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Qualitative Properties for a Fourth-Order Rational Difference Equation (VIII)

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Abstract

We investigate the dynamical behavior of the following fourth-order rational difference equation

$$x_{n+1} = \frac{x_{n-1}x_{n-2}^b x_{n-3} + x_{n-1} + x_{n-2}^b + x_{n-3} + a}{x_{n-1}x_{n-2}^b + x_{n-2}^b x_{n-3} + x_{n-1}x_{n-3} + 1 + a},\ n = 0, 1, 2, ...$$

where $a, b \in [0, \infty)$ and the initial values $x_{-3}, x_{-2}, x_{-1}, x_0 \in (0, \infty)$. We find that the successive lengths of positive and negative semicycles of nontrivial solutions of the above equation occur periodically. We also show that the positive equilibrium of the equation is globally asymptotically stable.

Mathematics Subject Classification: 39A10

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1 Introduction

Recently there has been a great interest in studying the qualitative properties of rational difference equations. For the systematical studies of rational and nonrational difference equations, one can refer to the monographs [1, 2] and the papers [3-10] and references therein.

The study of rational difference equations of order greater than one is quite challenging and rewarding because some prototypes for the development of the basic theory of the global behavior of nonlinear difference equations of order greater than one come from the results for rational difference equations.

However, there have not been any effective general methods to deal with the global behavior of rational difference equations of order greater than one so far. Therefore, the study of rational difference equations of order greater than one is worth further consideration.

G. Ladas [4] proposed to study the rational difference equation

$$x_{n+1} = \frac{x_n + x_{n-1}x_{n-2} + a}{x_n x_{n-1} + x_{n-2} + a}, \quad n = 0, 1, 2, ... \tag{1}$$

From then on, rational difference equations with the unique positive equilibrium $\bar{x} = 1$ have received considerable attention, one can refer to [3-10] and the references cited therein.

Recently, Li [10] investigated the global behavior of the following fourth-order rational difference equation

$$x_{n+1} = \frac{x_n x_{n-1} x_{n-3} + x_n + x_{n-1} + x_{n-3} + a}{x_n x_{n-1} + x_n x_{n-3} + x_{n-1} x_{n-3} + 1 + a}, \quad n = 0, 1, 2, ... \tag{2}$$

where $a \in [0, \infty)$ and initial values $x_{-3}, x_{-2}, x_{-1}, x_0 \in (0, \infty)$.

Xianyi Li and Ravi P. Agarwal [10] investigate the rule of trajectory structure and global asymptotic stability for a fourth-order rational difference equation

$$x_{n+1} = \frac{x_n^b + x_{n-2}x_{n-3}^b + a}{x_n^b x_{n-2} + x_{n-3}^b + a}, \quad n = 0, 1, 2, ... \tag{3}$$

where $a, b \in [0, \infty)$ and the initial values $x_{-3}, x_{-2}, x_{-1}, x_0 \in (0, \infty)$. In this note, we employ the method in Li [9, 10] to consider the following fourth-order rational difference equation

$$x_{n+1} = \frac{x_{n-1}x_{n-2}^b x_{n-3} + x_{n-1} + x_{n-2}^b + x_{n-3} + a}{x_{n-1}x_{n-2}^b + x_{n-2}^b x_{n-3} + x_{n-1}x_{n-3} + 1 + a}, \ n = 0, 1, 2, ... \tag{4}$$

where $a, b \in [0, \infty)$ and initial values $x_{-3}, x_{-2}, x_{-1}, x_0 \in (0, \infty)$.
By analyzing the length of positive and negative semicycles of nontrivial solutions of Eq. (4), we find that the lengths of positive and negative semicycles of nontrivial solutions of Eq. (4) occur periodically and can be expressed in the "form":

$..., 3^-, 1^+, 1^-, 2^+, 3^-, 1^+, 1^-, 2^+, 3^-, 1^+, 1^-, 2^+, ...$

or $..., 3^+, 1^-, 1^+, 2^-, 3^+, 1^-, 1^+, 2^-, 3^+, 1^-, 1^+, 2^- ...$ According to our knowledge, Eq. (4) has not been studied so far. Therefore, to study its qualitative properties is theoretically meaningful.

It is easy to see that the positive equilibrium $\bar{x}$ of Eq. (4) satisfies

$$\bar{x} = \frac{\bar{x}^{b+2} + \bar{x}^b + 2\bar{x} + a}{2\bar{x}^{1+b} + \bar{x}^2 + 1 + a} \tag{5}$$

from which one can see that Eq. (5) has a unique positive equilibrium $\bar{x} = 1$.

