solving the problem (2-2). And after the Remarks 1.3, $(u_r)_r$ steers the system (2-1) to the desired state (z_1^d, z_2^d) on Γ at time T.

It is optimal for (z_1, z_2) on ω_r but isn't it for (z_1^d, z_2^d) on Γ and we have the result

Proposition 3.1
If the system (2-1) is ω_r-approximately regionally controllable, then the sequence $(u_r)_r$ converges strongly to u^* solution of the problem (3-1).

Proof
Consider the problem (3-1) and let

$$\mathcal{U}_{ad} = \left\{ u \in U \text{ such that } (y_u(T), \frac{\partial y_u}{\partial t}(T)) = (z_1, z_2) \text{ on } \bar{\Omega} \right\}$$

For all $r > 0$, we have $\mathcal{U}_{ad} \subset \mathcal{U}_{ad}^r$. Then

$$\min_{u \in \mathcal{U}_{ad}^r} \mathcal{J}(u) \leq \min_{u \in \mathcal{U}_{ad}} \mathcal{J}(u)$$

ie $\| u_r \| \leq \| v^* \|$, where u_r is the unique solution of (3-4) and v^* is the unique solution of $\min\limits_{v \in \mathcal{U}_{ad}} \mathcal{J}(v)$.

One deduces that the sequence $(u_r)_r$ is bounded in the reflexive space U, then there exists a subsequence $(u_{r_k})_k$ which converges weakly to u^* in U.

1- The strict increasing sequence $(\mathcal{U}_{ad}^r)_r$ converges to $\bigcup\limits_{r>1} \mathcal{U}_{ad}^r = \mathcal{U}_{ad}^{\Gamma}$ in de Kuratowski's sense. Indeed, we have $\bigcup\limits_{r>1} \mathcal{U}_{ad}^r \subset \mathcal{U}_{ad}^{\Gamma}$ and let $u \in \mathcal{U}_{ad}^{\Gamma}$, then $\chi_{\Gamma}\gamma_0 H u = (z_1^d, z_2^d)$. But, as $r \longrightarrow +\infty$, the decreasing sequence $(\omega_r)_r$ converges to $\bigcap\limits_{r \geq 1} \bar{\omega}_r = \Gamma$ in Kuratowski's sense. Then u steers the system (2-1) to (z_1^d, z_2^d) on $\lim\limits_{r \longrightarrow +\infty} \omega_r = \Gamma$. Hence, $u \in \bigcup\limits_{r>1} \mathcal{U}_{ad}^r$ and $\mathcal{U}_{ad}^r$ converges to $\mathcal{U}_{ad}^{\Gamma}$ in Kuratowski's sense. that is to say, for all $v \in \mathcal{U}_{ad}^{\Gamma}$, there exists $v_r \in \mathcal{U}_{ad}^r$ which converges strongly to v.

2- Suppose that $(u_r)_r$ has at least two adherent point u_1^* and u_2^*, then there exists two subsequences $(u_{r_{k_1}})_{k_1}$ and $(u_{r_{k_2}})_{k_2}$ of $(u_r)_r$ such that $u_{r_{k_1}}$ (resp. $u_{r_{k_2}}$) converges weakly to u_1^*

(resp. u_2^*).
We have $\mathcal{J}(u_r) \leq \mathcal{J}(v_r)$ which gives $\mathcal{J}(u_{r_{k_1}}) \leq \mathcal{J}(v_{r_{k_1}})$.
Then, $\underline{\lim}\mathcal{J}(u_{r_{k_1}}) \leq \underline{\lim}\mathcal{J}(v_{r_{k_1}})$ and $\mathcal{J}(u_1^*) \leq \underline{\lim}\mathcal{J}(u_{r_{k_1}}) \leq \underline{\lim}\mathcal{J}(v_{r_{k_1}})$. Therefore, $\mathcal{J}(u_1^*) \leq \mathcal{J}(v)\ \forall\ v \in \mathcal{U}_{ad}^{\Gamma}$. Thus, u_1^* is solution of the problem (3-1) and by similar techniques, u_2^* is also solution of (3-1). Consequently $u_1^* = u_2^* = u^*$.
The sequence $(u_r)_r$ is bounded in a reflexive space U and has one adherent point u^* for the weak topology, then u_r converges weakly to u^*.

3- Let $l = \lim_{r \longrightarrow +\infty} \| u_r \|$, we prove that $l = \| u^* \|$.
For all $v \in U_{ad}^{\Gamma}$, there exists $v_r \in U_{ad}^{r}$ which converges to v and we have
$\| u^* \| \leq l \leq \| v \|$ for all $v \in U_{ad}^{\Gamma}$.
Hence, $\| u^* \| \leq l \leq \min_{v \in U_{ad}^{\Gamma}} \| v \|$ and finally $l = \| u^* \|$. Since u_r^* converges weakly to u^* and $\| u_r \|$ converges to $\| u^* \|$, then u_r converges strongly to u^* (see[1]).■

4 Numerical approach

In this section we give a numerical approach which gives explicit formulas for ϕ_0, ϕ_1 and the optimal control which steers the system (2-1) to the desired state (p^d, v^d) on ω or Γ at time T. This approach is developed for one dimensional system excited by one internal zone actuator.

Consider the system

$$\left\{ \begin{array}{ll} \dfrac{\partial^2 y(x,t)}{\partial t^2} - \dfrac{\partial^2 y(x,t)}{\partial x^2} = \chi_D f(x) u(t) &]0,a[\times]0,T[\\ y(x,0) = y_0(x), \quad \dfrac{\partial y}{\partial t}(x,0) = y_1(x) &]0,a[\end{array} \right. \tag{4-1}$$

and let take $T = 2ma$, for $m \in I\!N^*$ and $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$.
We have seen that the problem (2-2) turns up to solve the equation (2-9) which is equivalent to solve the minimization problem

$$\inf_{(\phi_1,\phi_0)\in\widehat{G}} R(\phi_1, \phi_0) \tag{4-2}$$

where R is given by

$$\begin{array}{rcl} R(\phi_1,\phi_0) & = & \dfrac{1}{2}\displaystyle\int_0^T < \phi, f >^2_{L^2(D)} dt + < \psi_0(T), \phi_1 > \\ & - & < \dfrac{\partial \psi_0}{\partial t}(T), \phi_0 > - < p^d, \phi_1 > + < v^d, \phi_0 > \end{array}$$

Using Green formula we obtain

$$\begin{array}{rcl} R(\phi_1,\phi_0) & = & \dfrac{1}{2}\displaystyle\int_0^T < \phi, f >^2_{L^2(D)} dt + < y_0, \dfrac{\partial \phi}{\partial t}(0) > \\ & - & < y_1, \phi(0) >< p^d, \dfrac{\partial \phi}{\partial t}(T) > + < v^d, \phi(T) > \end{array}$$

Let $w_i(x) = \sqrt{\frac{2}{a}}\sin(i\pi\frac{x}{a}),\ \lambda_i = -\frac{(i\pi)^2}{a^2}, i \in I\!N^*$ be the eigenfunctions and the eigenvalues of $\frac{\partial^2}{\partial x^2}$. And $(\phi(T), \frac{\partial\phi}{\partial t}(T)) = (\phi(0), \frac{\partial\phi}{\partial t}(0)), (T = 2ma)$ the problem (4-2) turns up to minimize the functional R with respect to $(\phi(0), \frac{\partial\phi}{\partial t}(0))$ and is given by

$$\begin{aligned} R(\phi(0), \frac{\partial\phi}{\partial t}(0)) &= \sum_{j=1}^{\infty}\frac{T}{4} < \phi(0), w_j >^2 < f, w_j >^2_{L^2(D)} - < y_1, w_j >< \phi(0), w_j > \\ &+ < \phi(0), w_j >< v^d, w_j > \\ &+ \sum_{j=1}^{\infty}\frac{-T}{4\lambda_j} < \frac{\partial\phi}{\partial t}(0), w_j >^2 < f, w_j >^2_{L^2(D)} + < y_0, w_j >< \frac{\partial\phi}{\partial t}(0), w_j > \\ &- < p^d, w_j >< \frac{\partial\phi}{\partial t}(0), w_j > \end{aligned} \tag{4-3}$$

