Int. Journal of Math. Analysis, Vol. 4, 2010, no. 42, 2075 - 2078

An Existence Result of Positive Solutions for a Class of Laplacian Systems

Ghasem A. Afrouzi

Department of Mathematics, Faculty of Basic Sciences
Mazandaran University, Babolsar, Iran
afrouzi@umz.ac.ir

Samira Ala

Department of Mathematics, Faculty of Basic Sciences
Islamic Azad University, Qaemshahr, Iran
golestan62@yahoo.com

Abstract. we mainly consider the existence of a positive solution of the following system

$$\begin{cases} -\Delta u + u = f(v) & in\Omega, \\ -\Delta v + v = g(u) & in\Omega, \\ u = v = 0 & on\partial\Omega \end{cases}$$

Where $\Omega \subset R^N$ is a bounded domain with smooth boundary $\partial\Omega$. we proved the existence of a positive solution when

$$\lim_{u\to+\infty} \frac{f[Mg(u)]}{u} = 0 \quad , \quad for\ every \quad M > 0$$

Mathematics Subject Classification: 35J60, 35B30, 35B40

Keywords: Nonlinear elliptic equation; Laplacian Systems; sub-super solution

1. **Introduction**

In this work, we study the existence of a positive solution for the system

$$(p) \begin{cases} -\Delta u + u = f(v) & in\Omega, \\ -\Delta v + v = g(u) & in\Omega, \\ u = v = 0 & on\partial\Omega \end{cases}$$

Where $\Omega \subset R^N$ is a bounded domain with smooth boundary $\partial\Omega$.
our motivation comes from [7], where the authors considered the existence of positive solutions for nonlinear elliptic equation $-\Delta u + u = f(x, u)$ in a bounded smooth domain $\Omega \subset R^N$ with a nonlinear boundary value condition. The existence results are obtained by the sub-supersolution method and the Mountain pass Lemma. And nonexistence is also considered.
In this paper, we consider the existence of a positive solution of the problem (p) based on the method of sub-supersolutions. First we give the following hypotheses:
(H_1) $\Omega \subset R^N$ is an open bounded domain with smooth boundary $\partial\Omega$.
(H_2) $f, g : [0, +\infty] \to R^+ \cup \{0\}$ are C^1, monotone functions such that

$$\lim_{u\to+\infty} f(u) = \lim_{u\to+\infty} g(u) = +\infty$$

(H_3) $(f + g)(0)$ is not identically zero.
(H_4) For any positive constant M

$$\lim_{u\to+\infty} \frac{f[Mg(u)]}{u} = 0$$

Let $W^{1,2}(\Omega) = \{u \in L^2(\Omega) : \nabla u \in L^2(\Omega)\}$ with the norm
$\|u\|_{W^{1,2}(\Omega)} = \left(\int_\Omega(|\nabla u|^2 + |u|^2)dx\right)^{\frac{1}{2}}$; then $W^{1,2}(\Omega)$ is a Banach space.

Definition 1.1. If $u, v \in W^{1,2}(\Omega)$, (u, v) is called a weak solution of the problem (p) if it satisfies

$$\int_\Omega \nabla u \cdot \nabla q dx + \int_\Omega uq dx = \int_\Omega f(v) q dx, \quad \forall\ q \in W^{1,2}(\Omega),$$

and

$$\int_\Omega \nabla v \cdot \nabla q dx + \int_\Omega vq dx = \int_\Omega g(u) q dx, \quad \forall\ q \in W^{1,2}(\Omega).$$

2. **Existence results**

Theorem 2.1. *Let $H_1 - H_4$ hold. Then (p) has one positive solution (u, v).*

Proof. By using a method of [5] we shall establish Theorem 2.1 by constructing a subsolution (Φ_1, Φ_2) and supersolution (z_1, z_2) of (p), such that $\Phi_1 \le z_1$ and $\Phi_2 \le z_2$. That is (Φ_1, Φ_2) and (z_1, z_2) satisfies

$$\begin{cases} \displaystyle\int_\Omega \nabla\Phi_1 \cdot \nabla q dx + \int_\Omega \Phi_1 q dx \le \int_\Omega f(\Phi_2) q dx, \\ \displaystyle\int_\Omega \nabla\Phi_2 \cdot \nabla q dx + \int_\Omega \Phi_2 q dx \le \int_\Omega g(\Phi_1) q dx, \end{cases}$$

