Int. Journal of Math. Analysis, Vol. 4, 2010, no. 42, 2089 - 2099

E-Valued Borel Exceptional Values of Meromorphic Functions

Subhas S. Bhoosnurmath and Veena L. Pujari

Department of Mathematics
Karnatak University
Dharwad-580003, India
ssbmath@gmail.com
vlpujari@gmail.com

Abstract

In this paper,we deal with the Borel exceptional values for the E-valued meromorphic functions and we introduce the order of multiplicity for Borel exceptional values and obtain a basic inequality.From this we deduce several well-known result and also obtain some new significant results in this direction.The results hold good for finite and infinite order meromorphic functions.

Mathematics Subject Classification: 30D35, 30D30

Keywords: Meromorphic function, Nevanlinna theorems , exceptional values, Banach space

1. Introduction and main result

In 1982 , H.J.W.Ziegler[8] succeeded in extending the Nevanlinna's first and second main theorem and Nevanlinna deficiency relation to finite dimensional spaces. Later C.G.Hu [1] , proved the Nevanlinna's theorems in infinite dimensional Banach space using the concept of compact projection.

In [3] , H.S.Gopalkrishna and S.S.Bhoosnurmath obtained much stronger results than those of Valiron [7] for meromorphic functions of all orders (finite or infinite) with the usual definitions of order and evB only.

In this paper , we define the Borel exceptional values for the E-valued meromorphic functions and extend several classical results of Nevanlinna theory to E-valued meromorphic functions.

Assume that E be a complex Banach space and $\mathbb{C}$ is the complex plane. Let $D = C_r = \{z/\,|z| < r\}$.

Definition 1.1 Let $f : D \to E$ be a meromorphic function and $a \in E \cup \{\infty\}$, if k is a positive integer , we denote by $\overline{n}_k(r,a,f)$, the number of distinct zeros of order $\leq k$ of $f-a$ in $|z|\leq r$ (each zero is counted only once irrespective of its multiplicity). $\overline{N}_k(r,a,f)$ is defined interms of $\overline{n}_k(r,a,f)$ as,

$$\overline{N}_k(r,a,f)=\int_0^r \frac{\overline{n}_k(r,a,f)-\overline{n}(o,a,f)}{t}dt+\overline{n}_k(0,a,f)logr.$$

Definition 1.2 For an E-valued meromorphic function f and $a\in E\cup\{\infty\}$, we define

$$\overline{\rho}_k(a,f)=\limsup_{r\to\infty}\frac{log^+\left[V(r,a,f)+\overline{N}_k(r,a,f)\right]}{logr}$$

$$\overline{\rho}(a,f)=\limsup_{r\to\infty}\frac{log^+\left[V(r,a,f)+\overline{N}(r,a,f)\right]}{logr}$$

$$\rho(a,f)=\limsup_{r\to\infty}\frac{log^+\left[V(r,a,f)+N(r,a,f)\right]}{logr}$$

where

$$V(r,a,f)=\frac{1}{2\pi}\int_{C_r} log\left|\frac{r}{\xi}\right|\Delta log\left\|f(\xi)-a\right\| d\sigma\wedge d\tau.$$

Definition 1.3 If f is an E-valued meromorphic function of order ρ $(0\leq\rho\leq\infty)$,$a\in E\cup\{\infty\}$ and k is a positive integer , we say that a is an
i) E-valued evB (exceptional value in the sense of Borel) for f for distinct zeros of order $\leq k$ if $\overline{\rho}_k(a,f)<\rho$.
ii) E-valued evB for f for distinct zeros if $\overline{\rho}(a,f)<\rho$.
iii) E-valued evB for f (for the whole aggregate of zeros) if $\rho(a,f)<\rho$.
Thus we call a is an E-valued evB for f for simple zeros if $\overline{\rho}_1(a,f)<\rho$ and an E-valued evB for f for distinct simple and double zeros if $\overline{\rho}_2(a,f)<\rho$.

Definition 1.4 Let f be a non-constant E-valued meromorphic function , a point $a\in E\cup\{\infty\}$ is called E-valued Picard exceptional value (evP) if $V(r,a)+N(r,a)=O\{logr\}$.