But the first term of (4-3) is independent of $< \frac{\partial\phi}{\partial t}(0), w_j >$ and the second term of (4-3) is independent of $< \phi(0), w_j >$. Hence we are leading to minimize with respect to $< \phi_0, w_j >_\omega$

$$\frac{T}{4} < \phi_0, w_j >^2_\omega < f, w_j >^2_{L^2(D)} - < y_1, w_j >< \phi_0, w_j >_\omega + < \phi_0, w_j >_\omega < v^d, w_j >$$

and to minimize with respect to $< \phi_1, w_j >_\omega$

$$-\frac{T}{4\lambda_j} < \phi_1, w_j >^2_\omega < f, w_j >^2_{L^2(D)} + < y_0, w_j >< \phi_1, w_j >_\omega - < p^d, w_j >< \phi_1, w_j >_\omega$$

which give

$$\begin{cases} < \phi_0, w_j >_\omega &= \frac{2}{T}\frac{(< y_1, w_j > \quad - \quad < v^d, w_j >)}{< f, w_j >^2_{L^2(D)}} \\ < \phi_1, w_j >_\omega &= \frac{2\lambda_j}{T}\frac{(< y_0, w_j > \quad - \quad < p^d, w_j >)}{< f, w_j >^2_{L^2(D)}} \end{cases} \tag{4-4}$$

Then we obtain

$$\phi_0 = \begin{cases} \frac{2}{T}\sum_{j=1}^{\infty}\frac{(< y_1, w_j > \quad - \quad < v^d, w_j >)}{< f, w_j >^2_{L^2(D)}} w_j(x) & x \in \omega \\ 0 & x \in \Omega\backslash\omega \end{cases} \tag{4-5}$$

$$\phi_1 = \begin{cases} \frac{2}{T}\sum_{j=1}^{\infty}\lambda_j\frac{(< y_0, w_j > \quad - \quad < p^d, w_j >)}{< f, w_j >^2_{L^2(D)}} w_j(x) & x \in \omega \\ 0 & x \in \Omega\backslash\omega \end{cases} \tag{4-6}$$

And the optimal control which steers the system (2-1) to the desired state (p^d, v^d) on ω at time T is given by

$$u^*(t) = -\sum_{i=1}^{+\infty}[< \phi_0, w_i > \cos\sqrt{-\lambda_i}t + \frac{< \phi_1, w_i >}{\sqrt{-\lambda_i}}\sin\sqrt{-\lambda_i}t] < w_i, f >_{L^2(D)} \tag{4-7}$$

We define a final error (depending on the subregion ω and the location of the actuator) by considering $\mathcal{E} = \|y_u(T) - p^d\|_{L^2(\omega)} + \|\frac{\partial y_u}{\partial t}(T) - v^d\|_{L^2(\omega)}$.
ϕ_0, ϕ_1 and u^* are given by the formulas (4-5), (4-6) and (4-7) truncated up to the order M. The good choice of M will be such that $\mathcal{E} \leq \varepsilon$. The general algorithm for computing the optimal control is the following.

Algorithm.

1. Choice of actuator location : $D \subset \Omega$, the subregion ω and the precision ε.
2. Choice of M (approximation order).
3. Computation of ϕ_0 and ϕ_1 by the formulas (4-5), (4-6) and u^* by (4-7).
4. Solving (4-1) gives $y_u(T)$ and $\frac{\partial y_u}{\partial t}(T)$.
5. If $\mathcal{E} \leq \varepsilon$ stop. Else return to 2.

5 Simulations

5.1 Internal subregion target

In this section we give a numerical example and the obtained results are related to the choice of the subregion, the desired state and the actuator location. Consider the one dimensional system excited by a pointwise actuator.

$$\begin{cases} \frac{\partial^2 y(x,t)}{\partial t^2} = \frac{\partial^2 y(x,t)}{\partial x^2} + u(t)\delta(x-b) &]0,1[\times]0,T[\\ y(0) = 0, \quad \frac{\partial y}{\partial t}(0) = 0 &]0,1[\\ y(0,t) = y(1,t) = 0 &]0,T[\end{cases} \tag{5-1}$$

We take $T = 2$, $\omega =]0.4, 0.64[$, $b = 0.82$ and the desired state is given by $p^d = A_1 sin(\pi x)$ and $v^d = (1 + B_1)sin(\pi x)$, where A_1 and B_1 are chosen for numerical considerations (in order to give the desired state with a reasonable amplitude). The different wave equations are solved by Newmark's method, the coefficient of which are chosen such that the numerical schema is stable. Applying the previous algorithm we obtain:

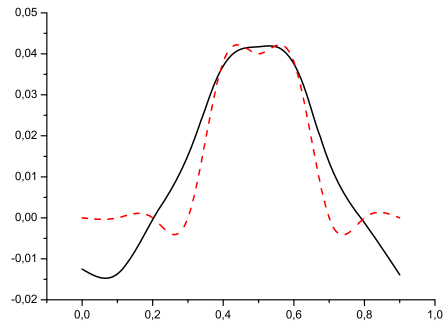

Figure 1: Desired and final position on ω.

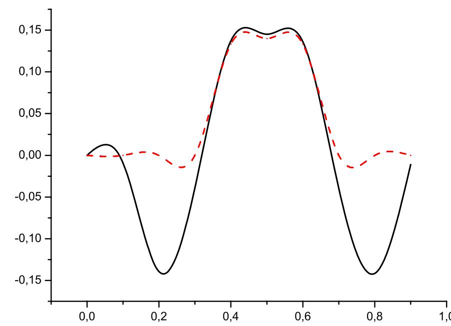

Figure 2: Desired and final speed on ω.

Figure 1 and figure 2 show how the reached position (solid lines) (resp. speed) is very close to the desired position (dotted lines) (resp. speed) on ω. The desired state is obtained with error $\mathcal{E} = 6.84 \times 10^{-5}$ and cost $\| u^* \|^2 = 1.4 \times 10^{-2}$.

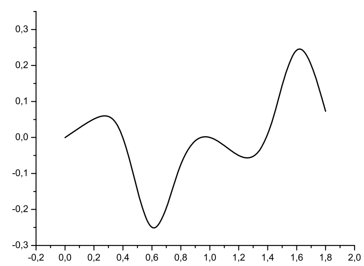

Figure 3: Evolution of the control function on [0, 2].

Relations amplitude-cost and amplitude-error

In this section we show numerically how both the cost *(table 1)* and error *(table 2)* grow with respect to the amplitude of the desired position.

Amplitude	$\|u^*\|^2_{L^2[0,T]}$
0.02	$2.34 * 10^{-3}$
0.05	$1.4 * 10^{-2}$
0.08	$2.46 * 10^{-2}$
0.09	$3.12 * 10^{-2}$

Table 1. Relation amplitude-cost.

Amplitude	$\mathcal{E}$
0.02	$6.39 * 10^{-5}$
0.05	$6.84 * 10^{-5}$
0.08	$4.96 * 10^{-4}$
0.09	$7.63 * 10^{-4}$

Table 2. Relation amplitude-error.

Relations subregion-cost and subregion-Error

In this section we show numerically how both the cost *(table 3)* and the error *(table 4)* grow with respect to the subregion area. It means that the larger the region is, the greater the cost and the error are.

Sous region ω	$\|u^*\|^2_{L^2[0,T]}$
$]0.48, 0.6[$	$3.58 * 10^{-4}$
$]0.4, 0.64[$	$1.4 * 10^{-2}$
$]0.32, 0.72[$	$4.21 * 10^{-2}$
$]0.24, 0.8[$	$5.15 * 10^{-2}$
$]0.08, 0.96[$	$5.50 * 10^{-2}$
$]0, 1[$	$5.58 * 10^{-2}$

Table 3. Relation subregion-cost.

Sous region ω	$\mathcal{E}$
$]0.48, 0.6[$	$4.7 * 10^{-5}$
$]0.4, 0.64[$	$6.84 * 10^{-5}$
$]0.32, 0.72[$	$3.87 * 10^{-4}$
$]0.24, 0.8[$	$6.07 * 10^{-4}$
$]0.08, 0.96[$	$8.03 * 10^{-3}$
$]0, 1[$	$1.89 * 10^{-2}$

Table 4. Relation subregion-error.

5.2 Boundary subregion target

Consider the two dimensional system excited by a pointwise actuator. The simulations are related to the choice of the boundary subregion, the desired boundary state (z_1^d, z_2^d) and the actuator location.