and

$$\begin{cases} \int_\Omega \nabla z_1 \cdot \nabla q dx + \int_\Omega z_1 q dx \geq \int_\Omega f(z_2) q dx, \\ \int_\Omega \nabla z_2 \cdot \nabla q dx + \int_\Omega z_2 q dx \geq \int_\Omega g(z_1) q dx, \end{cases}$$

for all $q \in W^{1,2}(\Omega)$ with $q \geq 0$.
$(0,0)$ is a subsolution of the problem (p), and $(0,0)$ is not a solution of the problem (p) by (H_3). We construct a supersolution of (p).
Let Φ_0 be the solution of

$$-\Delta \Phi_0 + \Phi_0 = 1 \quad in\ \Omega, \quad \Phi_0 = 0 \quad on\ \partial\Omega.$$

Let

$$(z_1, z_2) = \left(\frac{C}{\mu} \Phi_0, [g(C)] \Phi_0 \right),$$

where $\mu = \|\Phi_0\|_\infty$ and $C > 0$ is a Large number to be chosen later. we shall verify that (z_1, z_2) is a supersolution of (p). To this end, let $q \in W^{1,2}(\Omega)$ with $q \geq 0$. Then we have

$$\begin{aligned} \int_\Omega \nabla z_1 \cdot \nabla q dx + \int_\Omega z_1 q dx &= \frac{C}{\mu} \int_\Omega \nabla \Phi_0 \nabla q dx + \\ \frac{C}{\mu} \int_\Omega \Phi_0 q dx &= \frac{C}{\mu} \left(\int_\Omega \nabla \Phi_0 \cdot \nabla q dx + \int_\Omega \Phi_0 q dx \right) \\ &= \frac{C}{\mu} \int_\Omega q dx. \end{aligned}$$

By (H_4), we can choose C Large enough so that

$$C \geq \mu f(\mu g(C))$$

and therefore

$$\begin{aligned} \int_\Omega \nabla z_1 \cdot \nabla q dx + \int_\Omega z_1 q dx &\geq \int_\Omega f(\mu g(C)) q dx \geq \\ &\int_\Omega f(z_2) q dx. \end{aligned}$$

Next,

$$\begin{aligned}\int_\Omega \nabla z_2 \cdot \nabla q dx + \int_\Omega z_2 q dx &= g(C)\int_\Omega \nabla\Phi_0 \cdot \nabla q dx + \\ & g(C)\int_\Omega \Phi_0 q dx = g(C)\left(\int_\Omega \nabla\Phi_0 \cdot \nabla q dx + \int_\Omega \Phi_0 q dx\right) = \\ & g(C)\int_\Omega q dx \geq \int_\Omega g(C\mu^{-1}\Phi_0) q dx = \\ & \int_\Omega g(z_1) q dx,\end{aligned}$$

i.e. (z_1, z_2) is a supersolution of (p) with $z_i \geq 0$ for C Large, $i = 1, 2$. Thus, there exists a solution (u, v) of (p) with $0 \leq u \leq z_1$, $0 \leq v \leq z_2$. This completes the proof. □

References

[1] R. A. Adams, J. J. F. Fournier, sobolov spaces, Academic press, 2003.

[2] K. Chaib, Necessary and sufficient conditions of existence for a system involving the $p-$Laplacian $(1 < p < N)$, Journal of Differential Equations 189(2003)513-523.

[3] L. C. Evans, partial Differential Equations, American Mathematical society, 1998.

[4] D. J. Gua, Nonlinear Functional Analysis, shandong scientific and Technology press, shandong, 2002.

[5] D. D. Hai, R. Shivaji, An existence result on positive solutions of $p-$Laplacian systems, Nonlinear Anal. 56(2004)1007-1010.

[6] O. A. Ladyzhenskaya, N. Nuraltseva, Linear and Quasilinear Elliptic Equations, Academic press, New York, 1968.

[7] X. Song, W. Wang, P. Zhao, Positive solutions of elliptic equations with nonlinear boudary conditions, Nonlinear Anal. (2007)1-7.

Received: May, 2010