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Some Mappings in Connection

to Hadamard's Inequality

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Abstract

In this paper we point out to two new inequalities of the Hadamard's type and use a simple new technique in the proof.

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1. Introduction

Let $f : I \subset \Re \to \Re$ be a convex mapping of the interval I of real numbers and a , b$\in I$ with a<b. The following double inequality

(1) $$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}$$

is known in the literature as Hadamard's inequality. In [1], Fejer generalized the inequality (1) by proving that if $g : [a,b] \to \Re$ is nonnegative, integrable and symmetric to $x = \frac{a+b}{2}$, and if f is convex on [a,b], then

(2) $$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \le \int_a^b f(x)\,g(x)\,dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx\,.$$

Recently, Dragomir [2], proved the following results

Theorem 1.1. Let $f:[a,b]\to\Re$ be convex and let $H:[0,1]\to\Re$ be defined by

$$H(t)=\frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)dx.$$

Then,

(i) H is convex in [0,1]

(ii) We have

$$\inf_{t\in[0,1]} H(t)=H(0)=f\left(\frac{a+b}{2}\right),$$

$$\sup_{t\in[0,1]} H(t)=H(1)=\frac{1}{b-a}\int_a^b f(x)\,dx$$

(iii) H increases monotonically on [0,1] .

Theorem 1.2. Let $f:[a,b]\to\Re$ be convex and let $F:[0,1]\to\Re$ be defined by

$$F(t)=\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(tx+(1-t)y\right)dx\,dy .$$

Then,

(i) $F\left(\tfrac{1}{2}+t\right)=F\left(\tfrac{1}{2}-t\right)$ *for all* t *in* $[0,\tfrac{1}{2}]$

(ii) F is convex on [0,1]

(iii) We have

$$\sup_{t\in[0,1]} F(t)=F(0)=F(1)=\frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\inf_{t\in[0,1]} F(t)=F\left(\tfrac{1}{2}\right)=\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dx\,dy .$$

(iii) The following inequality is valid

$$f\left(\frac{a+b}{2}\right)\le F\left(\frac{1}{2}\right).$$

(v) F decreases monotonically on [0,1/2] and increases monotonically on [1/2,1].

(vi) We have the inequality

$$H(t)\le F(t)\qquad\text{for all } t\in[0,1].$$

We said that $(x,y)\ge 0$, *if* $x,y\ge 0$.

The object of this paper is to present the following new results via new simpler technique.

2. Main Results

We state and prove the following

Theorem 2.1. Let *f* be convex and let $M:[0,1]\times[0,1]\to\Re$ be defined by

$$M(t_1,t_2)=\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(t_1x+t_2y+(1-t_1-t_2)\frac{a+b}{2}\right)dxdy.$$

Then

(i) M is convex on [0,1] .

(ii) $\inf_{t,t\in[0,1]} M(t_1,t_2)= M(0,0)= f\left(\frac{a+b}{2}\right),$

$$\sup_{t_1,t_2\in[0,1]} M(t_1,t_2)=M(1,0)=M(0,1)=\frac{1}{b-a}\int_a^b f(x)\,dx.$$

(iii) $M(t_1,t_2)$ is increasing on [0,1] with respect to t_1 and t_2 .

(iv) $M(t_1,t_2)\geq M_1\left(\frac{t_1+t_2}{2}\right)$, where

$$M_1(t)=\int_a^b\int_a^b f\left(t(x+y)+(1-t)\frac{a+b}{2}\right)dx\,dy.$$

Proof. (i). Let $\alpha,\beta\geq 0$, with $\alpha+\beta=1$, and let $s_1,s_2,t_1,t_2\in[0,1]$. Then

$$M(\alpha(s_1,s_2)+\beta((t_1,t_2)))=M(\alpha s_1+\beta t_1,\alpha s_2+\beta t_2)$$

$$=\frac{1}{(b-a)^2}\int_a^b\int_a^b f((\alpha s_1+\beta t_1)x+(\alpha s_2+\beta t_2)y$$