We call a is an E-valued evP for f for zeros of order $\leq k$ if $V(r,a)+\overline{N}_k(r,a,f)=O\{logr\}$.

Definition 1.5 The order ρ of an E-valued meromorphic function f is defined by

$$\rho=\limsup_{r\to\infty}\frac{logT(r,f)}{logr}$$

and the lower order λ of f is defined by

$$\lambda=\liminf_{r\to\infty}\frac{logT(r,f)}{logr}$$

In [3] , H.S.Gopalkrishna and S.S.Bhoosnurmath proved the following result.
Theorem A Let f be a meromorphic function of order order ρ , $0 \leq \rho \leq \infty$ .If there exist distinct elements $a_1, a_2, \ldots, a_p$; $b_1, b_2, \ldots, b_q$; $c_1, c_2, \ldots, c_s$ in $\overline{\mathbb{C}}$ such that $a_1, a_2, \ldots, a_p$ are evB for f for distinct zeros of order $\leq k$, $b_1, b_2, \ldots, b_q$ are evB for f for distinct zeros of order $\leq l$ and $c_1, c_2, \ldots, c_s$ are evB for f for distinct zeros of order $\leq m$ where k, l and m are positive integers, then

$$\frac{pk}{k+1} + \frac{ql}{l+1} + \frac{sm}{m+1} \leq 2$$

It is natural to consider whether there exists a similar result , if meromorphic function f is replaced by E-valued meromorphic function f.In this paper we extend the above theorem to E-valued meromorphic function.This result gives several other significant results.

Theorem 1.1 Let $f(z)$ be an E-valued meromorphic function with the property of compact projection is of order ρ , $0 \leq \rho \leq \infty$ in $\mathbb{C}$.If there exist distinct elements $a^{[1]}, a, \ldots, a^{[p]}$; $b^{[1]}, b^{[2]}, \ldots, b^{[q]}$; $c^{[1]}, c^{[2]}, \ldots, c^{[s]}$ in $E \cup \{\infty\}$ such that $a^{[1]}, a^{[2]}, \ldots, a^{[p]}$ are E-valued evB for f for distinct zeros of order $\leq k$, $b^{[1]}, b^{[2]}, \ldots, b^{[q]}$ are E-valued evB for f for distinct zeros of order $\leq l$ and $c^{[1]}, c^{[2]}, \ldots, c^{[s]}$ are E-valued evB for f for distinct zeros of order $\leq m$ where k, l and m are positive integers, then

$$\frac{pk}{k+1} + \frac{ql}{l+1} + \frac{sm}{m+1} \leq 2 \tag{1}$$

2.Basic notions of Nevanlinna theory in Banach spaces

Let f be an E-valued meromorphic function in $|z| \leq r$. For any $a \in E \cup \{\infty\}$, $n(r, a, f) = n(r, a)$ denotes the number of a-points of f in $|z| \leq r$, counted with multiplicities and $n(r, \infty, f) = n(r, f)$ denote the number of poles of f in $|z| \leq r$. Then we have the counting function of finite or infinite a-points as

$$N(r, a) \equiv N(r, a, f) = n(0, a) log r + \int_0^r \frac{n(t, a) - n(0, a)}{t} dt$$

$$N(r, f) \equiv N(r, \infty, f) = n(0, f) log r + \int_0^r \frac{n(t, f) - n(0, f)}{t} dt$$

and

$$m(r, f) \equiv m(r, \infty, f) = \frac{1}{2\pi} \int_0^{2\pi} log^+ \left\| f(re^{i\phi}) \right\| d\phi,$$

$$m(r, a) \equiv m(r, a, f) = \frac{1}{2\pi} \int_0^{2\pi} log^+ \frac{1}{\| f(re^{i\phi}) - a \|} d\phi, (a \neq \infty),$$

$$T(r, f) = m(r, f) + N(r, f).$$

Where $log^{+}x = max\{logx, 0\}$. The volume function associated with E-valued meromorphic function f is given by

$$V(r,a,f) = \frac{1}{2\pi}\int_{C_r} log\left|\frac{r}{\xi}\right| \Delta log\|f(\xi)-a\|\, d\sigma \wedge d\tau.$$