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2}(x,t) - (\dfrac{\partial^2 y}{\partial x_1^2}(x,t) + \dfrac{\partial^2 y}{\partial x_2^2}(x,t)) = \delta(x-b)u(t) &]0,1[\times]0,1[\times]0,T[\\ y(x,0) = 0, \quad \dfrac{\partial y}{\partial t}(x,0) = 0 &]0,1[\times]0,1[\\ \dfrac{\partial y}{\partial \nu} = 0 &]0,T[\end{cases} \tag{5-2}$$

where $b = (0.31, 0.61)$, $\Gamma = \{1\}\times]0,1[$, $z_1^d = A_1\cos(\pi x_1)\cos(\pi x_2)$ and $z_2^d = B_1\cos(\pi x_1)\cos(\pi x_2)$ be the considered boundary subregion and the desired state on Γ, where $A_1 = B_1 = 0.02$.

The eigenfunctions and eigenvalues of $-(\dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2})$ are given by

$$\begin{array}{ll} \omega_{ij} = 2\cos(i\pi x_1)\cos(j\pi x_2) & i,j \geq 0 \\ \lambda_{ij} = (i^2 + j^2)\pi^2 & i,j \geq 0 \end{array}$$

By taking $T = 3$, $\omega_r =]0.82, 1[\times]0, 1[$ (figure 4) and applying the previous algorithm we obtain

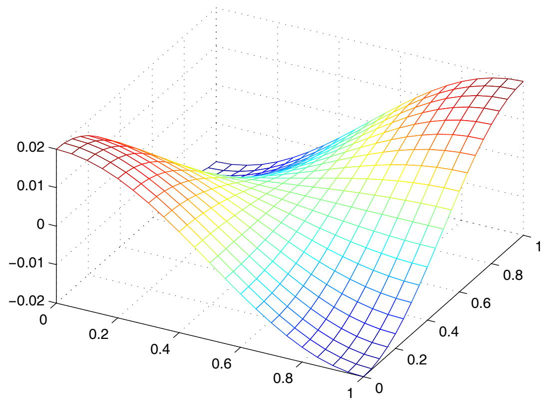

figure 4: Desired position on ω_r.

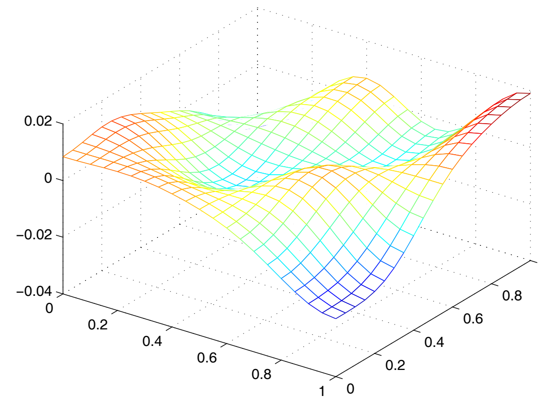

Figure 5: Reached position on ω_r.

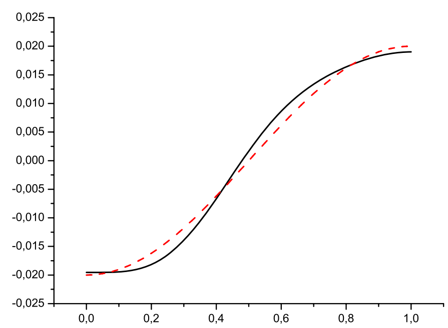

Figure 6: Desired (...) and reached (—) position on Γ.

Figure 6 shows how the reached position is very close to the desired position on Γ.

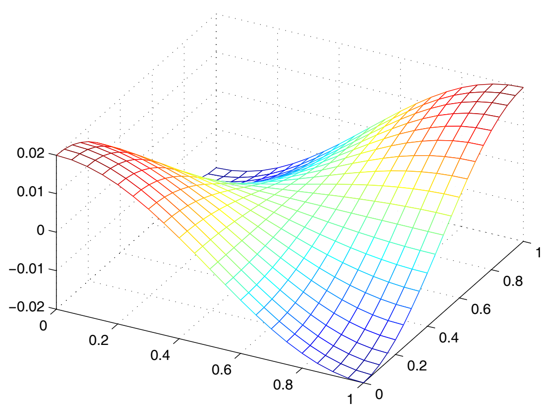

Figure 7: Desired speed on ω_r.

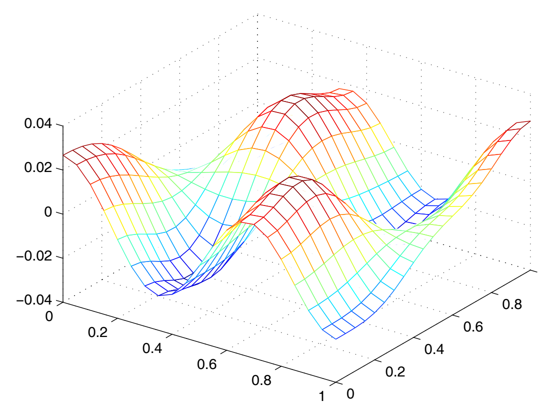

Figure 8: Reached speed on ω_r.

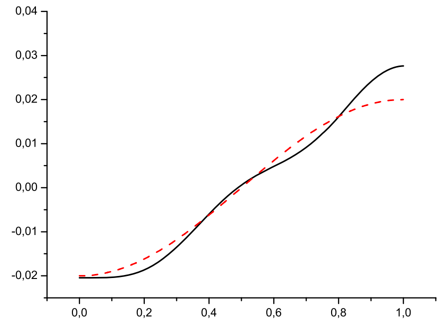

Figure 9: Desired (...) and reached (—) speed on Γ.

Figure 9 shows how the reached speed is very close to the desired speed on Γ. The desired state on Γ is obtained with error $\mathcal{E}_r = 4.17 \times 10^{-4}$ and cost $\mathcal{J}(u_r) = 1.59 \times 10^{-4}$.

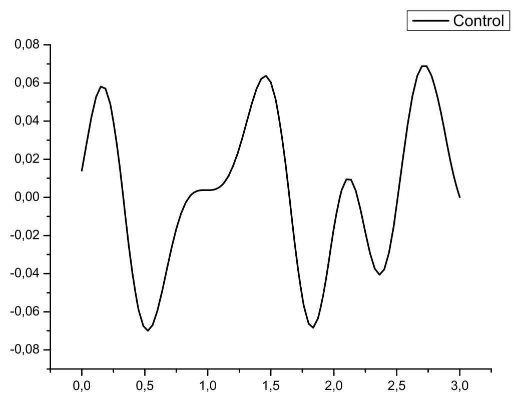

Figure 10: Evolution of the control function on the time interval [0, 3].

Part II. Regional controllability of semi linear systems

6 Problem statement

Consider the following semi linear hyperbolic system

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} + \mathcal{A}y + \mathcal{N}y = \chi_D f u & Q \\ y(x,0) = y_0(x), \ \dfrac{\partial y}{\partial t}(x,0) = y_1(x) & \Omega \end{cases} \tag{6-1}$$

where $\mathcal{A}$ is a second order elliptic linear operator, $\mathcal{N}$ is a nonlinear operator, $D \subset \Omega$, $f \in L^2(D)$, $u \in U = L^2(0,T)$ and $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$.

Let $z_u = (y_u, \dfrac{\partial y_u}{\partial t})$ denotes the solution of the equation (6-1) and assume that $z_u(T) \in \mathcal{E} = (L^2(\Omega))^2$.

In the sequel, we consider the regional controllability problem for system (6-1) excited by an internal zone actuator (f, D) stated as follows.

$$\left\{ \begin{array}{l} \text{For} \quad z^d = (z_1^d, z_2^d) \in L^2(\omega) \times L^2(\omega), \quad \text{find a control } u \in L^2(0,T) \text{ such that} \\ \\ \qquad\qquad y_u(T) = z_1^d \quad \text{and} \quad \dfrac{\partial y_u}{\partial t}(T) = z_2^d \quad \text{in} \quad \omega \end{array} \right. \tag{6-2}$$

Let denote $z = (y, \dfrac{\partial y}{\partial t})$, $A(z_1, z_2) = (z_2, \mathcal{A} z_1)$ for all $(z_1, z_2) \in D(A) = (H_0^1(\Omega) \cap H^2(\Omega)) \times L^2(\Omega)$, $Nz = (0, -\mathcal{N} z_1)$, $z_0 = (y_0, y_1)$ and $Bu = (0, \chi_D f u)$.
The system (6-1) may be written as

$$\left\{ \begin{array}{ll} \dfrac{\partial z}{\partial t} + Az = Nz + Bu & Q \\ z(0) = z_0 & \Omega \end{array} \right. \tag{6-3}$$

The associated linear system

$$\left\{ \begin{array}{ll} \dfrac{\partial z}{\partial t} + Az = Bu & Q \\ z(0) = z_0 & \Omega \end{array} \right. \tag{6-4}$$

and assume that $(-A)$ generates a strongly continuous semi-group $S(t)_{(t \geq 0)}$ on the state space $\mathcal{E}$.

