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A Note on Global Behaviuor of Solutions and Positive Nonoscilatory Solutions of Rational Difference Equation

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Abstract

In this paper we study the global behaviors of solutions of the rational difference equation

$$x_{n+1} = \frac{g(x_n, x_{n-1}, x_{n-2}, x_{n-3})x_{n-3}^{\alpha+1} + f(x_n, x_{n-1}, x_{n-2}, x_{n-3})}{g(x_n, x_{n-1}, x_{n-2}, x_{n-3})x_{n-3}^{\alpha} + f(x_n, x_{n-1}, x_{n-2}, x_{n-3})},\ n = 0, 1, 2, ...$$

where $f, g : (0,\infty)^4 \to (0,\infty)$ is arbitrary and differentiable, $\alpha \geq 0$.

We also study the positive nonoscillatory solutions of the rational difference equation

$$x_{n+1} = \frac{A_1}{(x_n x_{n-1})^\alpha} + \frac{A_2}{(x_{n-1}x_{n-2})^\alpha} + \ldots + \frac{A_{k-1}}{(x_{n-k+2}x_{n-k+1})^\alpha} - \frac{1}{x_{n-k}^{2\alpha}},\ n = 0, 1, 2, ...$$

where $A_1, A_2, \ldots, A_{k-1} \in [0,\infty)$ and $A = \sum_{i=1}^{k-1} A_i - 1 > 0, \alpha > 0$.

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1 Introduction

It is extremely difficult to understand throughly the global behaviours of solutions of rational difference equations although they have simple forms. We can refer to [1-3, 7-12], especcialy [1,3] for example to illustrate this.

The study of rational difference equations of order greater than one is quite challenging and rewarding because some prototypes for the development of the basic theory of the global behaviour of nonlinear difference equations of order greater than one come from the results for rational difference equations.

G. Ladas [7] proposed to study the rational difference equation

$$x_{n+1} = \frac{x_n + x_{n-1}x_{n-2} + a}{x_n x_{n-1} + x_{n-2} + a}, \quad n = 0, 1, 2, ... \tag{1}$$

From then on, rational difference equations with the unique positive equilibrium $\bar{x} = 1$ have received considerable attention, one can refer to [10-12] and the references cited therein.

Recently, Li [9] investigated the global behavior of the following fourth-order rational difference equation

$$x_{n+1} = \frac{x_n x_{n-1} x_{n-3} + x_n + x_{n-1} + x_{n-3} + a}{x_n x_{n-1} + x_n x_{n-3} + x_{n-1} x_{n-3} + 1 + a}, \quad n = 0, 1, 2, ... \tag{2}$$

where $a \in [0, \infty)$ and initial values $x_{-3}, x_{-2}, x_{-1}, x_0 \in (0, \infty)$.

In the note, we consider the following fourth-order rational difference equation

$$x_{n+1} = \frac{g(x_n, x_{n-1}, x_{n-2}, x_{n-3})x_{n-3}^{\alpha+1} + f(x_n, x_{n-1}, x_{n-2}, x_{n-3})}{g(x_n, x_{n-1}, x_{n-2}, x_{n-3})x_{n-3}^{\alpha} + f(x_n, x_{n-1}, x_{n-2}, x_{n-3})}, \; n = 0, 1, 2, ... \tag{3}$$

where $f, g : (0, \infty)^4 \to (0, \infty)$ is arbitrary and differentiable, $\alpha \geq 0$ and initial values $x_{-3}, x_{-2}, x_{-1}, x_0 \in (0, \infty)$.

It is easy to see that the positive equilibrium $\bar{x}$ of Eq. (3) satisfies

$$\bar{x} = \frac{g(\bar{x}, \bar{x}, \bar{x}, \bar{x})\bar{x}^{\alpha+1} + f(\bar{x}, \bar{x}, \bar{x}, \bar{x})}{g(\bar{x}, \bar{x}, \bar{x}, \bar{x})\bar{x}^{\alpha} + f(\bar{x}, \bar{x}, \bar{x}, \bar{x})}$$

from which one can see that Eq. (3) has a unique positive equilibrium $\bar{x} = 1$.

2 Global behaviour of solutions

We need the following definition.

