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Coefficient Bounds for Certain Subclasses of Close-to-Convex Functions

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Abstract

Let C' denote the subclass of normalized analytic and univalent functions f defined by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and satisfy $Re\left(\frac{zf'(z)}{g(z)}\right) > 0$, $z \in D = \{z : |z| < 1\}$, where $g \in C$(normalized convex function) and is given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. The aim of the present paper is to obtain sharp upper bound for the functional $|a_2a_4 - a_3^2|$ for functions $f(z) \in C'$.

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1 Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $D = \{z : |z| < 1\}$. Also let S, S^*, C denote the subclasses of A consisting of functions which are univalent, starlike and convex functions in D.

Janteng et al. [3] had established on the second Hankel determinant for functions f belongs to S^* and C. Shaharuddin et al. [1] obtained coefficient bounds for certain classes of close-to-convex functions.

In this paper we estimate the sharp upper bound for the functional $|a_2a_4 - a_3^2|$ for certain classes of close-to-convex functions.

Definition 1.1 *Let f be given by (1.1). Then $f \in S^*$ if and only if*

$$Re\left(\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in D. \tag{1.2}$$

Definition 1.2 *Let f be given by (1.1). Then $f \in C$ if and only if*

$$Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0, \quad z \in D. \tag{1.3}$$

Definition 1.3 *Let f be given by (1.1). Then $f \in C'$ if and only if*

$$Re\left(\frac{zf'(z)}{g(z)}\right) > 0, \quad g \in C. \tag{1.4}$$

Definition 1.4 *Let f be given by (1.1). Then $f \in Q$ if and only if*

$$Re\left(\frac{(zf'(z))'}{g'(z)}\right) > 0, \quad g \in C. \tag{1.5}$$

Definition 1.5 *Let $f \in Q^*$ if and only if*

$$Re\left(\frac{(zf'(z))'}{g'(z)}\right) > 0, \quad g \in S^*. \tag{1.6}$$

2 Preliminary Results

The following lemmas will be required in our investigation.

Let P be the family of all functions p analytic in D for which $Re(p(z)) > 0$ and

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots \qquad \text{for } z \in D. \tag{2.1}$$

Lemma 2.1 *[6] If $p \in P$ then $|c_k| \leq 2$ for each k.*

Lemma 2.2 *[2] The power series for $p(z)$ given by (2.1) converges in D to a function in P if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \dots & c_n \\ c_{-1} & 2 & c_1 & \dots & c_{n-1} \\ \vdots & & & & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \dots & 2 \end{vmatrix}, \quad n = 1, 2, 3, \dots$$

and $c_{-k} = \overline{c}_k$, are all non-negative. They are strictly positive except for $p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{it_k} z)$, $\rho_k > 0$, t_k real and $t_k \neq t_j$ for $k \neq j$; in this case $D_n > 0$ for $n < m - 1$ and $D_n = 0$ for $n \geq m$.

Lemma 2.3 *[4] Let the function $p \in P$ be given by the power series (2.1). Then,*

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{2.2}$$

for some x, $|x| \leq 1$, and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z \tag{2.3}$$

for some z, $|z| \leq 1$.

Lemma 2.4 *[1] Let $f \in S^*$. Then*

$$|a_2 a_4 - \frac{8}{9}a_3^2| \leq 1.$$

The result obtained is sharp.

Lemma 2.5 *[1] Let $f \in C$. Then*

$$|a_2 a_4 - a_3^2| \leq \frac{1}{8}.$$

The result obtained is sharp.

3 Main Result

Theorem 3.1 *Let $f \in C'$. Then*

$$|a_2 a_4 - a_3^2| \leq \frac{67}{108}.$$

Proof We refer to the method by Libera and Zlotkiewics [4, 5]. Since $f \in C'$, it follows from (1.2) that there exist $p \in P$ such that

$$zf'(z) = g(z)p(z). \tag{3.1}$$

Equating coefficients in (3.1) yields

$$\left.\begin{aligned} a_2 &= \frac{c_1}{2} + \frac{b_2}{2} \\ a_3 &= \frac{c_2}{3} + \frac{b_2 c_1}{3} + \frac{b_3}{3} \\ a_4 &= \frac{c_4}{4} + \frac{b_2 c_2}{4} + \frac{b_3 c_1}{4} + \frac{b_4}{4}. \end{aligned}\right\} \tag{3.2}$$

