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Positive Solutions for Fourth-Order Three-Point Boundary-Value Problems

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Abstract

In this paper, we study sufficient conditions for the existence of at least three positive solutions for fourth-order three-point BVP.

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1 Introduction

This paper concerns the existence of three positive solutions for the fourth-order three-point BVP

$$u''''(t) + a(t)f\big(u(t)\big) = 0, \quad t \in (0,1), \tag{1.1}$$

with the nonlinear boundary conditions

$$u(0) = \xi u(1), \quad u'(1) = \eta u'(0), u'''(0) = \alpha_1(u'''(\delta)), \quad u''(1) = \sqrt[p-1]{\beta_1}u''(\delta), \tag{1.2}$$

where $f : R \to [0, +\infty)$ and $a \in C((0,1) \to [0, +\infty))$, α_1, $\beta_1 \geq 0$, $\xi \neq 1$, $\eta \neq 1$ and $0 < \delta < 1$.

Recently, three-point BVP of the differential equations were presented and studied. However, three-point BVP (1.1), (1.2) have not received as much attention in the literature as Lidstone condition BVP

$$u''''(t) = a(t)f(u(t)), \quad t \in (0,1);\ u(0) = u(1) = u''(0) = u''(1) = 0. \qquad (1.3)$$

For the remainder of the paper, we assume that: (i) $0 < \int_0^1 a(s)ds < \infty$;

2 Background and definitions

Firstly, we state some background material from the theory of cones in Banach spaces.

Definition 2.1. Let X be a real Banach space. A nonempty closed set $P \subset X$ is said to be a cone provided that (i) $x \in P$ and $\lambda \geq 0$ implies $\lambda x \in X$,(ii) $x \in P$ and $-x \in P$ implies $x = 0$.

Every cone $P \subset X$ induces an ordering in X given by $x \leq y$ if and only if $y - x \in P$.

Definition 2.2. The map ψ is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E provided that $\psi : P \to [0, \infty)$ is continuous and $\psi(tx + (1-t)y) \geq t\psi(x) + (1-t)\psi(y)$, for all $x, y \in P$ and $0 \leq t \leq 1$. Similarly, we say the map β is a nonnegative continuous convex functional on a cone P of a real Banach space E provided that $\beta : P \to [0, \infty)$ is continuous and $\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$, for all $x, y \in P$ and $0 \leq t \leq 1$.

Let γ, β, θ be nonnegative, continuous, convex functionals on P and α, ψ be nonnegative, continuous, concave functionals on P. Then for nonnegative numbers h, a, b, d and c we define the following sets:

$$P(\gamma, c) = \{x \in P : \gamma(x) < c\}, P(\gamma, \alpha, a, c) = \{x \in P : a \leq \alpha(x),\ \gamma(x) \leq c\},$$
$$Q(\gamma, \beta, d, c) = \{x \in P : \beta(x) \leq d,\ \gamma(x) \leq c\},$$
$$P(\gamma, \theta, \alpha, a, b, c) = \{x \in P :\ a \leq \alpha(x),\ \theta(x) \leq b,\ \gamma(x) \leq c\},$$
$$Q(\gamma, \beta, \psi, h, d, c) = \{x \in P : h \leq \psi(x),\ \beta(x) \leq d,\ \gamma(x) \leq c\}.$$

Theorem 2.1. *Suppose P is a cone of X which is a real Banach space, γ, β, θ are nonnegative, continuous, convex functionals and α, ψ are nonnegative, continuous, concave functionals such that $\alpha(x) \leq \beta(x)$, $||x|| \leq M\gamma(x)$ for $x \in \overline{P(\gamma, c)}$ and some positive numbers c, M. Again, $T : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$ be a completely continuous operator and there are positive numbers h, d, a, b with $0 < d < a$ such that*

(i) $\{x \in P(\gamma, \theta, \alpha, a, b, c) : \alpha(x) > a\} \neq \emptyset$ *and* $x \in P(\gamma, \theta, \alpha, a, b, c)$ *implies* $\alpha(Tx) > a$.

(ii) $\{x \in Q(\gamma, \beta, \psi, h, d, c) : \beta(x) < d\} \neq \emptyset$ *and* $x \in Q(\gamma, \beta, \psi, h, d, c)$ *implies* $\beta(Tx) < d$.

(iii) $x \in P(\gamma, \alpha, a, c)$ *with* $\theta(Tx) > b$ *implies* $\alpha(Tx) > a$.

(iv) $x \in Q(\gamma, \beta, d, c)$ *with* $\psi(Tx) < h$ *implies* $\beta(Tx) < d$.