When $b = 0$, Eq. (4) is trivial. Hence, we assume in the sequel that $b > 0$. In the following, we state some main definitions used in this paper.

Definition 1.1. *A positive semicycle of a solution* $\{x_n\}_{n=-3}^{\infty}$ *consists of a "string" of terms* $\{x_l, x_{l+1}, ..., x_m\}$, *all greater than or equal to the equilibrium* $\overline{x}$, *with* $l \geq -3$ *and* $m \leq \infty$ *and such that*

$$\text{either } \ l = -3 \ \text{ or } \ l > -3 \ \text{ and } \ x_{l-1} < \overline{x}.$$

and

$$\text{either } \ m = \infty \ \text{ or } \ m < \infty \ \text{ and } \ x_{m+1} < \overline{x}.$$

A negative semicycle of a solution $\{x_n\}_{n=-3}^{\infty}$ *of Eq. (4) consists of a "string" of terms* $\{x_l, x_{l+1}, ..., x_m\}$, *all less than to* $\overline{x}$, *with* $l \geq -3$ *and* $m \leq \infty$ *and such that*

$$\text{either } \ l = -3 \ \text{ or } \ l > -3 \ \text{ and } \ x_{l-1} \geq \overline{x}.$$

and

$$\text{either } \ m = \infty \ \text{ or } \ m < \infty \ \text{ and } \ x_{m+1} \geq \overline{x}.$$

The length of a semicycle is the number of the total terms contained in it.

Definition 1.2. *A solution* $\{x_n\}_{n=-3}^{\infty}$ *of Eq. (4) is said to be enventually trivial if* x_n *eventually equal to* $\bar{x} = 1$*; otherwise, the solution is said to be nontrivial.*

2 Two Lemmas

Before to draw a qualitatively clear picture for the positive solutions of Eq. (4), we first establish two basic lemmas which will play a key role in the proof of our main results.

Lemma 2.1. *A positive solution* $\{x_n\}_{n=-3}^{\infty}$ *of Eq. (4) is eventually equal to* 1 *if and only if*

$$(x_{-3} - 1)(x_{-2} - 1)(x_{-1} - 1)(x_1 - 1) = 0 \tag{6}$$

Proof. Assume the (6) holds. Then according to Eq. (4), it is easy to see that the following conclusions hold:

(i) if $x_{-3} = 1$, then $x_n = 1$ for $n \geq 1$.

(ii) if $x_{-2} = 1$, then $x_n = 1$ for $n \geq 2$.

(iii) if $x_0 = 1$, then $x_n = 1$ for $n \geq 0$.

(iiii) if $x_1 = 1$, then $x_n = 1$ for $n \geq 1$.

Conversely, assume that

$$(x_{-3} - 1)(x_{-2} - 1)(x_0 - 1)(x_1 - 1) \neq 0 \tag{7}$$

Then one can show that $x_n \neq 1$ for any $n \geq 2$.
Assume the contrary that for some $N \geq 2$,

$$x_N = 1 \text{ and that } x_n \neq 1 \text{ for } -3 \leq n \leq N-1 \tag{8}$$

It is easy to see that

$$1 = x_N = \frac{x_{N-2}x_{N-3}^b x_{N-4} + x_{N-2} + x_{N-3}^b + x_{N-4} + a}{x_{N-2}x_{N-3}^b + x_{N-2}x_{N-4} + x_{N-4}x_{N-3}^b + 1 + a},$$

which implies $(x_{N-2}-1)(x_{N-3}^3-1)(x_{N-4}-1) = 0$. Obviously, this contradicts (8). □

Remark 2.1. *If the initial conditions do not satisfy Eq. (6), then, for any solution $\{x_n\}$ of Eq. (4), $x_n \neq 1$ for $n \geq -3$. Here, the solution is a nontrivial one.*

Lemma 2.2. *Let $\{x_n\}_{n=-3}^{\infty}$ be a nontrivial positive solution of Eq. (4). Then the following conclusions are true for $n \geq 0$.*
(a) $(x_{n+1}-1)(x_{n-1}-1)(x_{n-2}-1)(x_{n-3}-1) > 0$
(b) $(x_{n+1}-x_{n-1})(x_{n-1}-1) < 0$.
(c) $(x_{n+1}-x_{n-3})(x_{n-3}-1) < 0$.
(d) $(x_{n+1}-x_{n-2}^b)(x_{n-2}^b-1) < 0$.