$$+\left.(1-\alpha s_1-\beta t_1-\alpha s_2-\beta t_2)\frac{a+b}{2}\right)dx\,dy$$

$$=\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\alpha\left(s_1x+s_2y+(1-s_1-s_2)\frac{a+b}{2}\right)\right.$$

$$\left.+\beta\left(t_1x+t_2y+(1-t_1-t_2)\frac{a+b}{2}\right)\right)dx\,dy$$

$$\leq\alpha\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(s_1x+s_2y+(1-s_1-s_2)\frac{a+b}{2}\right)dxdy$$

$$+\ \beta\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(t_1x+t_2y+(1-t_1-t_2)\frac{a+b}{2}\right)dx\,dy$$

$$=\alpha M(s_1,s_2)+\beta M(t_1,t_2).$$

(ii) .

$$f\left(\frac{a+b}{2}\right)=\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{a+b}{2}\right)dx\,dy$$

$$=\frac{1}{(b-a)^2}$$

$$\times\int_a^b\int_a^b f\left(\frac{t_1x+t_2y+(1-t_1-t_2)\frac{a+b}{2}+t_1(a+b-x)+t_2(a+b-y)+(1-t_1-t_2)\frac{a+b}{2}}{2}\right)$$

$$\times dx\,dy$$

$$\leq\frac{1}{2}\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(t_1x+t_2y+(1-t_1-t_2)\frac{a+b}{2}\right)dx\,dy$$

$$+\ \frac{1}{2}\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(t_1(a+b-x)+t_2(a+b-y)+(1-t_1-t_2)\frac{a+b}{2}\right)dx\,dy$$

$$=\ \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(t_1x+t_2y+(1-t_1-t_2)\frac{a+b}{2}\right)dx\,dy$$

$$=M(t_1,t_2).$$

Therefore, we have

$$\inf_{t,t\in[0,1]}M(t_1,t_2)=\ f\left(\frac{a+b}{2}\right).$$

By the convexity of M we have

$$M(t_1,t_2)\leq\frac{1}{(b-a)^2}\left(t_1\int_a^b\int_a^b f(x)\,dx\,dy+t_2\int_a^b\int_a^b f(y)\,dx\,dy+(1-t_1-t_2)\,f\left(\frac{a+b}{2}\right)\int_a^b\int_a^b dx\,dy\right)$$

$$= \frac{1}{(b-a)}\left((t_1+t_2)\int_a^b f(x)\,dx\right)+(1-t-t)\,f\left(\frac{a+b}{2}\right).$$

Hence , by above we have

$$f\left(\frac{a+b}{2}\right)\le M(1,0)=\frac{1}{(b-a)}\int_a^b f(x)\,dx,$$

which implies that

$$M(t_1,t_2)\le \frac{1}{(b-a)}\int_a^b f(x)\,dx.$$

Therefore, we obtain

$$\sup_{t_1,t_2\in[0,1]} M(t_1,t_2)=\frac{1}{(b-a)}\int_a^b f(x)\,dx.$$

(iii) . Let $s_1,s_2,t_1,t_2\in[0,1]$ with $s_1\ge t_1,\ s_2\ge t_2$. By the convexity of M, we have

$$\frac{M(s_1,s_2)-M(t_1,t_2)}{(s_1,s_2)-(t_1,t_2)} = \frac{M(t_1,t_2)-M(0,0)}{(t_1,t_2)-(0,0)}\ge 0.$$

Therefore, we have $M(s_1,s_2)\ge M(t_1,t_2)$.