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ such thah

$$\beta(x_1) < d, \quad a < \alpha(x_2), \quad d < \beta(x_3), \quad \text{with} \quad \alpha(x_3) < a.$$

3 Related lemmas

Lemma 3.1. *Suppose $f \in C(R, R)$, then the three-point BVP*

$$-y'' = f(t), \quad t \in (0, 1); \quad y'(0) = \alpha_1 y'(\delta), \quad y(1) = \beta_1 y(\delta) \tag{3.4}$$

has a unique solution $y(t) = \int_0^1 g(t, s)f(s)ds$, $t \in (0, 1)$, where $M = (1 - \alpha_1)(1 - \beta_1) \neq 0$ and

$$g(t,s) = \frac{1}{M}\begin{cases} 1 - \beta_1\delta - t + \beta_1 t, & \text{if } 0 \leq s \leq t < \delta < 1 \\ & \text{or } 0 \leq s \leq \delta \leq t \leq 1, \\ 1 - \beta_1\delta + (1 - \beta_1)(\alpha_1 s - s - \alpha_1 t), & \text{if } 0 \leq t \leq s \leq \delta < 1, \\ 1 - \alpha_1 - \beta_1 s + \alpha_1\beta_1 s - t + \alpha_1 t + \beta_1 t + \alpha_1\beta_1 t, & \text{if } 0 \leq \delta \leq s \leq t \leq 1, \\ (1 - s)(t - \alpha_1), & \text{if } 0 < \delta \leq t \leq s \leq 1 \\ & \text{or } 0 \leq t < \delta \leq s \leq 1. \end{cases}$$

Lemma 3.2. *Suppose $f \in C(R,R)$, then the two-point BVP*

$$-y'' = f(t), \qquad t \in (0,1); \;\; y(0) = \xi y(1), \quad y'(1) = \eta y'(0) \tag{3.5}$$

has a unique solution $y(t) = \int_0^1 h(t,s)f(s)ds, \quad t \in [0,1]$, where $M_1 = (1-\xi)(1-\eta) \neq 0$ and

$$h(t,s) = \frac{1}{M_1}\begin{cases} s + \eta(t-s) + \xi\eta(1-t), & 0 \le s \le t \le 1, \\ t + \xi(s-t) + \xi\eta(1-s), & 0 \le t \le s \le 1. \end{cases}$$

Remark 3.3. If $u(t)$ is a solution of BVP (1.1) and (1.2). By Lemma 3.1 and (3.4), one has

$$u''(t) = -\int_0^1 g(t,s)a(s)f(u(s))ds. \tag{3.6}$$

By Lemma 3.2 and (3.5), one obtains

$$u(t) = \int_0^1 h(t,s)\int_0^1 g(s,\tau)a(\tau)f(u(\tau))d\tau ds. \tag{3.7}$$

Lemma 3.4. *Suppose* $0 \le \xi, \eta < 1$, $0 < t_1 < t_2 < 1$

$$\frac{h(t_1,s)}{h(t_2,s)} \ge \frac{t_1}{t_2}, \quad \frac{h(1,s)}{h(\delta,s)} \le \frac{1}{\delta}.$$

Lemma 3.5. *Suppose $\xi, \eta > 1$, $0 < t_1 < t_2 < 1$ and $\delta \in (0,1)$. Then, for $s \in [0,1]$,*

$$\frac{h(t_2,s)}{h(t_1,s)} \ge \frac{1-t_2}{1-t_1}, \quad \frac{h(0,s)}{h(\delta,s)} \le \frac{1}{1-\delta}.$$

4 Triple positive solutions to (1.1), (1.2)

Now let $E = C[0,1]$ with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$. Let the cones $P_1, P_2 \subset X$ defined by

$$P_1 = \{u \in X : u(t) \ge 0, \text{ is concave, nondecreasing on } (0,1)\},$$
$$P_2 = \{u \in X : u(t) \ge 0, \text{ is concave, nonincreasing on } (0,1)\}.$$

Next, choose $t_1, t_2, t_3 \in (0,1)$ and $t_1 < t_2$. Define nonnegative, continuous, concave functions α, ψ and nonnegative, continuous, convex functions β, θ, γ

on P_1 by

$$\gamma(u) = \max_{t\in[0,t_3]} u(t) = u(t_3), \quad u \in P_1, \psi(u) = \min_{t\in[\delta,1]} u(t) = u(\delta), \quad u \in P_1,$$

$$\beta(u) = \max_{t\in[\delta,1]} u(t) = u(1), \quad u \in P_1, \alpha(u) = \min_{t\in[t_1,t_2]} u(t) = u(t_1), \quad u \in P_1,$$

$$\theta(u) = \max_{t\in[t_1,t_2]} u(t) = u(t_2), \quad u \in P_1.$$

It is easy to verify that $\alpha(u) = u(t_1) \leq u(1) = \beta(u)$ and $\|u\| = u(1) \leq \frac{1}{t_3}u(t_3) = \frac{1}{t_3}\gamma(u)$ for $u \in P_1$.