Definition 2.1. *A solution $\{x_n\}_{n=-3}^{\infty}$ of Eq. (3) is said to be enventually trivial if x_n eventually equal to $\bar{x} = 1$; otherwise, the solution is said to be nontrivial.*

Before to draw a qualitatively clear picture for the positive solutions of Eq. (3), we first establish a basic lemma which will play a key role in the proof of our main results.

Lemma 2.1. *Let $\{x_n\}_{n=-3}^{\infty}$ be a nontrivial positive solution of Eq. (3). Then the following conclusions are true for $n \geq 0$:*

(a) $(x_{n+1} - 1)(x_{n-3} - 1) > 0$

(b) $(x_{n+1} - x_{n-3})(x_{n-3} - 1) < 0$.

Proof. It follows in light of Eq. (3) that

$$x_{n+1} - 1 = \frac{(x_{n-3} - 1)g(x_n, x_{n-1}, x_{n-2}, x_{n-3})x_{n-3}^{\alpha}}{g(x_n, x_{n-1}, x_{n-2}, x_{n-3})x_{n-3}^{\alpha} + f(x_n, x_{n-1}, x_{n-2}, x_{n-3})}, \quad n = 0, 1, 2, ...$$

and

$$x_{n+1} - x_{n-3} = \frac{(1 - x_{n-3})f(x_n, x_{n-1}, x_{n-2}, x_{n-3})}{g(x_n, x_{n-1}, x_{n-2}, x_{n-3})x_{n-3}^{\alpha} + f(x_n, x_{n-1}, x_{n-2}, x_{n-3})}, \quad n = 0, 1, 2, ...$$

□

Remark 2.1. *If the initial conditions of the Eq. (3) satisfy $x_k \neq 1, k = -3, -2, -1, 0$, then, for any solution $\{x_{4n+k}\}$ of Eq. (3), $x_{4n+k} \neq 1, k = -3, -2, -1, 0$ for $n \geq 0$. It mean that*

(a) If $x_k \neq 1, k = -3, -2, -1, 0$ then $x_{4n+k} \neq 1$ for $n \geq 0$.

(a) If $x_k > 1, k = -3, -2, -1, 0$ then $x_{4n+k} > 1$ for $n \geq 0$.

(a) If $x_k < 1, k = -3, -2, -1, 0$ then $x_{4n+k} < 1$ for $n \geq 0$.

Theorem 2.1. *Assume that $\alpha \in [0, \infty), f, g : (0, \infty)^4 \to (0, \infty)$ are two functions differentiable and positive. Then the positive equilibrium of Eq. (3) is globally asymptotically stable.*

Proof. We must prove that the positive equilibrium point $\bar{x} = 1$ of Eq. (3) is both locally asymptotically stable and globally attractive. The linearized equation of Eq. (3) about the positive equilibrium $\bar{x} = 1$ is

$$y_{n+1} = 0.y_n + 0.y_{n-1} + 0.y_{n-2} + \frac{g(\bar{x}, \bar{x}, \bar{x}, \bar{x})}{g(\bar{x}, \bar{x}, \bar{x}, \bar{x}) + f(\bar{x}, \bar{x}, \bar{x}, \bar{x})}y_{n-3}, \quad n = 0, 1, 2, ...$$

By virtue of ([7], Remark 1.3.7), $\bar{x}$ is locally asymptotically stable. It remains to verify that every positive solution $\{x_n\}_{n=-3}^{\infty}$ of Eq. (3) converges to $\bar{x} = 1$ as $n \to \infty$. Namely, we want to prove

$$\lim_{n \to \infty} x_n = \bar{x} = 1 \tag{4}$$

If the initial values of the solution satisfy $x_{-3} = x_{-2} = x_{-1} = x_0 = 1$, then Lemma 2.1 says the solution is eventually equal to 1 and, of course, (4) holds.

Therefore, we assume in the following that the initial values of the solution do not satisfy $x_{-3} = x_{-2} = x_{-1} = x_0 = 1$. This mean that