Also, since $g \in C$, it follows from (1.3) that there exists $p \in P$ such that

$$(zg'(z))' = g'(z)p(z). \tag{3.3}$$

Equating coefficients in (3.3) yields

$$\left.\begin{aligned} b_2 &= \frac{c_1}{2} \\ b_3 &= \frac{c_2}{6} + \frac{c_1^2}{6} \\ b_4 &= \frac{c_3}{12} + \frac{c_1c_2}{8} + \frac{c_1^3}{24}. \end{aligned}\right\} \tag{3.4}$$

From (3.2) and (3.4), it is easily established that

$$\begin{aligned} |a_2a_4 - a_3^2| &= \left| \left(\frac{b_2}{2} + \frac{c_1}{2}\right)\left(\frac{b_4}{4} + \frac{b_3c_1}{4} + \frac{b_2c_2}{4} + \frac{c_3}{4}\right) - \frac{1}{9}(c_2 + b_2c_1 + b_3)^2 \right| \\ &\leq \left| \frac{b_2b_4}{8} - \frac{b_3^2}{9} \right| + \left| \frac{19}{96}c_1c_3 - \frac{5}{192}c_1^2c_2 - \frac{4}{27}c_2^2 - \frac{17}{1728}c_1^4 \right| \end{aligned} \tag{3.5}$$

From lemma 2.5, $\left| \frac{b_2b_4}{8} - \frac{b_3^2}{9} \right| \leq \frac{1}{36}$.

Now assuming $c_1 = c$ $(0 \leq c \leq 2)$ and using lemma (2.3) we have

$$\left| \frac{19}{96}c_1c_3 - \frac{5}{192}c_1^2c_2 - \frac{4}{27}c_2^2 - \frac{17}{1728}c_1^4 \right| =$$
$$\left| \frac{61}{192}c^4 + \frac{41}{3456}(4-c^2)c^2x - \frac{(4-c^2)x^2}{3456}(43c^2 + 512) + \frac{19c}{192}(4-c^2)(1-|x|^2)z \right|$$

Application of triangle inequality gives,

$$\begin{aligned} &\left| \frac{19}{96}c_1c_3 - \frac{5}{192}c_1^2c_2 - \frac{4}{27}c_2^2 - \frac{17}{1728}c_1^4 \right| \\ &\leq \frac{61}{192}c^4 + \frac{41}{3456}(4-c^2)c^2\rho + \frac{(4-c^2)\rho^2}{3456}(43c^2 - 342c + 512) + \frac{19c}{192}(4-c^2) \\ &= F(\rho) \end{aligned} \tag{3.6}$$

For,

$$F'(\rho) = \frac{41}{3456}(4-c^2)c^2 + \frac{(4-c^2)2\rho}{3456}(43c^2 - 342c + 512) \geq 0$$

and thus is an increasing function implies $Max_{\rho \leq 1}F(\rho) = F(1)$. Now let

$$G(c) = F(1) = \frac{61}{192}c^4 + \frac{41c^2(4-c^2)}{3456} + \frac{(4-c^2)}{3456}(43c^2 + 512)$$

Trivially, one can show that G has attained maximum at $c = 0$. The upper bound for (3.6) corresponds to $\rho = 1$ and $c = 0$, in which case

$$\left|\frac{9}{19}c_1c_3 - \frac{5}{192}c_1^2c_2 - \frac{4}{27}c_2^2 - \frac{17}{1728}c_1^4\right| \leq \frac{16}{27}$$

Letting $c_1 = 0$, $c_2 = -1$ and $c_3 = -2$ in (3.5) shows that the result is $\frac{67}{108}$ and sharp.

Theorem 3.2 *Let $f \in Q$. Then*

$$|a_2a_4 - a_3^2| \leq \frac{79}{576}.$$

The result obtained is sharp.