In 2006 , C.G.Hu and Qijian Hu[2] proved the following theorems.
Theorem 2.1 (the E-valued Nevanlinna's first fundamental theorem)
Let $f(z)$ be an E-valued meromorphic mapping in C_R. Then for $0 < r < R$, $a \in E$, $f(z) \neq a$,

$$T(r,f) = V(r,a) + N(r,a) + m(r,a) + log\|c_q(a)\| + \epsilon(r,a).$$

Here $\epsilon(r,a)$ is a function such that

$$|\epsilon(r,a)| \leq log^{+}\|a\| + log2, \epsilon(r,0) \equiv 0,$$

and $c_q(a) \in E$ is the co-efficient of the first term in the Laurent series at the point a.

Theorem 2.2 (the E-valued Nevanlinna's second fundamental theorem)
Let $f(z)$ be a non-constant E-valued meromorphic mapping of compact projection in C_R and $a^{[k]} \in E \cup \{\infty\}$ $(k = 1, 2, \ldots, q)$ be $q \geq 3$ distinct finite or infinite points. Then

$$\sum_{k=1}^{q} m\left(r, a^{[k]}\right) + G(r,f) \leq T(r,f) - N_1(r) + S(r)$$

where $N_1(r) = N(r,0,f') + 2N(r,f) - N(r,f')$ and

$$G(r,f) = \int_0^r \frac{dt}{2\pi}\int_{C_t} \Delta log\|f'(\xi)\|\, d\sigma \wedge d\tau$$

If $R = +\infty$, then $S(r)$ satisfies

$$S(r) = O\{logT(r,f)\} + O(logr)$$

as $r \to +\infty$ without exception if $f(z)$ has finite order and otherwise as $r \to +\infty$ outside a set J of exceptional intervals of finite measure

$$\int_J dr < +\infty.$$

If $0 < R < +\infty$, then

$$S(r) = O\left\{log^{+}T(r,f)\right\} + 0\left\{log\frac{1}{R-r}\right\}$$

holds as $r \to R$ without exception if f has finite order

$$\rho = \limsup_{r\to R} \frac{logT(r,f)}{log(1/R-r)},$$

and otherwise as $r \to R$ outside of a set J exceptional intervals such that

$$\int_J d\frac{1}{R-r} < +\infty.$$

In all cases, the exceptional set J is independent of the choice of the finite points $a^{[k]} \in E$ and of their number.

Other interesting forms of the E-valued Nevanlinna's second fundamental theorem :

$$(q-1)T(r,f) + G(r,f) + N_1(r) \le \sum_{k=1}^{q+1} \left[V(r,a^{[k]}) + N(r,a^{[k]})\right] + S(r)$$

OR

$$(q-2)T(r,f) + G(r,f) \le \sum_{k=1}^{q} \left[V(r,a^{[k]}) + \overline{N}(r,a^{[k]})\right] + S(r)$$

with

$$\overline{N}(r,a) = \overline{n}(0,a)logr + \int_0^r \frac{\overline{n}(t,a) - \overline{n}(t,a)}{t} dt,$$

where $\overline{n}(t,a)$ denotes the number of solutions of $f(z) - a$ in $|z| \le t$, each solution counted only once.

The E-valued Picard's theorem and the E-valued Picard-Borel's theorem are the important applications of E-valued Nevanlinna's second fundamental theorem. These theorems are given by C.G.Hu [1] . We now give another significant application of the E-valued Nevanlinna's second fundamental theorem.

Lemma 2.1 Let $f(z)$ be a non-constant E-valued meromorphic function with compact projection is of finite order in $\mathbb{C}$. If , for a positive number λ , the integral

$$\int_{r_o}^{+\infty} \frac{V(r,a) + N(r,a)}{r^{\lambda+1}} dr, \quad (r_o > 0) \tag{2}$$

converges for three different $a \in E \cup \{\infty\}$, then the integral

$$\int_{r_o}^{+\infty} \frac{T(r,f) + G(r,f)}{r^{\lambda+1}} dr$$

converges , so that $f(z)$ is at most of convergence class of of order λ , and (2) converges for every $a \in E \cup \{\infty\}$.