$$(x_{-3}, x_{-2}, x_{-1}, x_0) \neq (1, 1, 1, 1).$$

Based on the Lemma 2.1 we see that one of the following sixteen cases must occur

1) $x_{-3} \leq 1,\ x_{-2} \leq 1,\ x_{-1} \leq 1, x_0 \leq 1.$
2) $x_{-3} \leq 1,\ x_{-2} \leq 1,\ x_{-1} \leq 1, x_0 \geq 1.$
3) $x_{-3} \leq 1,\ x_{-2} \leq 1,\ x_{-1} \geq 1, x_0 \geq 1.$
4) $x_{-3} \leq 1,\ x_{-2} \leq 1,\ x_{-1} \geq 1, x_0 \leq 1.$
5) $x_{-3} \leq 1,\ x_{-2} \geq 1,\ x_{-1} \leq 1, x_0 \leq 1.$
6) $x_{-3} \leq 1,\ x_{-2} \geq 1,\ x_{-1} \leq 1, x_0 \geq 1.$
7) $x_{-3} \leq 1,\ x_{-2} \geq 1,\ x_{-1} \geq 1, x_0 \geq 1.$
8) $x_{-3} \leq 1,\ x_{-2} \geq 1,\ x_{-1} \geq 1, x_0 \leq 1.$
9) $x_{-3} \geq 1,\ x_{-2} \leq 1,\ x_{-1} \leq 1, x_0 \geq 1.$
10) $x_{-3} \geq 1,\ x_{-2} \leq 1,\ x_{-1} \leq 1, x_0 \leq 1.$
11) $x_{-3} \geq 1,\ x_{-2} \leq 1,\ x_{-1} \geq 1, x_0 \geq 1.$
12) $x_{-3} \geq 1,\ x_{-2} \leq 1,\ x_{-1} \geq 1, x_0 \leq 1.$
13) $x_{-3} \geq 1,\ x_{-2} \geq 1,\ x_{-1} \leq 1, x_0 \geq 1.$
14) $x_{-3} \geq 1,\ x_{-2} \geq 1,\ x_{-1} \leq 1, x_0 \leq 1.$
15) $x_{-3} \geq 1,\ x_{-2} \geq 1,\ x_{-1} \geq 1, x_0 \geq 1.$
16) $x_{-3} \geq 1,\ x_{-2} \geq 1,\ x_{-1} \geq 1, x_0 \leq 1.$

If case 1) occurs from Lemma 2.1 a) we obtain $x_{4n+k} \leq 1, k = 0, -1, -2, -3$ for $n \geq 0$. From b) of the Lemma 2.1 we have

$$x_{4n+k} \leq x_{4(n+1)+k} \leq 1.$$

The sequences $\{x_{4n+k}\}_{n=0}^{\infty}, k = 0, -1, -2, -3$, are increasing with upper bound 1. So the limits $\lim\limits_{n\to\infty} x_{4n} = P,\ \lim\limits_{n\to\infty} x_{4n+1} = Q,\ \lim\limits_{n\to\infty} x_{4n+2} = R,\ \lim\limits_{n\to\infty} x_{4n+3} = L$ exist and are finite, too.

Nothing that

$$x_{4n+3} = \frac{g(x_{4n+2}, x_{4n+1}, x_{4n}, x_{4n-1})x_{4n-1}^{\alpha+1} + f(x_{4n+2}, x_{4n+1}, x_{4n}, x_{4n-1})}{g(x_{4n+2}, x_{4n+1}, x_{4n}, x_{4n-1})x_{4n-1}^{\alpha} + f((x_{4n+2}, x_{4n+1}, x_{4n}, x_{4n-1})}, \ n = 0, 1, 2, ...$$

$$x_{4n+2} = \frac{g(x_{4n+1}, x_{4n}, x_{4n-1}, x_{4n-2})x_{4n-2}^{\alpha+1} + f(x_{4n+1}, x_{4n}, x_{4n-1}, x_{4n-2})}{g(x_{4n+1}, x_{4n}, x_{4n-1}, x_{4n-2})x_{4n-2}^{\alpha} + f((x_{4n+1}, x_{4n}, x_{4n-1}, x_{4n-2})}, \ n = 0, 1, 2, ...$$

$$x_{4n+1} = \frac{g(x_{4n}, x_{4n-1}, x_{4n-2}, x_{4n-3})x_{4n-3}^{\alpha+1} + f(x_{4n}, x_{4n-1}, x_{4n-2}, x_{4n-3})}{g(x_{4n}, x_{4n-1}, x_{4n-2}, x_{4n-3})x_{4n-3}^{\alpha} + f(x_{4n}, x_{4n-1}, x_{4n-2}, x_{4n-3})}, \ n = 0, 1, 2, ...$$