Proof Since $f \in Q$, it follows from (1.5) that there exist $p \in P$ such that

$$(zf'(z))' = g'(z)p(z). \tag{3.7}$$

Equating coefficients in (3.7) yields

$$\left.\begin{aligned} a_2 &= \frac{c_1}{4} + \frac{b_2}{2} \\ a_3 &= \frac{b_3}{3} + \frac{2}{9}b_2c_1 + \frac{c_2}{9} \\ a_4 &= \frac{b_4}{4} + \frac{3}{16}b_3c_1 + \frac{b_2c_2}{8} + \frac{c_3}{16}. \end{aligned}\right\} \tag{3.8}$$

Also, since $g \in C$, it follows from (1.3) that there exists $p \in P$ such that

$$(zg'(z))' = g'(z)p(z) \text{ for some } z \in D. \tag{3.9}$$

Equating coefficients in (3.9) yields

$$\left.\begin{aligned} b_2 &= \frac{c_1}{2} \\ b_3 &= \frac{c_2}{6} + \frac{c_1^2}{6} \\ b_4 &= \frac{c_3}{12} + \frac{c_1c_2}{8} + \frac{c_1^3}{24}. \end{aligned}\right\} \tag{3.10}$$

From (3.8) and (3.10)

$$\begin{aligned} |a_2a_4 - a_3^2| &= \left|\left(\frac{b_2}{2} + \frac{c_1}{4}\right)\left(\frac{b_4}{4} + \frac{3}{16}b_3c_1 + \frac{b_2c_2}{8} + \frac{c_3}{16}\right) - \left(\frac{b_3}{3} + \frac{2}{9}b_2c_1 + \frac{c_2}{9}\right)^2\right| \\ &\leq \left|\frac{b_2b_4}{8} - \frac{b_3^2}{9}\right| + \left|\frac{7}{192}c_1c_3 + \frac{55}{10368}c_1^2c_2 - \frac{2}{81}c_2^2 - \frac{67}{10368}c_1^4\right| \end{aligned} \tag{3.11}$$

From lemma 2.5, $\left|\frac{b_2b_4}{8} - \frac{b_3^2}{9}\right| \leq \frac{1}{36}$.

Now assuming $c_1 = c$ $(0 \leq c \leq 2)$ and using lemma (2.3) we have

$$\left|\frac{7}{192}c_1c_3 + \frac{55}{10368}c_1^2c_2 - \frac{2}{81}c_2^2 - \frac{67}{10368}c_1^4\right|$$
$$= \left|-\frac{c^4}{1152} + \frac{59}{6912}(4-c^2)c^2x - \frac{(4-c^2)x^2}{20736}(61c^2+512) + \frac{7}{384}(4-c^2)c(1-|x|^2)z\right|$$

Application of triangle inequality gives,

$$\begin{aligned}
&\left|\frac{7}{192}c_1c_3 + \frac{55}{10368}c_1^2c_2 - \frac{2}{81}c_2^2 - \frac{67}{10368}c_1^4\right| \\
&\leq \frac{c^4}{1152} + \frac{59}{6912}(4-c^2)c^2\rho + \frac{(4-c^2)\rho^2}{20736}(61c^2+512) + \frac{7c}{384}(4-c^2)(1-\rho^2) \\
&= \frac{c^4}{1152} + \frac{59}{6912}(4-c^2)c^2\rho + \frac{(4-c^2)\rho^2}{20736}(61c^2-378c+512) + \frac{7c}{384}(4-c^2) \\
&= F(\rho) \qquad (3.12)
\end{aligned}$$

with $\rho = |x| < 1$. For,

$$F'(\rho) = \frac{59}{6912}(4-c^2)c^2 + \frac{(4-c^2)\rho}{10368}(61c^2-378c+512),$$

it can be shown that $F'(\rho) \geq 0$ and thus is an increasing function implying $Max_{\rho\leq 1}F(\rho) = F(1)$. Now let

$$G(c) = F(1) = \frac{c^4}{1152} + \frac{59}{6912}(4-c^2)c^2 + \frac{(4-c^2)}{20736}(61c^2+512)$$

Trivially, we can show that G has attained maximum at $c = 1$. The upper bound for (3.12) corresponds to $\rho = 1$ and $c = 1$, in which case

$$\left|\frac{7}{192}c_1c_3 + \frac{55}{10368}c_1^2c_2 - \frac{2}{81}c_2^2 - \frac{67}{10368}c^4\right| \leq \frac{14}{128}.$$

Letting $c_1 = 1$, $c_2 = -1$ and $c_3 = -2$ in (3.11) shows that the result is $\frac{79}{576}$ and sharp.