Let $L(.)$ and G_ω be the operators defined by

$$L(t)z(.) = \int_0^t S(t-s)z(s)ds \qquad \text{and} \qquad G_\omega u = \chi_\omega L(T)Bu\ .$$

Consider now the function

$$\Phi(z)(.) = S(.)z_0 + L(.)Nz(.) + L(.)BG_\omega^\dagger[z^d - \chi_\omega S(T)z_0 - \chi_\omega L(T)Nz(.)] \tag{6-5}$$

where $G_\omega^\dagger = (G_\omega^* G_\omega)^{-1}\, G_\omega^*$ is the generalized inverse of G_ω.

For $z^*(.)$ a fixed point of (6-5) such that $[z^d - \chi_\omega S(T)z_0 - \chi_\omega L(T)Nz^*(.)] \in ImG_\omega$. It is easy to show that if (6-4) is ω-approximately regionally controllable, then the control

$$u^* = G_\omega^\dagger[z^d - \chi_\omega S(T)z_0 - \chi_\omega L(T)Nz^*(.)] \tag{6-6}$$

drives the system (6-1) to z^d at time T.
Here we will study two important situations, the case of asymptotically linear systems and the analytical ones.

6.1 Asymptotically linear case

Here we deal with the problem (6-2) when the system (6-1) is assumed to verify

$$\lim_{|s|\longrightarrow+\infty} \frac{\mathcal{N}(s)}{s} = \alpha \quad (\alpha \geq 0) \ \textit{and} \ \mathcal{N}^{'} \in L^{\infty}(I\!R) \quad (\text{see}[12]) \tag{6-7}$$

The approach we shall use is an extension of the Hilbert Uniqueness Method used to establish controllability in the linear case (see[5]) and the semi-linear case (see[11]).

Let G be the set

$$G = \{(\phi_1, -\phi_0) \in \mathcal{C}^{\infty}(\Omega) \times \mathcal{C}^{\infty}(\Omega) \quad \text{such that} \quad \phi_0 = \phi_1 = 0 \quad \text{on} \quad \Omega\backslash\omega\}$$

For $(\phi_1, -\phi_0) \in G$, the system

$$\begin{cases} \dfrac{\partial^2\phi}{\partial t^2} + \mathcal{A}\phi = 0 & Q \\ \phi(x,T) = \phi_0(x), \dfrac{\partial\phi}{\partial t}(x,T) = \phi_1(x) & \Omega \\ \phi(\xi,t) = 0 & \Sigma \end{cases} \tag{6-8}$$

has a unique solution $\phi \in C(0,T,H_0^1(\Omega)) \cap C^1(0,T,L^2(\Omega))$ ([5]). And the solution of the system (6-1) can be written as

$$y = \psi_0 + \psi_1 + \psi_2$$

where ψ_0 and ψ_1 are respectively solutions of the systems

$$\begin{cases} \dfrac{\partial^2\psi_0}{\partial t^2} + \mathcal{A}\psi_0 = 0 & Q \\ \psi_0(x,0) = y_0(x), \quad \dfrac{\partial\psi_0}{\partial t}(x,0) = y_1(x) & \Omega \\ \psi_0(\xi,t) = 0 & \Sigma \end{cases} \tag{6-9}$$

and

$$\begin{cases} \dfrac{\partial^2\psi_1}{\partial t^2} + \mathcal{A}\psi_1 \quad = - < \phi, f >_{L^2(D)} \chi_D f & Q \\ \psi_1(x,0) = 0, \quad \dfrac{\partial\psi_1}{\partial t}(x,0) = 0 & \Omega \\ \psi_1(\xi,t) = 0 & \Sigma \end{cases} \tag{6-10}$$

which satisfy (see[5])

$$\psi_0, \psi_1 \in C(0,T,H_0^1(\Omega)) \cap C^1(0,T,L^2(\Omega))$$

and there exists a positive constant β_1 such that

$$\|\psi_0\|_{L^\infty(0,T,H_0^1(\Omega))} + \|\frac{\partial\psi_0}{\partial t}\|_{L^\infty(0,T,L^2(\Omega))} \leq \beta_1\|(y_0,y_1)\|_{H_0^1(\Omega)\times L^2(\Omega)} \tag{6-11}$$

$$\|\psi_1\|_{L^\infty(0,T,H_0^1(\Omega))} + \|\frac{\partial\psi_1}{\partial t}\|_{L^\infty(0,T,L^2(\Omega))} \leq \beta_1\|(\phi_0,\phi_1)\|_{H_0^1(\Omega)\times L^2(\Omega)} \tag{6-12}$$

and ψ_2 is solution of the system

$$\begin{cases} \dfrac{\partial^2\psi_2}{\partial t^2} + \mathcal{A}\psi_2 + \mathcal{N}(\psi_0+\psi_1+\psi_2) = 0 & Q \\ \psi_2(x,0)=0, \dfrac{\partial \psi_2}{\partial t}(x,0)=0 & \Omega \\ \psi_2(\xi,t)=0 & \Sigma \end{cases} \tag{6-13}$$

and since $\mathcal{N}' \in L^\infty(I\!R)$, the mapping $\psi \longrightarrow \mathcal{N}(\psi_0(t)+\psi_1(t)+\psi(t))$ is lipschitz continuous from $L^2(\Omega) \longrightarrow L^2(\Omega)$ for a.e in $[0,T]$. Then (6-13) has a unique solution (see[7]):

$$\psi_2 \in C(0,T,H_0^1(\Omega)) \cap C^1(0,T,L^2(\Omega))$$

Let now define the nonlinear operator

$$\mu(\phi_1,-\phi_0) = \mathcal{P}(\psi_1(T), \frac{\partial\psi_1}{\partial t}(T)) + \mathcal{P}(\psi_2(T), \frac{\partial\psi_2}{\partial t}(T)) \tag{6-14}$$

Then, the problem of regional controllability of (6-1) turns up to solve the equation

$$\mu(\phi_1,-\phi_0) = \chi_\omega^*(z_1^d, z_2^d) - \mathcal{P}(\psi_0(T), \frac{\partial\psi_0}{\partial t}(T)) \tag{6-15}$$

The equation (6-15) is equivalent to the equation

$$\Lambda(\phi_1,-\phi_0) \quad = \chi_\omega^*(z_1^d, z_2^d) - \mathcal{P}(\psi_2(T), \frac{\partial\psi_2}{\partial t}(T)) - \mathcal{P}(\psi_0(T), \frac{\partial\psi_0}{\partial t}(T)) \tag{6-16}$$

For a large constant $\beta_2 > 0$ consider the set

$$\mathcal{G} = \{(\phi_1,-\phi_0) \in G \quad \text{such that} \quad \|(\phi_0,\phi_1)\|_{H_0^1(\Omega)\times L^2(\Omega)} \leq \beta_2 \|(\phi_1,-\phi_0)\|_G\}$$

then a solution of (6-15) is a fixed point of the nonlinear operator

$$\tilde{\mu}(\phi_1,-\phi_0) = \Lambda^{-1}\chi_\omega^*(z_1^d, z_2^d) - \Lambda^{-1}\mathcal{K}_\omega(\phi_1,-\phi_0) - \Lambda^{-1}\mathcal{P}(\psi_0(T), \frac{\partial\psi_0}{\partial t}(T)) \tag{6-17}$$

where $\mathcal{K}_\omega$ is the operator defined by

$$\begin{array}{cccc} \mathcal{K}_\omega : & \mathcal{G} & \longrightarrow & \widehat{G}^* \\ & (\phi_1,-\phi_0) & \longrightarrow & \mathcal{P}(\psi_2(T), \dfrac{\partial\psi_2}{\partial t}(T)) \end{array}$$

Theorem 6.1
If the system (6-4) is ω-approximately regionally controllable, then the equation (6-17) has a unique fixed point $(\phi_1,-\phi_0)$ and the control $u^*(t) = -\langle\phi(t), f\rangle_{L^2(D)}$ drives the system (6-1) to z^d in ω at time T, where ϕ is solution of the system (6-8).