Proof. By the E-valued Nevanlinna second fundamental theorem with $q = 3$,

$$T(r, f) + G(r, f) \leq \sum_{i=1}^{3} \left[V(r, a^{[i]}) + N(r, a^{[i]})\right] - N_1(r) + S(r)$$

Since $N_1(r)$ is a non-negative number and can be neglected and f is of finite order, so $S(r) = O(logr)$. Therefore ,

$$T(r, f) + G(r, f) \leq \sum_{i=1}^{3} \left[V(r, a^{[i]}) + N(r, a^{[i]})\right] + O(logr)$$

$$\int_{r_0}^{r} \frac{T(r, f) + G(r, f)}{r^{\lambda+1}} dr \leq \sum_{i=1}^{3} \int_{r_0}^{r} \frac{V(r, a^{[i]}) + N(r, a^{[i]})}{r^{\lambda+1}} dr + \int_{r_0}^{r} \frac{O(logr)}{r^{\lambda+1}} dr, (r > r_0)$$

letting $r \to +\infty$, we get

$$\int_{r_0}^{+\infty} \frac{T(r, f) + G(r, f)}{r^{\lambda+1}} dr \leq \sum_{i=1}^{3} \int_{r_0}^{+\infty} \frac{V(r, a^{[i]}) + N(r, a^{[i]})}{r^{\lambda+1}} dr + \int_{r_0}^{+\infty} \frac{O(logr)}{r^{\lambda+1}} dr$$

using (2) and observing $\int_{r_0}^{+\infty} \frac{O(logr)}{r^{\lambda+1}} dr < +\infty$ we get

$$\int_{r_o}^{+\infty} \frac{T(r, f) + G(r, f)}{r^{\lambda+1}} dr < \infty$$

Hence the integral $\int_{r_o}^{+\infty} \frac{T(r,f)+G(r,f)}{r^{\lambda+1}} dr$ converges.This completes the lemma. We shall now prove our main result :

Proof of the Theorem 1.1 By the E-valued Nevanlinna's second fundamental theorem , we have

$$(q - 2)T(r, f) + G(r, f) \leq \sum_{k=1}^{q} \left[V(r, a^{[k]}) + \overline{N}(r, a^{[k]})\right] + S(r, f) \qquad (3)$$

Given that $a^{[1]}, a^{[2]} \dots , a^{[p]}$; $b^{[1]}, b^{[2]}, \dots , b^{[q]}$; $c^{[1]}, c^{[2]}, \dots , c^{[s]}$ are distinct elements in $E \cup \{\infty\}$. So (3) can be written as ,

$$(p+q+s-2)T(r, f)+G(r, f) \leq \sum_{i=1}^{p} \left[V(r, a^{[i]}) + \overline{N}(r, a^{[i]})\right] + \sum_{j=1}^{q} \left[V(r, b^{[j]}) + \overline{N}(r, b^{[j]})\right]$$

$$+ \sum_{t=1}^{s} \left[V(r, c^{[t]}) + \overline{N}(r, c^{[t]})\right] + S(r, f) \qquad (4)$$

By hypothesis , we have

$$\overline{\rho}_k(a^{[i]}, f) < \rho \quad for \quad i = 1, 2, \ldots, p \quad ,$$
$$\overline{\rho}_l(b^{[j]}, f) < \rho \quad for \quad j = 1, 2, \ldots, q \quad ,$$
$$\overline{\rho}_m(c^{[t]}, f) < \rho \quad for \quad t = 1, 2, \ldots, s.$$

We choose the positive number $\lambda < \rho$ such that

$$\overline{\rho}_k(a^{[i]}, f) < \lambda \quad for \quad i = 1, 2, \ldots, p \quad ,$$
$$\overline{\rho}_l(b^{[j]}, f) < \lambda \quad for \quad j = 1, 2, \ldots, q \quad ,$$
$$\overline{\rho}_m(c^{[t]}, f) < \lambda \quad for \quad t = 1, 2, \ldots, s.$$