Theorem 4.1. *Suppose* $0 \leq \xi$, $\eta < 1$, $\alpha_1 < 1$, $0 \leq \beta_1 < 1$ *and there exist numbers* $0 < a < b < c$ *such that* $0 < a < b < \frac{t_2}{t_1}b \leq c$ *and* $f(w)$ *satisfies the following conditions:*

$$f(w) < \frac{a}{C}, \quad 0 \leq w \leq a, \tag{4.8}$$

$$f(w) > \frac{b}{B}, \quad b \leq w \leq \frac{t_2}{t_1}b, \tag{4.9}$$

$$f(w) \leq \frac{c}{A}, \quad 0 \leq w \leq \frac{1}{t_3}c, \tag{4.10}$$

where

$$A = \int_0^1 h(t_3,s)\int_0^1 g(s,\tau)a(\tau)d\tau ds, B = \int_0^1 h(t_1,s)\int_{t_1}^{t_2} g(s,\tau)a(\tau)d\tau ds,$$

$$C = \int_0^1 h(1,s)\int_0^1 g(s,\tau)a(\tau)d\tau ds.$$

Then (1.1), (1.2) *has at least three positive solutions* $u_1, u_2, u_3 \in \overline{P_1(\gamma,c)}$ *such that*

$$u_1(t_1) > b, \quad u_2(1) < a, \quad u_3(t_1) < b \tag{4.11}$$

with $u_3(1) > a$, $u_i(\delta) \leq c$ *for* $i = 1, 2, 3$.

Proof. We begin by defining the completely continuous operator $T : P_1 \to X$ by (3.7) as

$$(Tu)(t) = \int_0^1 h(t,s)\int_0^1 g(s,\tau)a(\tau)f(u(\tau))d\tau ds$$

Obviously (1.1), (1.2) has a solution $u = u(t)$ if and only if the operator T has a fixed point on P_1.

Firstly, we prove $T : \overline{P_1(\gamma, c)} \subset \overline{P_1(\gamma, c)}$. For $u \in P_1$, by Remark 3.1, it is easy to check that $Tu \geq 0$. Moreover,

$$(Tu)'(t) = (1-\xi)\Big(\eta \int_0^t \int_0^1 g(s,\tau)a(\tau)f(u(\tau))d\tau d + \int_t^1 \int_0^1 g(s,\tau)a(\tau)f(u(\tau))d\tau ds\Big) \geq 0$$

and $(Tu)''(t) = -\int_0^1 g(t,s)a(s)f(u(s))ds \leq 0$. So, we have $TP_1 \subset P_1$.

For $u \in \overline{P_1(\gamma, c)}$, $0 \leq u(t) \leq \|u\| \leq \frac{1}{t_3}\gamma(u) \leq \frac{1}{t_3}c$. By (4.10), it follows that

$$\begin{aligned}\gamma(Tu) &= \max_{0\leq t\leq t_3}(Tu)(t) = (Tu)(t_3) = \int_0^1 h(t_3,s)\int_0^1 g(s,\tau)a(\tau)f(u(\tau))d\tau ds\\ &\leq \int_0^1 h(t_3,s)\int_0^1 g(s,\tau)a(\tau)d\tau\ \phi_p(c/A)ds = \frac{c}{A}\int_0^1 h(t_3,s)\int_0^1 g(s,\tau)a(\tau)d\tau ds = c.\end{aligned}$$

Thus, $T : \overline{P_1(\gamma, c)} \subset \overline{P_1(\gamma, c)}$. Secondly, by taking

$$u_1(t) = b + \varepsilon_1 \quad \text{for } 0 < \varepsilon_1 < \frac{t_2}{t_1}b - b;\, u_2(t) = a - \varepsilon_2 \text{ for } 0 < \varepsilon_2 < a - \delta a,$$

It is immediate that

$$u_1(t) \in \{P(\gamma,\theta,\alpha,b,\frac{t_2}{t_1}b,c):\ \alpha(u) > b\} \neq \emptyset, u_2(t) \in \{Q(\gamma,\beta,\psi,\delta a,a,c):\ \beta(u) < a\} \neq \emptyset.$$

Next, we verify the remaining conditions of Theorem 2.1. Now the proof is divided into four steps.