(iv). Since

$$\begin{aligned} M(t_1,t_2) &= \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(t_1x+t_2y+(1-t_1-t_2)\frac{a+b}{2}\right)dx\,dy \\ &= \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(t_2y+t_1x+(1-t_2-t_1)\frac{a+b}{2}\right)dy\,dx \\ &= M(t_2,t_1),\ \text{then} \end{aligned}$$

$$\begin{aligned} M(t_1,t_2) &= \frac{M(t_1,t_2)+M(t_2,t_1)}{2} \\ &= \frac{1}{(b-a)^2}\times \end{aligned}$$

$$\int_a^b\int_a^b \frac{f\left(t_1x+t_2y+(1-t_1-t_2)\frac{a+b}{2}\right)+f\left(t_2x+t_1y+(1-t_2-t_1)\frac{a+b}{2}\right)}{2}dx\,dy$$

$$\begin{aligned} &\ge \int_a^b\int_a^b f\left(\left(\frac{t_1+t_2}{2}\right)(x+y)+\left(1-\frac{t_1+t_2}{2}\right)\frac{a+b}{2}\right)dx\,dy \\ &= M_1\left(\frac{t_1+t_2}{2}\right). \end{aligned}$$

Theorem 2.2. Let $f:[a,b]\to\Re$ be convex and let $N:[a,b]\times[a,b]]\to\Re$ be defined by

$$N(t_1,t_2)=\int_a^b\int_a^b\int_a^b f\left(t_1x+t_2y+(1-t_1-t_2)z\right)dx\,dy\,dz\,.$$

Then

(i) N is convex on $[0,1]$

(ii) $$\inf_{t_1,t_2\in[0,1]} N(t_1,t_2)=N\left(\tfrac{1}{3},\tfrac{1}{3}\right)=\frac{1}{(b-a)^3}\int_a^b\int_a^b\int_a^b f\left(\frac{x+y+z}{3}\right)dx\,dy\,dz\,,$$

$$\sup_{t_1,t_2\in[0,1]} N(t_1,t_2)=N(1,0)=N(0,1)=\frac{1}{b-a}\int_a^b f(x)\,dx\,.$$

(iii) $$f\left(\frac{a+b}{2}\right)\le N\left(\frac{1}{3},\frac{1}{3}\right)$$

(iv) $N(t_1,t_2)$ is increasing on $[\tfrac{1}{3},1]$ for both t_1 and t_2.

(v) $M(t_1,t_2)\le N(t_1,t_2)$.

Proof. (i). It is exactly as (i) in theorem 2.1.

(ii). Let us write

$$\begin{aligned}
t_1x+t_2y+(1-t_1-t_2)z &= p_1\\
t_1x+t_2z+(1-t_1-t_2)y &= p_2\\
t_1y+t_2x+(1-t_1-t_2)z &= p_3\\
t_1y+t_2z+(1-t_1-t_2)x &= p_4\\
t_1z+t_2x+(1-t_1-t_2)y &= p_5\\
t_1z+t_2y+(1-t_1-t_2)x &= p_6\,.
\end{aligned}$$

$$\begin{aligned}
N\left(\tfrac{1}{3},\tfrac{1}{3}\right) &= \frac{1}{(b-a)^3}\int_a^b\int_a^b\int_a^b f\left(\frac{x+y+z}{3}\right)dx\,dy\,dz\\
&= \frac{1}{(b-a)^3}\int_a^b\int_a^b\int_a^b f\left(\frac{1}{6}\sum\nolimits_{i=1}^6 p_i\right)dxdy\,dz\\
&\le \frac{1}{(b-a)^3}\int_a^b\int_a^b\int_a^b \frac{1}{6}\left(\sum\nolimits_{i=1}^6 f(p_i)\right)dx\,dy\,dz\\
&\le \frac{1}{(b-a)}\frac{1}{6}6\int_a^b\int_a^b\int_a^b f(p_1)\,dx\,dy\,dz\\
&= \frac{1}{(b-a)}\int_a^b\int_a^b\int_a^b f\left(t_1x+t_2y+(1-t_1-t_2)z\right)dx\,dy\,dz
\end{aligned}$$