Proof. It follows in light of Eq. (4) that

$$x_{n+1} - 1 = \frac{(x_{n-1}-1)(x_{n-2}^b-1)(x_{n-3}-1)}{x_{n-1}x_{n-2}^b + x_{n-2}^b x_{n-3} + x_{n-1}x_{n-3} + 1 + a}, \quad n = 0, 1, 2, ...$$

and

$$x_{n+1} - x_{n-1} = \frac{(1-x_{n-1})[a + (1+x_{n-1}) + (x_{n-2}^b + x_{n-3})]}{x_{n-1}x_{n-2}^b + x_{n-2}^b x_{n-3} + x_{n-1}x_{n-3} + 1 + a}, \quad n = 0, 1, 2, ...$$

and

$$x_{n+1} - x_{n-3} = \frac{(1-x_{n-3})[a + (1+x_{n-3}) + (x_{n-1} + x_{n-2}^b)]}{x_{n-1}x_{n-2}^b + x_{n-2}^b x_{n-3} + x_{n-1}x_{n-3} + 1 + a}, \quad n = 0, 1, 2, ...$$

and

$$x_{n+1} - x_{n-2}^b = \frac{(1-x_{n-2}^b)[a + (1+x_{n-2}^b) + (x_{n-1} + x_{n-2})]}{x_{n-1}x_{n-2}^b + x_{n-2}^b x_{n-3} + x_{n-1}x_{n-3} + 1 + a}, \quad n = 0, 1, 2, ...$$

□

3 Main results

First we analyze the structure of the semicycles of nontrivial solutions of Eq. (4). Here we confine us to consider the situation of the strictly oscillatory solution of Eq. (4).

Theorem 3.1. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a strictly oscillatory solution of Eq. (4). Then the "rule for the trajectory structure" of nontrivial solution of Eq. (4) is*
or $..., 3^-, 1^+, 1^-, 2^+, 3^-, 1^+, 1^-, 2^+, 3^-, 1^+, 1^-, 2^+, ...$
or $..., 3^+, 1^-, 1^+, 2^-, 3^+, 1^-, 1^+, 2^-, 3^+, 1^-, 1^+, 2^- ...$

Proof. By Lemma 2.2 (a) and the character of the strictly oscillatory, one can see the lengths of a positive and a negative semicycle is at most 3. So, for some integer $p \geq 0$, one of the following four cases must occur:
Case 1: $x_{p-3} > 1,\ x_{p-2} < 1,\ x_{p-1} > 1$ and $x_p > 1$.
Case 2: $x_{p-3} > 1,\ x_{p-2} < 1,\ x_{p-1} > 1$ and $x_p < 1$.
Case 3: $x_{p-3} > 1,\ x_{p-2} < 1,\ x_{p-1} < 1$ and $x_p > 1$.
Case 4: $x_{p-3} > 1,\ x_{p-2} < 1,\ x_{p-1} < 1$ and $x_p < 1$.
If case 1 and 4 occur, it follows from Lemma 2.2 (a) that
$x_{p+1} < 1,\ x_{p+2} < 1,\ x_{p+3} < 1,\ x_{p+4} > 1,\ x_{p+5} < 1,\ x_{p+6} > 1,\ x_{p+7} > 1,\ x_{p+8} < 1,\ x_{p+9} < 1,\ x_{p+10} < 1,\ x_{p+11} > 1,\ x_{p+12} < 1,\ x_{p+13} > 1,\ x_{p+14} > 1,\ x_{p+15} < 1,\ x_{p+16} < 1,$
$x_{p+17} < 1,\ x_{p+18} > 1,\ x_{p+19} < 1, ...$
This shows that the rule for the numbers of terms of positive and negative semicycles of the solution of Eq. (4) to occur successively is $..., 3^-, 1^+, 1^-, 2^+, 3^-, 1^+, 1^-, 2^+, ...$
If case 2 and 3 occur, it follows from Lemma 2.2 (a) that
$x_{p+1} < 1,\ x_{p+2} > 1,\ x_{p+3} > 1,\ x_{p+4} > 1,\ x_{p+5} < 1,\ x_{p+6} > 1,\ x_{p+7} < 1,\ x_{p+8} < 1,\ x_{p+9} > 1,\ x_{p+10} > 1,\ x_{p+11} > 1,\ x_{p+12} < 1,\ x_{p+13} > 1,\ x_{p+14} < 1,\ x_{p+15} < 1,\ x_{p+16} > 1,$
$x_{p+17} > 1,\ x_{p+18} > 1,\ x_{p+19} < 1,\ x_{p+20} > 1, ...$
This shows the rule for the numbers of terms of positive and negative semicycles of the solution of Eq. (4) to successively occur is $..., 3^+, 1^-, 1^+, 2^-, 3^+, 1^-, 1^+, 2^-, ...$
It means that the rule for the lengths of positive and negative semicycles of the solution of Eq. (4) to occur successively is:
or $..., 3^-, 1^+, 1^-, 2^+, 3^-, 1^+, 1^-, 2^+, 3^-, 1^+, 1^-, 2^+, ...$
or $..., 3^+, 1^-, 1^+, 2^-, 3^+, 1^-, 1^+, 2^-, 3^+, 1^-, 1^+, 2^- ...$
The proof is complete. □

Next, we state the second main result in this note.