$$x_{4n} = \frac{g(x_{4n-1}, x_{4n-2}, x_{4n-3}, x_{4n-4})x_{4n-4}^{\alpha+1} + f(x_{4n-1}, x_{4n-2}, x_{4n-3}, x_{4n-4})}{g(x_{4n-1}, x_{4n-2}, x_{4n-3}, x_{4n-4})x_{4n-4}^{\alpha} + f(x_{4n-1}, x_{4n-2}, x_{4n-3}, x_{4n-4})}, \ n = 0, 1, 2, ...$$

and taking the limit on both sides of those above equalities one can see that

$$T = \frac{g(R,Q,P,T)T^{\alpha+1} + f(R,Q,P,T)}{g(R,Q,P,T)T^{\alpha} + f(R,Q,P,T)} \Rightarrow T = 1.$$

$$R = \frac{g(Q,P,T,R)R^{\alpha+1} + f(Q,P,T,R)}{g(Q,P,T,R)R^{\alpha+1} + f(Q,P,T,R)} \Rightarrow R = 1.$$

$$Q = \frac{g(P,T,R,Q)Q^{\alpha+1} + f(P,T,R,Q)}{g(P,T,R,Q)Q^{\alpha} + f(P,T,R,Q)} \Rightarrow Q = 1.$$

$$P = \frac{g(T,R,Q,P)P^{\alpha+1} + f(T,R,Q,P)}{g(T,R,Q,P)P^{\alpha} + f(T,R,Q,P)} \Rightarrow P = 1.$$

Thus, $P = Q = R = L = 1$, we have (4).
If case 2) occurs: $x_{-3} \leq 1,\ x_{-2} \leq 1,\ x_{-1} \leq 1, x_0 \geq 1$. By the similar arguments, we have $\{x_{4n+1}\}_{n=0}^{\infty}, \{x_{4n+2}\}_{n=0}^{\infty}, \{x_{4n+3}\}_{n=0}^{\infty}$ are increasing with upper bound 1, and $\{x_{4n}\}_{n=0}^{\infty}$ is decreasing with lower bound 1. So, the limits

$$\lim_{n\to\infty} x_{4n} = P,\ \lim_{n\to\infty} x_{4n+1} = Q,\ \lim_{n\to\infty} x_{4n+2} = R,\ \lim_{n\to\infty} x_{4n+3} = L$$

exist and are finite, too. By the above similar method, we have $P = Q = R = L = 1$, too. All of the rest cases we obtain the same result $\lim_{n\to\infty} x_n = \bar{x} = 1$. □

3 Positive nonoscillatory solutions

In the paper we also show that the following difference equation

$$x_{n+1} = \frac{A_1}{(x_n x_{n-1})^{\alpha}} + \frac{A_2}{(x_{n-1}x_{n-2})^{\alpha}} + \ldots + \frac{A_{k-1}}{(x_{n-k+2}x_{n-k+1})^{\alpha}} - \frac{1}{x_{n-k}^{2\alpha}},\ n = 0,1,2,... \tag{5}$$

where $A_1, A_2, \ldots, A_{k-1} \in [0,\infty)$ and $A = \sum_{i=1}^{k-1} A_i - 1 > 0, \alpha > 0$, initial values $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in (0,\infty)$, has positive nonoscillatory solutions which converge to the positive equilibrium

$$\bar{x}^{2\alpha+1} = \sum_{i=1}^{k-1} A_i - 1 = A$$

$$\bar{x} = \sqrt[2\alpha+1]{\sum_{i=1}^{k-1} A_i - 1} = \sqrt[2\alpha+1]{A}$$

Note that the linearized equation of Eq. (5) about the positive equilibrium $\bar{x} = \sqrt[2\alpha+1]{A}$ is

$$y_{n+1} = -\frac{\alpha A_1}{\bar{x}^{2\alpha+1}} y_n - \frac{\alpha(A_1 + A_2)}{\bar{x}^{2\alpha+1}} y_{n-1} - \cdots - \frac{\alpha(A_{k-2} + A_{k-1})}{\bar{x}^{2\alpha+1}} y_{n-k+2} - $$
$$- \frac{\alpha A_{k-1}}{\bar{x}^{2\alpha+1}} y_{n-k+1} + \frac{2\alpha}{\bar{x}^{2\alpha+1}} y_{n-k}, \; n = 0, 1, 2, ...$$

$$\bar{x}^{2\alpha+1} y_{n+1} + \alpha A_1 y_n + \alpha(A_1 + A_2) y_{n-1} + \cdots + \alpha(A_{k-2} + A_{k-1}) y_{n-k+2} + \tag{6}$$
$$+\alpha A_{k-1} y_{n-k+1} - 2\alpha y_{n-k} = 0$$