Theorem 3.3 *Let $f \in Q^*$. Then*

$$|a_2a_4 - a_3^2| \leq \frac{11}{24}.$$

The result obtained is sharp.

Proof Since $f \in Q^*$, it follows from (1.5) that there exist $p \in P$ such that

$$(zf'(z))' = g'(z)p(z) \text{ for some } z \in D. \tag{3.13}$$

Equating coefficients in (3.13) yields

$$\left.\begin{aligned} a_2 &= \frac{b_2}{2} + \frac{c_1}{4} \\ a_3 &= \frac{b_3}{3} + \frac{2}{9}b_2c_1 + \frac{c_2}{9} \\ a_4 &= \frac{b_4}{4} + \frac{3}{16}b_3c_1 + \frac{b_2c_2}{8} + \frac{c_3}{16}. \end{aligned}\right\} \tag{3.14}$$

Also, since $g \in S^*$, it follows from (1.1) that there exists $p \in P$ such that

$$zg'(z) = g(z)p(z) \text{ for some } z \in D. \tag{3.15}$$

Equating coefficients in (3.15) yields

$$\left.\begin{aligned} b_2 &= c_1 \\ b_3 &= \frac{c_2}{2} + \frac{c_1^2}{2} \\ b_4 &= \frac{c_3}{3} + \frac{c_1c_2}{2} + \frac{c_1^3}{6}. \end{aligned}\right\} \tag{3.16}$$

From (3.14) and (3.16)

$$|a_2a_4 - a_3^2| = \left|\frac{b_2b_4}{8} - \frac{b_3^2}{9}\right| + \left|\frac{13}{192}c_1c_3 + \frac{361}{10368}c_1^2c_2 - \frac{443}{10368}c_1^4 - \frac{4}{81}c_2^2\right| \tag{3.17}$$

From lemma 2.4, $\left|\frac{b_2b_4}{8} - \frac{b_3^2}{9}\right| \le \frac{1}{8}$.

Now assuming $c_1 = c$ $(0 \le c \le 2)$ and using lemma (2.3) we have

$$\left|\frac{13}{192}c_1c_3 + \frac{361}{10368}c_1^2c_2 - \frac{443}{10368}c_1^4 - \frac{4}{81}c_2^2\right|$$
$$= \left|-\frac{215}{10368}c^4 + \frac{551}{20736}c^2x(4-c^2) - \frac{(4-c^2)x^2}{20736}(95c^2+1024) + \frac{13c}{384}(4-c^2)(1-|x|^2)z\right|$$

Application of triangle inequality gives,

$$\left|\frac{13}{192}c_1c_3 + \frac{361}{10368}c_1^2c_2 - \frac{443}{10368}c_1^4 - \frac{4}{81}c_2^2\right|$$
$$\le \frac{215}{10368}c^4 + \frac{551}{20736}c^2\rho(4-c^2) + \frac{13c}{384}(4-c^2)(1-\rho^2) + \frac{(4-c^2)\rho^2}{20736}(95c^2+1024)$$
$$= F(\rho) \tag{3.18}$$

with $\rho = |x| < 1$. For

$$F'(\rho) = \frac{551}{20736}c^2(4-c^2) + \frac{(4-c^2)2\rho}{20736}(95c^2 - 702c + 1024),$$

it can be shown that $F'(\rho) \geq 0$ and thus is an increasing function implying $Max_{\rho \leq 1} F(\rho) = F(1)$. Now let

$$G(c) = F(1) = \frac{215}{10368}c^4 + \frac{551}{20736}c^2(4-c^2) + \frac{(4-c^2)}{20736}(95c^2 + 1024).$$

Trivially, one can show that G has attained maximum at $c = 1.9$. The upper bound for (3.18) corresponds to $\rho = 1$ and $c = 1.9$, in which case

$$\left|\frac{13}{192}c_1c_3 + \frac{361}{10368}c_1^2c_2 - \frac{443}{10368}c_1^4 - \frac{4}{81}c_2^2\right| \leq \frac{1}{3}.$$

Letting $c_1 = 1.9$, $c_2 = 1$ and $c_3 = 1.14$ in (3.17) shows that the result is $\frac{11}{24}$ and sharp.

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