By the definition 1.3 , we have

$$\overline{\rho}_k(a^{[i]}, f) = \limsup_{r\to\infty} \frac{log^+ \left[V(r, a^{[i]}) + \overline{N}_k(r, a^{[i]})\right]}{logr} < \lambda \quad ,$$

$$\overline{\rho}_l(b^{[j]}, f) = \limsup_{r\to\infty} \frac{log^+ \left[V(r, b^{[j]}) + \overline{N}_l(r, b^{[j]})\right]}{logr} < \lambda \quad ,$$

$$\overline{\rho}_m(c^{[t]}, f) = \limsup_{r\to\infty} \frac{log^+ \left[V(r, c^{[t]}) + \overline{N}_m(r, c^{[t]})\right]}{logr} < \lambda$$

for $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q$ and $t = 1, 2, \ldots, s$. Then

$$V(r, a^{[i]}) + \overline{N}_k(r, a^{[i]}) = O(r^\lambda) \quad , \quad i = 1, 2, \ldots, p$$
$$V(r, b^{[j]}) + \overline{N}_l(r, b^{[j]}) = O(r^\lambda) \quad , \quad j = 1, 2, \ldots, q$$
$$V(r, c^{[t]}) + \overline{N}_m(r, c^{[t]}) = O(r^\lambda) \quad , \quad t = 1, 2, \ldots, s$$

it follows that

$$\int_{r_o}^{+\infty} \frac{V(r, a^{[i]}) + \overline{N}_k(r, a^{[i]})}{r^{1+\lambda}} dr < \infty, \int_{r_o}^{+\infty} \frac{V(r, b^{[j]}) + \overline{N}_l(r, b^{[j]})}{r^{1+\lambda}} dr < \infty,$$

$$\int_{r_o}^{+\infty} \frac{V(r, c^{[t]}) + \overline{N}_m(r, c^{[t]})}{r^{1+\lambda}} dr < \infty \tag{5}$$

for $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q$ and $t = 1, 2, \ldots, s$.
If $a \in E \cup \{\infty\}$ and d is a positive integer, we have

$$\overline{N}(r, a, f) \le \frac{1}{d+1} \left\{ d\overline{N}_d(r, a, f) + N(r, a, f) \right\} \tag{6}$$

Using (6) in (4) , we get

$$(p+q+s-2)T(r,f)+G(r,f) \leq \sum_{i=1}^{p} V(r,a^{[i]})+\frac{k}{k+1}\sum_{i=1}^{p}\overline{N}_k(r,a^{[i]})+\frac{1}{k+1}\sum_{i=1}^{p}N(r,a^{[i]})$$

$$+\sum_{j=1}^{q} V(r,b^{[j]})+\frac{l}{l+1}\sum_{j=1}^{q}\overline{N}_l(r,b^{[j]})+\frac{1}{l+1}\sum_{j=1}^{q}N(r,b^{[j]})$$

$$+\sum_{t=1}^{s} V(r,c^{[t]})+\frac{m}{m+1}\sum_{t=1}^{s}\overline{N}_m(r,c^{[t]})+\frac{1}{m+1}\sum_{t=1}^{s}N(r,c^{[t]})+S(r,f)$$

Since , we know that

$$N(r,a,f) \leq T\left(r,\frac{1}{f-a}\right) = T(r,f)-V(r,a)+O(1) \quad [by \quad Theorem \quad 2.1]$$

So,

$$(p+q+s-2)T(r,f)+G(r,f) \leq \sum_{i=1}^{p} V(r,a^{[i]})+\frac{k}{k+1}\sum_{i=1}^{p}\overline{N}_k(r,a^{[i]})+$$

$$\frac{1}{k+1}\sum_{i=1}^{p}\left[T(r,f)-V(r,a^{[i]})+O(1)\right]+\sum_{j=1}^{q}V(r,b^{[j]})+\frac{l}{l+1}\sum_{j=1}^{q}\overline{N}_l(r,b^{[j]})$$

$$+\frac{1}{l+1}\sum_{j=1}^{q}\left[T(r,f)-V(r,b^{[j]})+O(1)\right]+\sum_{t=1}^{s}V(r,c^{[t]})+\frac{m}{m+1}\sum_{t=1}^{s}\overline{N}_m(r,c^{[t]})$$