The characteristic polynomial associated with Eq. (6) is

$$p(t) = \bar{x}^{2\alpha+1} t^{k+1} + \alpha A_1 t^k + \alpha(A_1 + A_2) t^{k-1} + \cdots + \alpha(A_{k-2} + A_{k-1}) t^2 + \alpha A_{k-1} t - 2\alpha = 0 \tag{7}$$

Since $p(0) = -2\alpha < 0, p(1) = \bar{x}^{2\alpha+1} + 2\alpha(A_1 + A_1 + A_2 + A_{k-2} + A_{k-1} + A_{k-1} - 1) > 0$.

$$p'(t) = (k+1)\bar{x}^{2\alpha+1} t^k + k\alpha A_1 t^{k-1} + \cdots + 2\alpha(A_{k-2} + A_{k-1}) t + \alpha A_{k-1} > 0,$$

for $t \in (0, 1)$ and $\sum_{i=1}^{k-1} A_i - 1 > 0$.
It follow that for $\alpha \in (0, \infty)$ and $\sum_{i=1}^{k-1} A_i - 1 = A > 0$, there is a unique positive root $t_0 \in (0, 1)$ of polynomial belonging to the inteval (0,1). It mean that

$$p(t_0) = \bar{x}^{2\alpha+1} t_0^{k+1} + \alpha A_1 (t_0^k + t_0^{k-1}) + \cdots + \alpha A_{k-1} (t_0^2 + t_0) - 2\alpha = 0 \tag{8}$$

This fact motivated us to believe that there are solutions of Eq. (5) which have the following asymptotics

$$x_n = \bar{x} + a t_0^n + o(t_0^n) \tag{9}$$

where $a \in \mathbb{R}$ and t_0 is the above mentioned root of Eq. (7). We solve the open problem, showing that such a solition exits, developing Berg's ideas in [4] which are based on the asymptotics. The asymptotics for solutions of difference equation have been investigated by L. Berg and S. Stević, see, for example [4-6], [8-12] and the reference therein. The problem is solved by constructing appropriate sequences y_n and z_n with

$$y_n \le x_n \le z_n \tag{10}$$

for sufficiently large n. In [4-6] some methods can be found for the construction of these bounds, see, also [7,8].

For (9) we expect that for $k \geq 2$ such solutions have the first three members in their asymptotics in the following form

$$\varphi_n = \bar{x} + at_0^n + bt_0^{2n} \tag{11}$$

This is proved by developing Berg's ideas in [4] which are based on asymptotics. We need the following result in the proof of main theorem. The proof of the following theorem can be found in [10-12].

Theorem 3.1. *Let $f : I^{k+2} \to I$ be a continuous and nondecreasing function in each argument on the interval $I \subset \mathbb{R}$, and let $\{y_n\}$ and $\{z_n\}$ be sequences with $y_n \leq z_n$ for $n \geq n_0$ and such that*

$$y_{n-k} \leq f(n, y_{n-k+1}, ..., y_{n+1}), f(n, z_{n-k+1}, ..., z_{n+1}) \leq z_{n-k} \text{ for } n > n_0 + k - 1 \tag{12}$$

Then there is a solution of the following difference equation

$$x_{n-k} = f(n, x_{n-k+1}, ..., x_{n+1}) \tag{13}$$

with property (10) for $n \geq n_0$.

Theorem 3.2. *For each $\alpha > 0$ and $\sum_{i=1}^{k-1} A_i - 1 = A > 0$ and $A_1, A_2, \ldots, A_{k-1} \in [0, \infty)$ there is a nonoscillatory solution of Eq. (5) converging to the positive equilibrium*

$$\bar{x} = \sqrt[2\alpha+1]{\sum_{i=1}^{k-1} A_i - 1} = \sqrt[2\alpha+1]{A}, \text{ as } n \to \infty.$$

Proof. First, Eq. (5) can be written in the form

$$x_{n-k} = \left[\frac{A_1}{(x_n x_{n-1})^\alpha} + \frac{A_2}{(x_{n-1} x_{n-2})^\alpha} + \ldots + \frac{A_{k-1}}{(x_{n-k+2} x_{n-k+1})^\alpha} - x_{n+1}\right]^{\frac{-1}{2\alpha}}$$