Proof
$\psi_2 \in C(0,T,H_0^1(\Omega)) \cap C^1(0,T,L^2(\Omega))$ and there exists $\beta_3 > 0$ such that for all $t > 0$

$$\|\mathcal{P}(\frac{\partial\psi_2}{\partial t}(t), -\psi_2(t))\|_{\widehat{G}^*} \leq \beta_3 \left[\|\psi_2(t)\|_{H_0^1(\Omega)} + \|\frac{\partial\psi_2(t)}{\partial t}\|_{L^2(\Omega)} \right] \tag{6-18}$$

Then $\mathcal{P}(\dfrac{\partial\psi_2}{\partial t}, -\psi_2) \in C(0, T, \widehat{G}^*)$.
From ([5]), there exist $\varepsilon > 0$ and $\beta_4 > 0$ such that

$$\|\psi_2\|_{L^\infty(0,T,H_0^1(\Omega))} + \|\frac{\partial\psi_2}{\partial t}\|_{L^\infty(0,T,L^2(\Omega))} \leq \varepsilon(\|(y_0, y_1)\|_{H_0^1(\Omega)\times L^2(\Omega)} + \|(\phi_0, \phi_1)\|_{H_0^1(\Omega)\times L^2(\Omega)}) + \beta_4$$

Moreover, since $(\phi_1, -\phi_0) \in \mathcal{G}$, then for all $t > 0$

$$\|\mathcal{P}(\frac{\partial\psi_2}{\partial t}(t), -\psi_2(t))\|_{\widehat{G}^*} \leq \varepsilon\left[\|(y_0, y_1)\|_{H_0^1(\Omega)\times L^2(\Omega)} + \beta_2\|(\phi_1, -\phi_0)\|_G + \frac{\beta_4}{\varepsilon}\right] \tag{6-19}$$

Applying (6-19) with $\varepsilon = [2\beta_2\|\Lambda^{-1}\|_{\mathcal{L}(G^*,G)}]^{-1}$ and for some constant $\beta_5 > 0$, we have

$$\begin{aligned}\|\tilde{\mu}(\phi_1, -\phi_0)\|_G &\leq \|\Lambda^{-1}\mathcal{K}_\omega(\phi_1, -\phi_0)\|_G + \|\Lambda^{-1}\chi_\omega^*(z_1^d, z_2^d) - \Lambda^{-1}\mathcal{P}(\psi_0(T), \frac{\partial\psi_0}{\partial t}(T))\|_G \\ &\leq \frac{1}{2}\|(\phi_1, -\phi_0)\|_G + \beta_5\end{aligned}$$

Moreover, from (6-18) and (6-19), $\mathcal{K}_\omega$ is a compact operator, then $\tilde{\mu}$ is also compact and there exists $M \geq 2\beta_5$ such that

$$\|\tilde{\mu}(\phi_1, -\phi_0)\|_G \leq M \qquad \forall(\phi_1, -\phi_0) \in G \text{ such that } \|(\phi_1, -\phi_0)\|_G \leq M$$

Hence, by applying the Schauder's fixed point theorem (see[8]) the operator (6-17) has at least one fixed point and the proof is completed. ∎

Remark 6.2

1. *The approach used here is a natural generalization of the one developed for the linear case. Indeed, when $\mathcal{N} = 0$, the operator μ coincides with the isomorphism Λ given by (2-8)*

2. *The problem (6-2) can be solved by similar techniques when the system is excited by a boundary actuator.*

6.2 Analytical case

In the following, we consider the problem (6-2) for the system (6-3) with $z_0 = 0$ and assume that $(-A)$ generates an analytic semi-group $S(t)_{(t\geq 0)}$ on the state space $\mathcal{E}$.
Moreover, let $A_1 = A + aI$ where a is a real such that $Re\sigma(A_1) > \delta > 0$ while $Re\sigma(A_1)$ indicates the real part of the spectrum of A_1. Then for $0 \leq \alpha < 1$, $\mathcal{E}^\alpha = D(A_1^\alpha)$ defines a dense Banach space on $\mathcal{E}$ endowed with the graph norm

$$\|.\|_{\mathcal{E}^\alpha} = \|A_1^\alpha(.)\|_{\mathcal{E}}$$

and $\|S(t)\|_{\mathcal{L}(\mathcal{E},\mathcal{E}^\alpha)} = c.t^{-\alpha}\exp(a-\delta)t = g_1(t)$ ([8]).
Assume that $g_1 \in L^q(0,T)$, $q \geq 1$ and let $r, s \geq 1$ be such that $\frac{1}{q} = 1 + \frac{1}{r} - \frac{1}{s}$ and that N is well defined from $L^r(0,T;\mathcal{E}^\alpha) \longrightarrow L^s(0,T;\mathcal{E})$ verifying

$$\left\{\begin{array}{ll} N(0) = 0 & \\ \|Nx - Ny\|_{L^s(0,T;\mathcal{E})} & \leq k(\|x\|,\|y\|)\|x-y\|_{L^r(0,T;\mathcal{E}^\alpha)} \quad ; \forall x,y \in L^r(0,T;\mathcal{E}^\alpha) \\ \text{with} \quad k : I\!\!R^+ \times I\!\!R^+ & \longrightarrow I\!\!R^+ \text{ such that } \lim\limits_{\theta_1,\theta_2\to 0} k(\theta_1,\theta_2) = 0 \end{array}\right. \tag{6-20}$$

This hypothesis are verified by many important class of semi linear hyperbolic systems. For example the equation governing the flow of neutrons in a nuclear reactor

$$\left\{\begin{array}{l} y' - Ky'' - ay = -by^2 + f \\ \\ y(t) = 0 \quad \text{sur} \quad [0,1] \\ \\ y(0) = 0 \end{array}\right. \tag{6-21}$$

whith $f(t), y(t) \in L^2([0,1]), 0 \leq t \leq T, k, a, b > 0$. The operators A and N corresponding are

$$\begin{array}{llllcll} A: & D(A) & \longrightarrow & X & D(A) & = & H_0^1([0,1]) \cap H^2([0,1]) \\ & y & \longrightarrow & \frac{-1}{K}y' + \frac{a}{K}y & X & = & L^2([0,1]) \\ N: & X^{\frac{1}{2}} & \longrightarrow & X & X^{\frac{1}{2}} & = & D(A^{\frac{1}{2}}) = H_0^1([0,1]) \\ & y & \longrightarrow & \frac{b}{K}y^2 & & & \end{array}$$

The assumption is satisfied with $r = 2,\ s = 1$ and $k(\theta_1,\theta_2) = \frac{b}{K}(\theta_1+\theta_2)$. Various examples are given and discussed in (see[4]).
Consider now the functions

$$\tilde{\Phi}(z,u) = L(.)Nz + L(.)Bu \tag{6-22}$$

and

$$\tilde{\psi}_\omega(z^d,u) = G_\omega^\dagger(z^d - \chi_\omega L(T)Nz_u) \tag{6-23}$$

In the following ImG_ω is endowed with the semi-norm:

$$\|z^d\|_{ImG_\omega} = \|G_\omega^\dagger z^d\|_{L^2(0,T)} \tag{6-24}$$

then we have the following result.

Theorem 6.3
We assume that the system (6-4) is ω-approximately regionally controllable, the hypothesis (6-20) holds and

$$\|L(.)Bu\|_{L^r(0,T,\mathcal{E}^\alpha)} \leq \beta\|u\|_{L^2(0,T)} \tag{6-25}$$

$$\|\chi_\omega S(.)\|_{\mathcal{L}(\mathcal{E},ImG_\omega)} = g_2 \in L^d(0,T) \quad \textit{such that } \frac{1}{d} + \frac{1}{s} = 1 \tag{6-26}$$

Then

1. *There exist $m > 0$ and $\rho > 0$, such that*
   $\forall z^d \in B(0,\rho) \subset ImG_\omega$*, there exists* $u^* \in B(0,m)$ *the unique solution of the problem (6-2)*

2. *The mapping* $z^d \longrightarrow u^*(z^d)$ *from* $B(0,\rho) \longrightarrow L^2(0,T)$ *is Lipschitz.*

Proof

1- Since the system (6-4) is ω-approximately controllable, then $G_\omega^\dagger$ is injective and consequently (6-24) is a norm.