$$+\frac{1}{m+1}\sum_{t=1}^{s}\left[T(r,f)-V(r,c^{[t]})+O(1)\right]+S(r,f)$$

Therefore

$$\left(p+q+s-2-\frac{p}{k+1}-\frac{q}{l+1}-\frac{s}{m+1}\right)T(r,f)+G(r,f) \leq \left(1-\frac{1}{k+1}\right)\sum_{i=1}^{p}V(r,a^{[i]})+$$

$$\frac{k}{k+1}\sum_{i=1}^{p}\overline{N}_k(r,a^{[i]})+\left(1-\frac{1}{l+1}\right)\sum_{j=1}^{q}V(r,b^{[j]})+\frac{l}{l+1}\sum_{j=1}^{q}\overline{N}_l(r,b^{[j]})$$

$$+\left(1-\frac{m}{m+1}\right)\sum_{t=1}^{s}V(r,c^{[t]})+\frac{m}{m+1}\sum_{t=1}^{s}\overline{N}_m(r,c^{[t]})+S(r,f)$$

$$\left(\frac{pk}{k+1}+\frac{ql}{l+1}+\frac{sm}{m+1}-2\right)T(r,f)+G(r,f) \leq \frac{k}{k+1}\sum_{i=1}^{p}V(r,a^{[i]})+\frac{k}{k+1}\sum_{i=1}^{p}\overline{N}_k(r,a^{[i]})$$

$$+\frac{l}{l+1}\sum_{j=1}^{q}V(r,b^{[j]})+\frac{l}{l+1}\sum_{j=1}^{q}\overline{N}_l(r,b^{[j]})+\frac{m}{m+1}\sum_{t=1}^{s}V(r,c^{[t]})+$$

$$\frac{m}{m+1}\sum_{t=1}^{s}\overline{N}_m(r,c^{[t]})+S(r,f) \qquad (7)$$

We have

$$\int_{r_o}^{r}\frac{S(x,f)}{x^{1+\lambda}}dx=o\left(\int_{r_o}^{r}\frac{T(x,f)}{x^{1+\lambda}}dx\right) \quad , \quad (r>r_o)$$

Clearly it implies that

$$\int_{r_o}^{r}\frac{S(x,f)}{x^{1+\lambda}}dx=o\left(\int_{r_o}^{r}\frac{T(x,f)+G(x,f)}{x^{1+\lambda}}dx\right) \quad , \quad (r>r_o)$$

Hence (7) yields that

$$\left(\frac{pk}{k+1}+\frac{ql}{l+1}+\frac{sm}{m+1}-2+o(1)\right)\int_{r_o}^{r}\frac{T(x,f)+G(x,f)}{x^{1+\lambda}}dx\leq$$

$$\frac{k}{k+1}\sum_{i=1}^{p}\int_{r_o}^{r}\frac{V(x,a^{[i]})+\overline{N}_k(x,a^{[i]})}{x^{1+\lambda}}dx+\frac{l}{l+1}\sum_{j=1}^{q}\int_{r_o}^{r}\frac{V(x,b^{[j]})+\overline{N}_l(x,b^{[j]})}{x^{1+\lambda}}dx+$$

$$\frac{m}{m+1}\sum_{t=1}^{s}\int_{r_o}^{r}\frac{V(x,c^{[t]})+\overline{N}_m(x,c^{[t]})}{x^{1+\lambda}}dx$$

letting $r\to+\infty$, we get

$$\left(\frac{pk}{k+1}+\frac{ql}{l+1}+\frac{sm}{m+1}-2+o(1)\right)\int_{r_o}^{+\infty}\frac{T(x,f)+G(x,f)}{x^{1+\lambda}}dx\leq$$

$$\frac{k}{k+1}\sum_{i=1}^{p}\int_{r_o}^{+\infty}\frac{V(x,a^{[i]})+\overline{N}_k(x,a^{[i]})}{x^{1+\lambda}}dx+\frac{l}{l+1}\sum_{j=1}^{q}\int_{r_o}^{+\infty}\frac{V(x,b^{[j]})+\overline{N}_l(x,b^{[j]})}{x^{1+\lambda}}dx+$$

$$\frac{m}{m+1}\sum_{t=1}^{s}\int_{r_o}^{+\infty}\frac{V(x,c^{[t]})+\overline{N}_m(x,c^{[t]})}{x^{1+\lambda}}dx$$