Setting

$F(x_{n-k}, x_{n-k+1}, \ldots, x_n, x_{n+1}) =$

$$= \left[\frac{A_1}{(x_n x_{n-1})^\alpha} + \frac{A_2}{(x_{n-1} x_{n-2})^\alpha} + \ldots + \frac{A_{k-1}}{(x_{n-k+2} x_{n-k+1})^\alpha} - x_{n+1}\right]^{\frac{-1}{2\alpha}} - x_{n-k} \tag{14}$$

We expect the solutions of Eq. (5) have the asymptotic appropriation (11).

$F(\varphi_{n-k}, \varphi_{n-k+1}, \ldots, \varphi_n, \varphi_{n+1}) =$

$$= \left\{ \frac{A_1}{[(\bar{x} + at^n + bt^{2n})(\bar{x} + at^{n-1} + bt^{2n-2})]^\alpha} + \frac{A_2}{[(\bar{x} + at^{n-1} + bt^{2n-2})(\bar{x} + at^{2n-2} + bt^{2n-4})]^\alpha} + \right.$$

$$\cdots + \frac{A_{k-1}}{[(\bar{x}+at^{n-k+2}+bt^{2n-2k+4})(\bar{x}+at^{n-k+1}+bt^{2n-2k+2})]^{\alpha}} - (\bar{x}+at^{n+1}+bt^{2n+2})\Bigg\}^{\frac{-1}{2\alpha}} -$$

$$-\bar{x} - at^{n-k} - bt^{2n-2k}.$$

$$F = \frac{a}{2\alpha t^k}[\bar{x}^{2\alpha+1}t^{k+1}\alpha A_1(t^k+t^{k-1}) + \alpha A_2(t^{k-1}+t^{k-2}) + \cdots + \alpha A_{k-1}(t^2+t) - 2\alpha]t^n + \tag{15}$$

$+\dfrac{b}{2\alpha t^{2k}}[\bar{x}^{2\alpha+1}t^{2k+2}\alpha A_1(t^{2k}+t^{2k-2})+\alpha A_2(t^{2k-2}+t^{2k-4})+\cdots+\alpha A_{k-1}(t^4+t^2)-$
$2\alpha]t^{2n}+$
$+\dfrac{a^2}{4\bar{x}}\Bigg\{\dfrac{1+2\alpha}{2}\Big[A_1(t^{-1}+1)+A_2(t^{-1}+t^{-2})+\cdots+A_{k-1}(t^{-k+2}+t^{-k+1})+\dfrac{\bar{x}^{2\alpha+1}}{\alpha}t\Big]^2-$
$-\Big[(\alpha+1)A_1t^{-2}+(\alpha+1)A_1+2\alpha A_1t^{-1}+(\alpha+1)A_2t^{-4}+(\alpha+1)A_2t^{-2}+2\alpha A_1t^{-3}+$
$\cdots$
$+(\alpha+1)A_{k-1}t^{-2k+2}+(\alpha+1)A_{k-1}t^{-2k}+2\alpha A_{k-1}t^{-2k+1}\Big]\Bigg\}t^{2n}+o(t^{2n}).$

Here

$$\bar{x}^{2\alpha+1} = \sum_{i=1}^{k-1} A_i - 1 = A$$

As mentioned earlier exits $t_0 \in (0,1)$ such that

$$p(t_0) = \bar{x}^{2\alpha+1}t_0^{k+1}+\alpha A_1(t_0^k+t_0^{k-1})+\alpha A_2(t_0^{k-1}+t_0^{k-2})+\cdots+\alpha A_{k-1}(t_0^2+t_0)-2\alpha = 0$$

$$p(t_0^2) = \bar{x}^{2\alpha+1}t_0^{2k+2}+\alpha A_1(t_0^{2k}+t_0^{2k-2})+\alpha A_2(t_0^{2k-2}+t_0^{2k-4})+\cdots+\alpha A_{k-1}(t_0^4+t_0^2)-2\alpha < 0$$