2- We have $\lim\limits_{\theta_1,\theta_2 \longrightarrow 0} k(\theta_1,\theta_2) = 0$, then there exists $\gamma > 0$ such that

$$C_1 := \quad \|g_2\|_{L^q(0,T)} \sup_{\theta_1,\theta_2<\gamma} k(\theta_1,\theta_2) < C_2$$

and

$$C_2 := \quad (\beta\|g_2\|_{L^d(0,T)} + \|g_1\|_{L^q(0,T)}) \sup_{\theta_1,\theta_2<\gamma} k(\theta_1,\theta_2) < 1$$

But, it is well known (see[4]) that there exists

$$m := \frac{\gamma}{\beta}(1 - \|g_1\|_{L^q(0,T)} \sup_{\theta\leq\gamma} k(\theta,0)) \tag{6-27}$$

such that for all $u \in B(0,m)$, $\tilde{\Phi}(.,u)$ has only one fixed point $z \in B(0,\gamma) \subset L^r(0,T,\mathcal{E}^\alpha)$ solution of (6-3) and the mapping $u \longrightarrow z_u$ is Lipshitz with report $\dfrac{\beta}{1-C_1}$. Then , for $z^d \in ImG_\omega$ we have

$$\begin{aligned}
\|\tilde{\psi}_\omega(z^d,u) - \tilde{\psi}_\omega(z^d,v)\|_{L^2(0,T)} &= \|G_\omega^\dagger \chi_\omega L(T)(Nz_v - Nz_u)\|_{L^2(0,T)} \\
&= \|\chi_\omega L(T)(Nz_v - Nz_u)\|_{ImG_\omega} \\
&\leq \|g_2\|_{L^d(0,T)}\|Nz_v - Nz_u\|_{L^s(0,T,\mathcal{E})} \\
&\leq \frac{\beta}{1-C_1}\|g_2\|_{L^d(0,T)} \sup_{\theta_1,\theta_2<\gamma} k(\theta_1,\theta_2)\|u-v\|
\end{aligned}$$

Consequently

$$\|\tilde{\psi}_\omega(z^d,u) - \tilde{\psi}_\omega(z^d,v)\|_{L^2(0,T)} \leq C_3\|u-v\| \tag{6-28}$$

where

$$C_3 := \frac{\beta}{1-C_1}\|g_2\|_{L^d(0,T)} \sup_{\theta_1,\theta_2<\gamma} k(\theta_1,\theta_2) < 1$$

which shows that $\tilde{\psi}_\omega$ is contractor.

Also

$$\begin{aligned}
\|\tilde{\psi}_\omega(z^d,u)\| &= \|z^d - \chi_\omega L(T)Nz_u\| \leq \|z^d\| + \|\chi_\omega L(T)Nz_u\| \\
&\leq \|z^d\| + \|g_2\|_{L^d(0,T)} k(\|z_u\|,0)\|z_u\| \\
&\leq \|z^d\| + \|g_2\|_{L^d(0,T)} \sup_{\theta\leq\gamma} k(\theta,0)\gamma
\end{aligned}$$

Thus, if $u \in B(0,m)$ and $\|z^d\| \leq m - \|g_2\|_{L^d(0,T)} \sup_{\theta\leq\gamma} k(\theta,0)\gamma$, then $\tilde{\psi}_\omega(z^d,u) \in B(0,m)$.

From (6-27) we obtain

$$\|z^d\| \leq \frac{\gamma}{\beta}(1-(\|g_1\|_{L^q(0,T)}+\beta\|g_2\|_{L^d(0,T)}) \sup_{\theta\leq\gamma} k(\theta,0)) =: \rho \tag{6-29}$$

Consequently, if $z^d \in B(0,\rho) \subset ImG_\omega$, then $\tilde{\psi}_\omega(z^d,.)$ has a unique fixed point in $B(0,m)$ solution of the problem (6-2).

3- For $z^d, y^d \in B(0,\rho)$ we have

$$\begin{aligned} u^*(z^d)-u^*(y^d) &= \tilde{\psi}_\omega(z^d,u^*(z^d)) - \tilde{\psi}_\omega(y^d,u^*(y^d)) \\ &= \tilde{\psi}_\omega(z^d,u^*(z^d)) - \tilde{\psi}_\omega(z^d,u^*(y^d)) + \tilde{\psi}_\omega(z^d,u^*(y^d)) - \tilde{\psi}_\omega(y^d,u^*(y^d)) \end{aligned}$$

but

$$\begin{aligned} \|\tilde{\psi}_\omega(z^d,u^*(z^d)) - \tilde{\psi}_\omega(z^d,u^*(y^d))\| &\leq C_3\|u^*(z^d)-u^*(y^d)\| \\ \|\tilde{\psi}_\omega(z^d,u^*(y^d)) - \tilde{\psi}_\omega(y^d,u^*(y^d))\| &= \|z^d-y^d\| \end{aligned}$$

hence

$$\|u^*(z^d)-u^*(y^d)\| \leq \frac{1}{1-C_3}\|z^d-y^d\|$$

which shows that the mapping $z^d \longrightarrow u^*(z^d)$ from $B(0,\rho) \longrightarrow L^2(0,T)$ is Lipshitz. ■

Proposition 6.4 *The sequence of controls*

$$\left\{ \begin{array}{rcl} u_{n+1} &=& G_\omega^\dagger(z^d - \chi_\omega L(T)Nz_{u_n}) \\ u_0 &=& 0 \end{array} \right. \tag{6-30}$$

converges in $L^2(0,T)$ to u^* solution of the problem (6-2).

Proof

The proof is obtained using (6-28) and (6-24).

Remark 6.5

1. *We can consider the regional boundary problem*

$$\left\{ \begin{array}{l} \text{For } z^d = (z_1^d, z_2^d) \in H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) \\ \text{find a control } u \in L^2(0,T) \text{ such that} \\ \gamma_0(y_u(T), \dfrac{\partial y_u}{\partial t}(T)) = (z_1^d, z_2^d) \quad \text{on } \Gamma \end{array} \right. \tag{6-31}$$

2. *It can be solved by choosing ω_Γ a part of Ω, such that $\Gamma = \partial\Omega \cap \partial\omega_\Gamma$. And through the Remarks 1.3, the controls given by the Theorem 6.3 and the Proposition 6.4 ensures the controllability of the desired state (z_1^d, z_2^d) on the boundary subregion Γ.*

3. *In the semi-linear case there isn't necessary a sequence $(\omega_r)_r$, but just $\omega_\Gamma \in \Omega$ for solve the boundary problem (6-31). Because it doesn't expect the optimality of control.*

7 Numerical approach

Here we shall give a numerical approach that leads to the computation of the control solution of the problem (6-2) when the system (6-3) is analytic. From (6-30) the control verifies

$$G_\omega^\dagger(z^d - \chi_\omega z_{u_n}) = u_{n+1} - u_n \tag{7-1}$$

and the problem turns up to compute $G_\omega^\dagger$.

Since the system (6-4) is ω-approximately regionally controllable, then $(G_\omega^* G_\omega)$ is invertible and the operator $G_\omega^\dagger = (G_\omega^* G_\omega)^{-1} G_\omega^*$ is well defined and may be written as $G_\omega^\dagger y = \sum_{i\geq 1} \tilde{z}_i w_i(t)$, where $w_i(t) = \left(\frac{2}{T}\right)^{\frac{1}{2}} \sin\left(\frac{i\pi t}{T}\right)$, with $i \geq 1$ and the problem is to calculate the component $\tilde{z}_i$.

If Φ_n are the eigenfunctions of A, then the operator $G_\omega : L^2(0,T) \longrightarrow L^2(\omega) \times L^2(\omega)$ may be written as

$$G_\omega u = \begin{bmatrix} \displaystyle\sum_{n\geq 1} \frac{1}{n\pi} \langle \chi_D f, \Phi_n \rangle \langle u(.), \sin n\pi(T-.)\rangle \Phi_n|_\omega \\ \\ \displaystyle\sum_{n\geq 1} \langle \chi_D f, \Phi_n \rangle \langle u(.), \cos n\pi(T-.)\rangle \Phi_n|_\omega \end{bmatrix}$$

Hence, for all $v \in L^2(0,T)$ and $z = (z_1, z_2) \in L^2(\omega) \times L^2(\omega)$ we have

$$\begin{aligned} \langle G_\omega v, (z_1, z_2)\rangle &= \langle \sum_{n\geq 1} \frac{1}{n\pi} \langle \chi_D f, \Phi_n \rangle \langle \Phi_n, z_1\rangle_\omega \langle \sin n\pi(T-.), v(.)\rangle \\ &+ \sum_{n\geq 1} \langle \chi_D f, \Phi_n \rangle \langle \Phi_n, z_2\rangle_\omega \langle \cos n\pi(T-.), v(.)\rangle \end{aligned} \tag{7-2}$$

and $G_\omega^* : L^2(\omega) \times L^2(\omega) \longrightarrow L^2(0,T)$ is given by

$$\begin{aligned} G_\omega^*(z_1, z_2) &= \sum_{m\geq 1} \frac{1}{m\pi} \langle \chi_D f, \Phi_m \rangle \langle \Phi_m, \tilde{\chi}_\omega^* z_1\rangle_\Omega \sin m\pi(T-.) \\ &+ \sum_{m\geq 1} \langle \chi_D f, \Phi_m \rangle \langle \Phi_m, \tilde{\chi}_\omega^* z_2\rangle_\Omega \cos m\pi(T-.) \end{aligned} \tag{7-3}$$