Step 1: We prove that

$$u \in P(\gamma,\theta,\alpha,b,\frac{t_2}{t_1}b,c) \quad \text{implies} \quad \alpha(Tu) > b. \tag{4.12}$$

In fact, $u(t) \geq u(t_1) = \alpha(u) \geq b$ for $t_1 \leq t \leq t_2$, and $u(t) \leq u(t_2) = \theta(u) \leq \frac{t_2}{t_1}b$ for $t_1 \leq t \leq t_2$. Thus using (4.9), one gets

$$\begin{aligned}\alpha(Tu) &= \min_{t_1\leq t\leq t_2}(Tu)(t) = (Tu)(t_1) = \int_0^1 h(t_1,s)\int_0^1 g(s,\tau)a(\tau)f(u(\tau))d\tau ds\\ &\geq \int_0^1 h(t_1,s)\int_{t_1}^{t_2} g(s,\tau)a(\tau)f(u(\tau))d\tau ds > \int_0^1 h(t_1,s)\int_{t_1}^{t_2} g(s,\tau)a(\tau)d\tau\ \phi_p(b/B)ds\\ &= \frac{b}{B}\int_0^1 h(t_1,s)\int_{t_1}^{t_2} g(s,\tau)a(\tau)d\tau ds = \frac{b}{B}B = b.\end{aligned}$$

Step 2: We show that

$$u \in Q(\gamma, \beta, \psi, \delta a, a, c) \quad \text{implies} \quad \beta(Tu) < a. \tag{4.13}$$

In fact, $0 \le u(t) \le u(1) = \beta(u) \le a$ for $0 \le t \le 1$, Thus using (4.8), one arrives at

$$\begin{aligned}\beta(Tu) &= \max_{\delta \le t \le 1}(Tu)(t) = (Tu)(1) = \int_0^1 h(1,s)\int_0^1 g(s,\tau)a(\tau)f(u(\tau))d\tau ds \\ &< \int_0^1 h(1,s)\int_0^1 g(s,\tau)a(\tau)d\tau\ \phi_p(a/C)ds = \frac{a}{C}\int_0^1 h(1,s)\int_0^1 g(s,\tau)a(\tau)d\tau ds = a.\end{aligned}$$

Step 3: We verify that

$$u \in Q(\gamma, \beta, a, c) \quad \text{with} \quad \psi(Tu) < \delta a \quad \text{implies} \quad \beta(Tu) < a. \tag{4.14}$$

By Lemma 3.4,

$$\begin{aligned}\beta(Tu) &= \max_{\delta \le t \le 1}(Tu)(t) = (Tu)(1) = \int_0^1 h(1,s)\int_0^1 g(s,\tau)a(\tau)f(u(\tau))d\tau ds \\ &= \int_0^1 \frac{h(1,s)}{h(\delta,s)}h(\delta,s)\int_0^1 g(s,\tau)a(\tau)f(u(\tau))d\tau ds \le \frac{1}{\delta}(Tu)(\delta) = \frac{1}{\delta}\psi(Tu) < a.\end{aligned}$$

Step 4: We prove that

$$u \in P(\gamma, \alpha, b, c) \quad \text{with} \quad \theta(Tu) > \frac{t_2}{t_1}b \quad \text{implies} \quad \alpha(Tu) > b. \tag{4.15}$$

By Lemma 3.4,

$$\begin{aligned}\alpha(Tu) &= \min_{t_1 \le t \le t_2}(Tu)(t) = (Tu)(t_1) = \int_0^1 h(t_1,s) int_0^1 g(s,\tau)a(\tau)f(u(\tau))d\tau ds \\ &= \int_0^1 \frac{h(t_1,s)}{h(t_2,s)}h(t_2,s)\int_0^1 g(s,\tau)a(\tau)f(u(\tau))d\tau ds \ge \frac{t_1}{t_2}(Tu)(t_2) = \frac{t_1}{t_2}\theta(Tu) > b.\end{aligned}$$

Therefore, the hypotheses of Theorem 2.1 are satisfied and there exist three positive solutions x_1, x_2, x_3 for BVP (1.1), (1.2) satisfying (4.11). □

References

[1] R. I. Avery, *A generalization of the Leggett-Williams fixed-point theorem*, Math. Sci. Res. Hot-line, 2 (1998) 9-14.

[2] L. E. Bobisud, D. O'Regan, W. D. Royalty, *Existence and nonexistence for a singular boundary-value problem*, Appl. Anal. 28 (1998), 245-256.

[3] L. E. Bobisud, D. O'Regan, W. D. Royalty, *Solbability of some nonlinear boundary-value problems*, Nonl. Anal. 12 (1998), 855-869.

[4] L. E. Bobisud, *Existence of solutions for a singular boundary-value problem*, Appl. Anal. 35 (1990), 43-57.

[5] L. E. Bobisud, Y. S. Lee, *Existence for a class of nonliear boundary-value problems*, Appl. Anal. 38 (1990), 45-67.

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