Therefore, we have

$$\inf_{t_1,t_2\in[0,1]} N(t_1,t_2)=N\left(\tfrac{1}{3},\tfrac{1}{3}\right).$$

By the convexity of N, we have

$$N(t_1,t_2) \le \frac{1}{(b-a)^3}\left(\int_a^b\int_a^b\int_a^b \big(t_1 f(x) + t_2 f(y) + (1-t_1-t_2) f(z)\big) dx\, dy\, dz \right)$$

$$= \frac{1}{(b-a)}\int_a^b \big(t_1 f(x) + t_2 f(y) + (1-t_1-t_2) f(z)\big) dx\, dy\, dz$$

$$= \frac{1}{(b-a)}\int_a^b f(x)\, dx.$$

Hence, we obtain

$$\sup_{t_1,t_2\in[0,1]} N(t_1,t_2) = \frac{1}{(b-a)}\int_a^b f(x)\, dx.$$

(iii).

$$f\left(\frac{a+b}{2}\right) = \frac{1}{(b-a)^3}\int_a^b\int_a^b\int_a^b f\left(\frac{a+b}{2}\right) dx\, dy\, dz$$

$$= \frac{1}{(b-a)^3}\int_a^b\int_a^b\int_a^b f\left(\frac{\frac{x+y+z}{3} + \frac{(a+b-x)+(a+b-y)+(a+b-z)}{3}}{2}\right) dx\, dy\, dz$$

$$\le \frac{1}{2}\frac{1}{(b-a)^3}\int_a^b\int_a^b\int_a^b f\left(\frac{x+y+z}{3}\right) dx\, dy\, dz$$

$$+\frac{1}{2}\frac{1}{(b-a)^3}\int_a^b\int_a^b\int_a^b f\left(\frac{(a+b-x)+(a+b-y)+(a+b-z)}{3}\right) dx\, dy\, dz$$

$$= \frac{1}{(b-a)^3}\int_a^b\int_a^b\int_a^b f\left(\frac{x+y+z}{3}\right) dx\, dy\, dz.$$

(iv) . Let $s_1, s_2, t_1, t_2 \in [\frac{1}{3}, 1]$ with $s_1 \ge t_1,\ s_2 \ge t_2$. By the convexity of M, we have

$$\frac{M(s_1,s_2) - M(t_1,t_2)}{(s_1,s_2)-(t_1,t_2)} \ge \frac{M(t_1,t_2) - M(\frac{1}{3},\frac{1}{3})}{(t_1,t_2)-(\frac{1}{3},\frac{1}{3})} \ge 0.$$

Therefore, we have $M(s_1,s_2) \ge M(t_1,t_2).$

(v).

$$M(t_1,t_2)=\frac{1}{(b-a)^3}\int_a^b\int_a^b\int_a^b f\left(t_1x+t_2y+(1-t_1-t_2)\frac{a+b}{2}\right)dx\,dy\,dz$$

$$=\frac{1}{(b-a)^3}\times$$

$$\int_a^b\int_a^b\int_a^b f\left(\frac{t_1x+t_2y+(1-t_1-t_2)z+t_1x+t_2y+(1-t_1-t_2)(a+b-z)}{2}\right)$$

$$\times\,dx\,dy\,dz$$

$$\le\frac{1}{2}\frac{1}{(b-a)^3}\int_a^b\int_a^b\int_a^b f\big(t_1x+t_2y+(1-t_1-t_2)z\big)dx\,dy\,dz$$

$$+\frac{1}{2}\frac{1}{(b-a)^3}\int_a^b\int_a^b\int_a^b f\big(t_1(a+b-x)+t_2(a+b-y)+(1-t_1-t_2)(a+b-z)\big)dxdy\,dz$$

$$=\frac{1}{(b-a)^3}\int_a^b\int_a^b\int_a^b f\big(t_1x+t_2y+(1-t_1-t_2)z\big)dx\,dy\,dz\,.$$

Corollary 2.3. The following two refinements are valid

$$f\left(\frac{a+b}{2}\right)\le M(t_1,t_2)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{f(a)+f(b)}{2},$$

$$f\left(\frac{a+b}{2}\right)\le N\left(\tfrac{1}{3},\tfrac{1}{3}\right)\le N(t_1,t_2)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{f(a)+f(b)}{2}\;.$$

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