Then, for $j \geq 1$

$$\begin{aligned} \langle G_\omega^* \left(z^d - \chi_\omega(y_u(T), \frac{\partial y_u}{\partial t}(T))\right), w_j\rangle &= \sum_{m\geq 1} \frac{\mathcal{R}(m)}{m\pi} \left[\langle \Phi_m, z_1^d\rangle_\omega - \langle \Phi_m, y_u(T)\rangle_\omega\right] \Theta(j,m) \\ &+ \sum_{m\geq 1} \mathcal{R}(m) \left[\langle \Phi_m, z_2^d\rangle_\omega - \langle \Phi_m, \frac{\partial y_u}{\partial t}(T)\rangle_\omega\right] \mathcal{D}(j,m) \\ &=: \mathcal{Y}_j \end{aligned} \tag{7-4}$$

and

$$\begin{aligned}\langle G_\omega^* G_\omega w_i, w_j \rangle &= \sum_{m\geq 1} \frac{1}{m\pi} \mathcal{R}(m) \sum_{n\geq 1} \left[\frac{\Pi(m,n)\mathcal{R}(n)\Theta(i,n)}{n\pi}\right] \Theta(j,m) \\ &+ \sum_{m\geq 1} \mathcal{R}(m) \sum_{n\geq 1} \left[\Pi(m,n)\mathcal{R}(n)\mathcal{D}(i,n)\right] \mathcal{D}(j,m) =: \tilde{A}_{ij}\end{aligned} \tag{7-5}$$

where

$$\begin{cases} \mathcal{R}(m) = \langle f, \Phi_m \rangle_D & ; \quad \Theta(i,n) = \langle w_i, \sin n\pi(T-.)\rangle \\ \\ \Pi(n,m) = \langle \Phi_n, \Phi_m \rangle_\omega & ; \quad \mathcal{D}(i,n) = \langle w_i, \cos n\pi(T-.)\rangle \end{cases}$$

$\tilde{z}_j$ are solutions of the system

$$\sum_{j\geq 1} \tilde{A_{ij}} \mathcal{Z}_j = \mathcal{Y}_i \quad ; \quad i = 1, 2, ... \tag{7-6}$$

where $(\tilde{A}_{ij})$ are given by (7-5) and $\mathcal{Y}_j$ by (7-4) and from (7-1) the control u_n^* is approximated by

$$u_{n+1}^* \simeq \sum_{i=1}^{M} \tilde{z}_i w_i + u_n^* \qquad M \in I\!N^* \tag{7-7}$$

Let $z_{u_n^*}$ be the solution of the system (6-3) excited by u_n^* and

$$\mathcal{Y}_j = \left(\langle G_\omega^*(z^d - \chi_\omega z_{u_{n+1}^*}(T)), w_j\rangle\right) \quad ; \quad (j \leq M) \tag{7-8}$$

and we obtain the following algorithm

Algorithm.

Step 1 :
- Let (z_1^d, z_2^d), the region ω and the actuator location D.
- Choose the truncation order M.

Step 2 : Repeat
- Solve the system (7-6).
- Computation of the control u_{n+1}^* by the formula (7-7).
- Solve the system (6-3).
- Solve (7-8) to obtain $\mathcal{Y}_j$. Until $(\|z^d - \chi_\omega z_{u_{n+1}^*}(T)\| < \varepsilon)$.

Step 3 : Let $z^*(T)|_\omega = \chi_\omega z_{u_{n+1}^*}(T)$ the estimated state in ω.

8 Simulations

In this part, we give a numerical illustrating example and the simulations are related to the choice of the subregion, the desired state and the actuator location.

8.1 Internal subregion target

Consider the one dimensional system excited by a zone actuator located in D.

$$\begin{cases} \dfrac{\partial^2 y(x,t)}{\partial t^2} - \dfrac{\partial^2 y(x,t)}{\partial x^2} + \displaystyle\sum_{i=1}^{m} |\langle y(t), w_i\rangle| \langle y(t), w_i\rangle w_i(x) + \chi_D(x)\mathbb{1}(x)u(t) = 0 &]0,1[\times]0,T[\\ y(x,0) = 0, \quad \dfrac{\partial y}{\partial t}(x,0) = 0 &]0,1[\\ y(0,t) = y(1,t) = 0 &]0,T[\end{cases} \tag{8-1}$$

where $w_i(x) = \sqrt{2}\sin(i\pi x)$, $i \in I\!N^*$ and $D =]0,52; 0,63[$.

Example 1

Let $z_1^d = A_1 \sin(\pi x)$ and $z_2^d = A_1(1 + B_1)\sin(\pi x)$ be the desired state in $\omega =]0.41, 0.79[$.
For numerical considerations A_1 and B_1 are chosen in order to give a desired state with a reasonable amplitude. By taking $A_1 = B_1 = 0.004$, $T = 1$ and applying the previous algorithm, the desired state (dotted lines) is obtained with error

$$\|y_{u^\star}(T) - z_1^d\|_{L^2(\omega)} + \|\frac{\partial y_{u^\star}}{\partial t}(T) - z_2^d\|_{L^2(\omega)} = 1.18 \times 10^{-4} \text{and cost } \|u^\star\|_{L^2(0,T)} = 1.17 \times 10^{-5}.$$

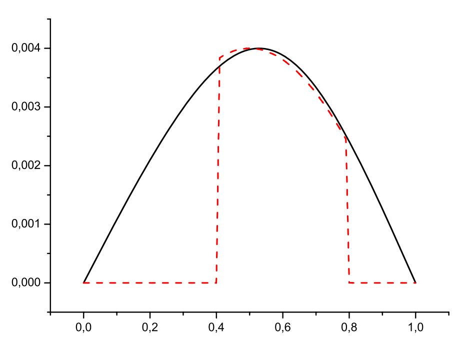

Figure 11: Desired and final position on ω.

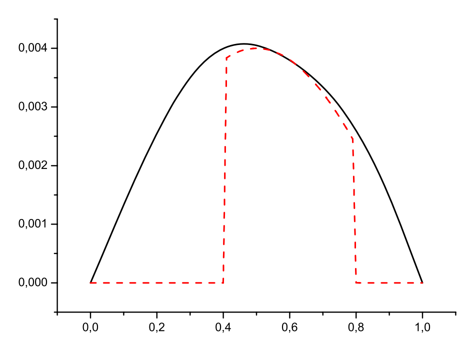

Figure 12: Desired and final speed on ω.

Example 2

Let $z_1^d = A_1 \times x(x-1)$ and $z_2^d = B_1 \times x(x-1)$ be the desired state in $\omega =]0.41, 0.79[$, $A_1 =$

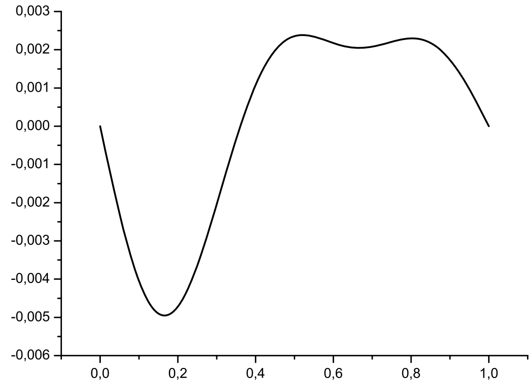

Figure 13: Evolution of the control function on the time interval [0,1].

$B_1 = 0.01$ and $M = 4$. Taking $T = 1$ the desired state (dotted lines) is obtained with error $\|y_{u^\star}(T) - z_1^d\|_{L^2(\omega)} + \|\dfrac{\partial y_{u^\star}}{\partial t}(T) - z_2^d\|_{L^2(\omega)} = 5.76 \times 10^{-5}$ and cost $\|u^\star\|_{L^2(0,T)} = 5.21 \times 10^{-6}$.

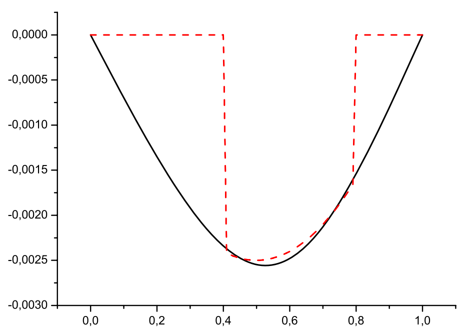

Figure 14: Desired and final position on ω.