Theorem 3.2. *Assume that* $a, b \in [0, \infty)$. *Then the positive equilibrium of Eq. (4) is globally asymptotically stable.*

Proof. When $b = 0$, equation (4) is trivial. So, we only consider the case $b > 0$, and prove the positive equilibrium point $\bar{x}$ of Eq. (4) is both locally asymptotically stable and globally attractive. The linearized equation of Eq. (4) about the positive equilibrium $\bar{x} = 1$ is

$$y_{n+1} = 0.y_n + 0.y_{n-1} + 0.y_{n-2} + 0.y_{n-3}, \quad n = 0, 1, 2, ...$$

By virtue of ([2], Remark 1.3.7), $\bar{x}$ is locally asymptotically stable. It remains to verify that every positive solution $\{x_n\}_{n=-3}^{\infty}$ of Eq. (4) converges to $\bar{x} = 1$ as $n \to \infty$. Namely, we want to prove

$$\lim_{n\to\infty} x_n = \bar{x} = 1 \tag{9}$$

If the initial values of the solution satisfy (6), then Lemma 2.1 says the solution is eventually equal to 1 and, of course, (9) holds. Therefore, we assume in the following that the initial values of the solution do not satisfy (6). Then, Remark 2.1 we know, for any solution $\{x_n\}_{n=-3}^{\infty}$ of Eq. (4), $x_n \neq 1$ for $n \geq -3$.

If the solution is nonoscillatory about the positive equilibrium point $\bar{x} = 1$ of Eq. (4), then we know from Lemma 2.2 (a) that the solution is actually an eventually positive one. According to Lemma 2.2 (b), we see that $\{x_{2n}\}$, $\{x_{2n-1}\}$ are eventually decreasing and bounded from below by 1. So, the limits $\lim\limits_{n\to\infty} x_{2n} = L$, $\lim\limits_{n\to\infty} x_{2n-1} = M$ exist and are finite. Note

$$x_{2n+1} = \frac{x_{2n-1}x_{2n-2}^b x_{2n-3} + x_{2n-1} + x_{2n-2}^b + x_{2n-3} + a}{x_{2n-1}x_{2n-2}^b + x_{2n-2}^b x_{2n-3} + x_{2n-1}x_{2n-3} + 1 + a}, \quad n = 0, 1, 2, ...$$

$$x_{2n+2} = \frac{x_{2n}x_{2n-1}^b x_{2n-2} + x_{2n} + x_{2n-1}^b + x_{2n-2} + a}{x_{2n}x_{2n-1}^b + x_{2n-1}^b x_{2n-2} + x_{2n}x_{2n-2} + 1 + a}, \quad n = 0, 1, 2, ...$$

Take the limits on both sides of the above equalities, we obtain

$$M = \frac{M.L^b.M + M + L^b + M + a}{M.L^b + M.L^b + M^2 + 1 + a},$$

$$L = \frac{L^b.M.P + L^b + M + P + a}{L^b.M + M.P + L^b.P + 1 + a},$$

We have $M = L = 1$, which shows (9) is true. Thus, it suffices to prove that (9) holds for the solution to be the strictly oscillatory.

Assume now $\{x_n\}_{n=-3}^{\infty}$ to be strictly oscillatory about the possitive equilibrium point $\bar{x} = 1$ of Eq. (4). By virtue of Thoerem 3.1, one understands that the lengths of positive and negative semicycles which occur successively is

or $..., 3^-, 1^+, 1^-, 2^+, 3^-, 1^+, 1^-, 2^+, 3^-, 1^+, 1^-, 2^+, ...$

or $..., 3^+, 1^-, 1^+, 2^-, 3^+, 1^-, 1^+, 2^-, 3^+, 1^-, 1^+, 2^-...$

First, we investigate the case where the rule for the lengths of positive and negative semicycles which occur successively is