Thus , if $\frac{pk}{k+1}+\frac{ql}{l+1}+\frac{sm}{m+1}>2$, then using (5), we get

$$\int_{r_o}^{+\infty}\frac{T(r,f)+G(r,f)}{r^{1+\lambda}}dr<\infty$$

which would imply $\lim_{r\to\infty}\frac{T(r,f)}{r^{\lambda}}=0$, so that the order of $f\leq\lambda$.But, we have $\lambda<\rho$, which is contradiction. So we should have

$$\frac{pk}{k+1}+\frac{ql}{l+1}+\frac{sm}{m+1}\leq 2.$$

This completes the proof of the theorem.

Consequences of Theorem 1.1

(i) For $k \geq 3$, (1) yields $p \leq 8/3$.By the Theorem 1.1,it follows that there exist at most two elements in $E \cup \{\infty\}$ which are E-valued evB for distinct zeros of order $\leq k$ if $k \geq 3$.

On the other hand , for $k = 3$, $p = 1$, $l = 4$ and $q = 1$, (1) yields $s \leq \frac{9}{20}\left(1 + \frac{1}{m}\right)$ so that $s = 0$ for $m = 1$.Thus , it follows that if there exist distinct elements a, b in $E \cup \{\infty\}$ such that a is an E-valued evB for f for distinct zeros of order ≤ 3 and b is an E-valued evB for f for distinct zeros of order ≤ 4, then there exists no elements of $E \cup \{\infty\}$ which is an E-valued evB for f for simple zeros.

(ii) For $k = 2$,(1) yields $\frac{2p}{3} + \frac{ql}{l+1} \leq 2$ so that $p \leq 3$ and if $p = 3$ then $q = 0$ whatever l be. Thus , there exist at most three elements in $E \cup \{\infty\}$ which are E-valued evB for f for distinct simple and double zeros and if there exist three such elements then there is no other element of $E \cup \{\infty\}$ which is an E-valued evB for f for simple zeros.

Again , for $k = 2$ and $p = 1$, (1) yields $q \leq \frac{4}{3}\left(1 + \frac{1}{l}\right)$ so that $q \leq 2$ for $l = 1$ and $q \leq 1$ for $l = 3$. Hence it follows that if there exists an element of $E \cup \{\infty\}$ which is an E-valued evB for f for distinct simple and double zeros then there exist at most two other elements of $E \cup \{\infty\}$ which are E-valued evB for f for simple zeros and there exists at most one other element of $E \cup \{\infty\}$ which is E-valued evB for f for distinct zeros of order ≤ 3.

On the other hand , $k = 2$, $p = 1$, $l = 3$, and $q = 1$, (1) yields $s \leq \frac{7}{12}\left(1 + \frac{1}{m}\right)$ so that $s = 0$ for $m = 2$. Hence , it follows that if there exist distinct elements a , b in $E \cup \{\infty\}$ such that a is an E-valued evB for f for distinct zeros of order ≤ 2 and b is an E-valued evB for f for distinct zeros of order ≤ 3 then there exists no other element of $E \cup \{\infty\}$ which is an E-valued evB for f for distinct zeros of order ≤ 2.

For $k = 2$, $p = 1$, $l = 6$ and $q = 1$, (1) yields $s \leq \frac{10}{21}\left(1 + \frac{1}{m}\right)$ so that $s = 0$ when $m = 1$.Hence , if there exist distinct elements a, $b \in E \cup \{\infty\}$ such that a is an E-valued evB for f for distinct simple and double zeros and b is an E-valued evB for f for distinct zeros of order ≤ 6 then there exisis no other element of $E \cup \{\infty\}$ which is an E-valued evB for f for simple zeros.

(iii) For $k = 1$, (1) yields $p \leq 4$, hence it follows that there exist at most four elements in $E \cup \{\infty\}$ which are E-valued evB for f for simple zeros.

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Received: May, 2010