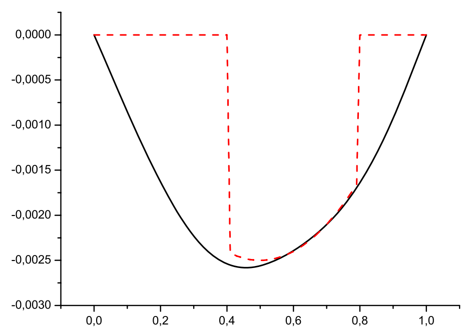

Figure 15: Desired and final speed on ω.

8.2 Boundary subregion target

Consider the two dimensional system excited by a pointwise actuator.

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} + \mathcal{A}y + \displaystyle\sum_{i,j=1}^{M} |\langle y(t), \Phi_{ij}\rangle| \langle y(t), \Phi_{ij}\rangle \Phi_{ij}(x) + \delta(x-b)u(t) = 0 &]0,1[\times]0,1[\times]0,T[\\ y(x,0) = y_0(x), \dfrac{\partial y}{\partial t}(x,0) = y_1(x) & (]0,1[\times]0,1[) \\ \dfrac{\partial y}{\partial \nu_A}(\xi, t) = 0 &]0,T[\end{cases}$$

Let $\Gamma = \{0\}\times]0,1[$ be the considered subregion, $z_1^d(x_1,x_2) = A(2x_1-1)x_2(x_2-1)$ and $z_2^d(x_1,x_2) = B(2x_2-1)(1-x_1)(x_1-1)$ the desired state on Γ.

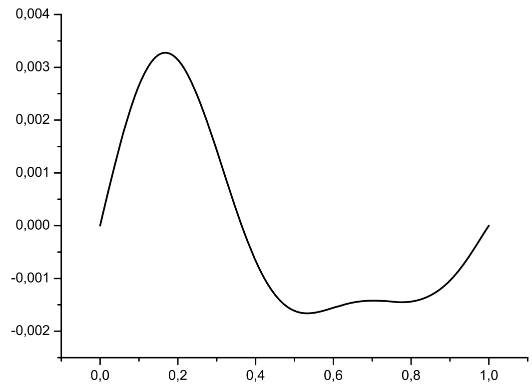

Figure 16: Evolution of the control function on the time interval [0,1].

The eigenfunctions and eigenvalues of $\mathcal{A} = -(\dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2})$ are given by

$$\Phi_{ij} = 2\cos(i\pi x_1)\cos(j\pi x_2); \;\; \lambda_{ij} = (i^2 + j^2)\pi^2; \;\; i, j \geq 0$$

By taking $A = B = 0.01$, $b = (0.2, 0.64)$, $T = 3$, $\omega_\Gamma =]0, 0.25[\times]0, 1[$ (figure 18), and applying the previous algorithm we obtain

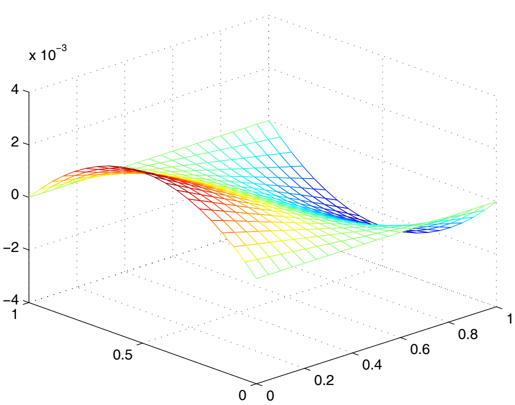

Figure 17: Desired position on ω_Γ.

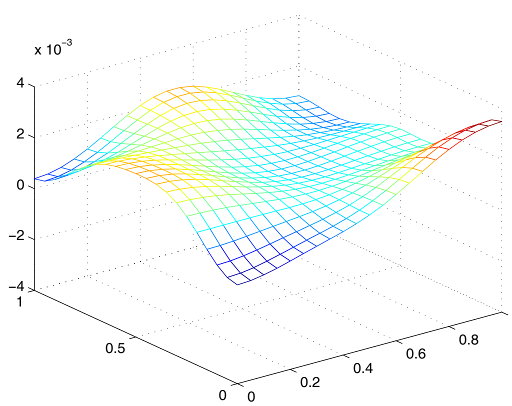

Figure 18: Reached position on ω_Γ.

Figure 19 shows how the reached position is very close to the desired position on Γ.

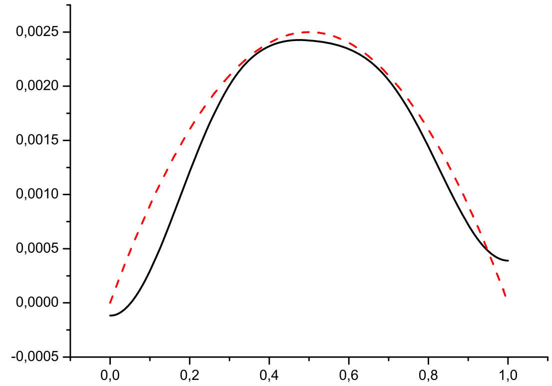

Figure 19: Desired (...) and reached (—) position on Γ.

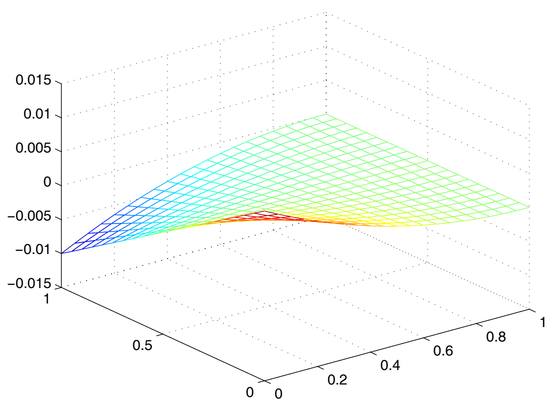

Figure 20: Desired speed on ω_Γ.

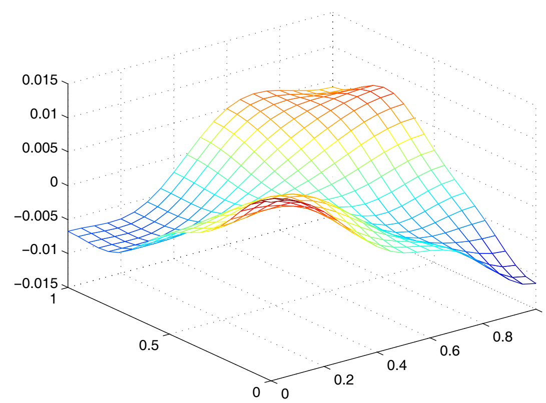

Figure 21: Reached speed on ω_Γ.

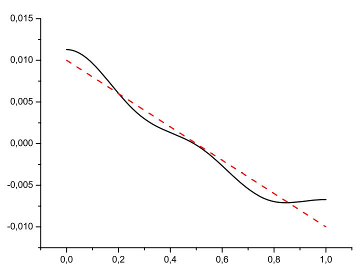

Figure 22: Desired (...) and reached (—) speed on Γ.

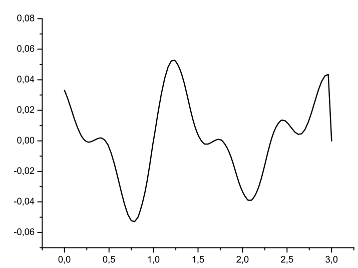

Figure 23: Control function.

Figure 22 shows how the reached speed is very close to the desired speed on Γ.

The desired state on Γ is obtained with error

$$\|y_{u^\star}(T) - z_1^d\|_{L^2(\Gamma)} + \|\frac{\partial y_{u^\star}}{\partial t}(T) - z_2^d\|_{L^2(\Gamma)} = 2.62 \times 10^{-4} \text{ and cost } \|u^\star\|_{L^2(0,T)} = 6.86 \times 10^{-4}.$$

9 Conclusion

This work deals with regional controllability for hyperbolic systems. Both linear and semi linear cases are examined and the developed approaches allow the optimal control which steers such a system to a desired state defined only on a subregion of the evolution system domain and we examine the two situations internal subregion or a boundary one. Also the given simulations show the efficiency of the derived numerical approaches, illustrate perfectly the established results and lead to some conjectures.

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Received: June, 2009