$..., 3^-, 1^+, 1^-, 2^+, 3^-, 1^+, 1^-, 2^+, 3^-, 1^+, 1^-, 2^+, ...$

For the sake of convenience, we denote by $\{x_p, x_{p+1}, x_{p+2}\}^-, \{x_{p+3}\}^+, \{x_{p+4}\}^-, \{x_{p+5},\ x_{p+6}\}^+$ the terms of negative and positive semicycles of length seven respectively. So, the rule for the negative and positive semicycles to occur successively can be periodically expressed as follows:

$$\{x_{p+7n}, x_{p+7n+1}, x_{p+7n+2}\}^-, \{x_{p+7n+3}\}^+, \{x_{p+7n+4}\}^-, \{x_{p+7n+5},\ x_{p+7n+6}\}^+, n = 0, 1, 2, \dots$$

We have easily the following inequalities

$$\frac{1}{x_{p+7(n-1)+3}} < x_{p+7n} < x_{p+7n+2} < x_{p+7n+4} < x_{p+7n+8} < \frac{1}{x_{p+7(n+1)+3}} < 1$$

We can see that $\{\frac{1}{x_{p+7(n+1)+3}}\}$ is increasing with upper bound 1. So, the limits

$$\lim_{n\to\infty} x_{p+7n} = \lim_{n\to\infty} x_{p+7n+2} = \lim_{n\to\infty} x_{p+7n+4} = \lim_{n\to\infty} x_{p+7n+8} = \lim_{n\to\infty} \frac{1}{x_{p+7(n+1)+3}} = L$$

exist and finite.
Noting that

$$x_{p+7n+4} = \frac{x_{p+7n+2}x^b_{p+7n+1}x_{p+7n} + x_{p+7n+2} + x^b_{p+7n+1} + x_{p+7n} + a}{x_{p+7n+2}x^b_{p+7n+1} + x^b_{p+7n+1}x_{p+7n} + x_{p+7n}x_{p+7n+2} + 1 + a}$$

Taking the limits on both sides of this equality, we obtain

$$L = \frac{L^{b+2} + 2L + L^b + a}{L^2 + 2L^{b+1} + 1 + a} \to L = 1.$$

Next, we also have

$$1 < x_{p+7n+5} < x_{p+7n+3} < x_{p+7(n-1)+6} < \frac{1}{x_{p+7(n-1)+2}}$$

Taking the limits on both sides of this equality, we obtain

$$\lim_{n\to\infty} x_{p+7n+5} = \lim_{n\to\infty} x_{p+7n+3} = \lim_{n\to\infty} x_{p+7n+6} = 1$$

Next, we investigate the case where the rule for the lengths of positive and negative semicycles which occur successively is

$..., 3^+, 1^-, 1^+, 2^-, 3^+, 1^-, 1^+, 2^-, 3^+, 1^-, 1^+, 2^-...$

The following results can be easily obtained from Lemma 2.2 (b), (c)

$$1 < \frac{1}{x_{p+7n+10}} < x_{p+7n+8} < x_{p+7n+4} < x_{p+7n+2} < x_{p+7n} < \frac{1}{x_{p+7(n-1)+3}}$$

We can see that $\{\frac{1}{x_{p+7(n+1)+3}}\}$ is decreasing with lower bound 1. Therefore,

$$\lim_{n\to\infty}\frac{1}{x_{p+7n+10}} = \lim_{n\to\infty} x_{p+7n+8} = \lim_{n\to\infty} x_{p+7n+4} = \lim_{n\to\infty} x_{p+7n+2} = \lim_{n\to\infty} x_{p+7n} = L$$

From the equation

$$x_{p+7n+4} = \frac{x_{p+7n+2}x^b_{p+7n+1}x_{p+7n} + x_{p+7n+2} + x^b_{p+7n+1} + x_{p+7n} + a}{x_{p+7n+2}x^b_{p+7n+1} + x^b_{p+7n+1}x_{p+7n} + x_{p+7n}x_{p+7n+2} + 1 + a}, n = 0, 1, 2, \ldots$$

Taking the limits on both sides of this equation, we have

$$L = \frac{L^{b+2} + 2L + L^b + a}{L^2 + 2L^{b+1} + 1 + a} \to L = 1.$$

From the inequalities:

$$\frac{1}{x_{p+7n+1}} < x_{p+7n+3} < x_{p+7n+5} < 1$$

Taking the limits on both sides of this equality, we have

$$\lim_{n\to\infty} x_{p+7n+3} = \lim_{n\to\infty} x_{p+7n+5} = 1$$

From the inequality:

$$\frac{1}{x_{p+7n+4}} < x_{p+7n+6} < 1$$

Taking the limits on both sides of this equality, we obtain

$$\lim_{n\to\infty} x_{p+7n+6} = 1$$

The proof is complete. □

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