Posing in (15), $t = t_0$, we obtain
$F = \dfrac{bp(t_0^2)t_0^{2n}}{2\alpha t_0^{2k}} + \dfrac{1}{4\bar{x}}\Big\{\dfrac{1+2\alpha}{2}\Big[A_1(t_0^{-1}+1)+A_2(t_0^{-1}+t_0^{-2})+\cdots+A_{k-1}(t_0^{-k+2}+$
$t_0^{-k+1})+\dfrac{\bar{x}^{2\alpha+1}}{\alpha}t_0\Big]^2-\Big[(\alpha+1)A_1t_0^{-2}+(\alpha+1)A_1+2\alpha A_1t_0^{-1}+(\alpha+1)A_2t_0^{-4}+(\alpha+$
$1)A_2t_0^{-2}+2\alpha A_1t_0^{-3}+\cdots+(\alpha+1)A_{k-1}t_0^{-2k+2}+(\alpha+1)A_{k-1}t_0^{-2k}+2\alpha A_{k-1}t_0^{-2k+1}\Big]\Big\}t_0^{2n}+$
$o(t_0^{2n})$
$= (Bb+C)t_0^{2n}+o(t_0^{2n}).$
where

$B = \dfrac{p(t_0^2)}{2\alpha t_0^{2k}} < 0$

$C = \dfrac{1}{4\bar{x}}\Big\{\dfrac{1+2\alpha}{2}\Big[A_1(t_0^{-1}+1)+A_2(t_0^{-1}+t_0^{-2})+\cdots+A_{k-1}(t_0^{-k+2}+t_0^{-k+1})+$
$\dfrac{\bar{x}^{2\alpha+1}}{\alpha}t_0\Big]^2-$

$$-\Big[(\alpha+1)A_1t_0^{-2}+(\alpha+1)A_1+2\alpha A_1t_0^{-1}+(\alpha+1)A_2t_0^{-4}+(\alpha+1)A_2t_0^{-2}+$$
$$+2\alpha A_1t_0^{-3}+\cdots+(\alpha+1)A_{k-1}t_0^{-2k+2}+(\alpha+1)A_{k-1}t_0^{-2k}+2\alpha A_{k-1}t_0^{-2k+1}\Big]\Big\}.$$

Setting

$$H_{t_0}(q)=Bq+C=0\Rightarrow q_0=-\frac{C}{B},$$
$$H'_{t_0}(q)=B<0.$$

We obtain that there are $q_1<q_0$ and $q_2>q_0$ such that $H_{t_0}(q_1)>0, H_{t_0}(q_2)<0, q_1<q_0<q_2$. We assume that $a\neq 0$. If $\widehat{\varphi}_n=\overline{x}+at_0^n+q_0t_0^{2n}$, we obtain

$$F(\widehat{\varphi}_{n-k},\widehat{\varphi}_{n-k+1},\ldots,\widehat{\varphi}_n,\widehat{\varphi}_{n+1})\underset{n\to\infty}{\sim}[q_0B+C]t_0^{2n}.$$

With the notations

$$y_n=\overline{x}+at_0^n+q_1t_0^{2n}$$
$$z_n=\overline{x}+at_0^n+q_2t_0^{2n}.$$

We get

$$F(y_{n-k},y_{n-k+2},\ldots,y_n,y_{n+1})\sim[q_1B+C]t_0^{2n},$$
$$F(z_{n-k},z_{n-k+2},\ldots,z_n,z_{n+1})\sim[q_2B+C]t_0^{2n}.$$

These relations show that inequalities (12) are satisfied for sufficiently large n, where $f=F+x_{n-k}$ and F is given by (14). Because the function $f(x_{n-k+1},x_{n-k+2},\ldots,x_n,x_{n+1})$ is continuous and nondecreasing on the interval $[\bar{x},+\infty)$. We easily have $f(\bar{x},\bar{x},\bar{x},\bar{x})=\bar{x}$. We can apply Theorem 3.1 with $I=[\bar{x},+\infty)$ and see that there is an $n_0\geq 0$ and a solution of Eq. (5) with the asymptotics $x_n=\widehat{\varphi}_n+o(t_0^{2n})$ for $n\geq n_0$, where $b=q_0$ in $\widehat{\varphi}_n$. In particular, the solution converge monotonically to the positive equilibrium $\bar{x}=\sqrt[2\alpha+1]{A}$, for $n\geq n_0$. Hence, the solution x_{n+n_0+k} is also such a solution when $n\geq -k$.
The proof is complete.

$\square$

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