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On Two Inequalities Similar to Hardy-Hilbert's Integral Inequality

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Abstract. Two new inequalities similar to Hardy-Hilbert's inequality are given.

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1. Introduction

Let $f, g \geq 0$ satisfy

$$0 < \int_0^\infty f^2(t)\,dt < \infty \quad and \quad 0 < \int_0^\infty g^2(t)\,dt < \infty\ ,$$

then

(1) $$\int_0^\infty\int_0^\infty \frac{f(x)\,g(y)}{x+y}\,dx\,dy < \pi\left(\int_0^\infty f^2(t)\,dt\int_0^\infty g^2(t)\,dt\right)^{1/2}\ ,$$

where the constant factor π is the best possible (cf. Hardy et al. [3]).Inequality (1) is well known as Hilbert's integral inequality. This inequality had been extended by Hardy [2] as follows

If p>1, $\frac{1}{p}+\frac{1}{q}=1$, $f, g \geq 0$ satisfy

$$0 < \int_0^\infty f^p(t)\,dt < \infty \quad and \quad \int_0^\infty g^q(t)\,dt < \infty ,$$

then

(2) $$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(t)\,dt\right)^{1/p}\left(\int_0^\infty g^q(t)\,dt\right)^{1/q},$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequality (2) is called Hardy-Hilbert's integral inequality and is important in analysis and application (cf. Mitrinovic et al. [4]).

B. Yang gave the following extension of (2) as follows :

Theorem [5]. *If* $\lambda > 2 - \min\{p,q\}$, $f, g \geq 0$, satisfy

$$0 < \int_0^\infty t^{1-\lambda} f^p(t)\,dt < \infty \quad and \quad \int_0^\infty t^{1-\lambda} g^q(t)\,dt < \infty,$$

then

(3) $$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\,dx\,dy < k_\lambda(p)\left(\int_0^\infty t^{1-\lambda} f^p(t)\,dt\right)^{1/p}\left(\int_0^\infty t^{1-\lambda} g^q(t)\,dt\right)^{1/q},$$

where the constant factor $k_\lambda(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$ is the best possible, B is the beta function.

The object of this paper is that to give some new inequalities similar to Hardy-Hilbert's inequality.

New Results

We need the following two results for our aim

Theorem A [1]. Let f be nonnegative integrable function. Define

$$F(x) = \int_a^x f(t)\,dt .$$

Then

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)\,dx, \qquad p > 1.$$

Lemma 1. If 0<p<1/2, then

$$\int_0^\infty \frac{u^{p-1}}{|1-u|^{2p}}\,du = 2B(p,1-2p).$$

Proof.

$$\int_0^\infty \frac{u^{p-1}}{|1-u|^{2p}}\,du = \int_0^1 \frac{u^{p-1}}{(1-u)^{2p}}\,du + \int_1^\infty \frac{u^{p-1}}{(u-1)^{2p}}\,du$$

$$= 2\int_0^1 \frac{u^{p-1}}{(1-u)^{2p}}\,du$$

$$= 2\int_0^1 u^{p-1}(1-u)^{1-2p-1}\,du = 2\,B(p,1-2p).$$

The following are our main results

Theorem 1. Let $f,g \ge 0,\ \frac{1}{p}+\frac{1}{q}=1,\ 1<p<\alpha/2,\ 1<q<\beta/2,\ \alpha,\beta>0.$ Define

$$F(x)=\int_0^x f(t)\,dt, \quad G(y)=\int_0^y g(t)\,dt.$$

Then

$$\int_0^\infty\int_0^\infty \frac{x^{\frac{1}{\beta}-1} y^{\frac{1}{\alpha}-1} F^{\frac{1}{\alpha}+\frac{1}{p}}(x)\, G^{\frac{1}{\beta}+\frac{1}{q}}(y)}{|x-y|^{\frac{2}{\alpha}+\frac{2}{\beta}}}\,dx\,dy \le K_{p,\alpha}^{1/p} K_{q,\beta}^{1/q} \left(\int_0^\infty f^{\frac{p}{\alpha}+1}(x)\,dx\right)^{1/p} \times \left(\int_0^\infty g^{\frac{q}{\beta}+1}(y)\,dy\right),$$

where

$$K_{p,\alpha} = 2(1+\alpha/p)^{p/\alpha+1}\,B(p/\alpha,1-2p/\alpha).$$

Proof.

$$\int_0^\infty\int_0^\infty \frac{y^{\frac{1}{\alpha}-1} x^{\frac{1}{\beta}-1} F^{\frac{1}{\alpha}+\frac{1}{p}}(x)\, G^{\frac{1}{\beta}+\frac{1}{q}}(y)}{|x-y|^{\frac{2}{\alpha}+\frac{2}{\beta}}}\,dx\,dy$$

$$= \int_0^\infty\int_0^\infty \frac{y^{\frac{1}{\alpha}-\frac{1}{p}} F^{\frac{1}{\alpha}}(x)}{x^{\frac{1}{p}}|x-y|^{\frac{2}{\alpha}}} \times \frac{x^{\frac{1}{\beta}-\frac{1}{q}} G^{\frac{1}{\beta}}(y)}{y^{\frac{1}{q}}|x-y|^{\frac{2}{\beta}}}\,dx\,dy$$

$$\le \left(\int_0^\infty \int_0^\infty \frac{y^{\frac{p}{\alpha}-1} F^{\frac{p}{\alpha}}(x)}{x|x-y|^{\frac{2p}{\alpha}}} dx\, dy \right)^{1/p} \left(\int_0^\infty \int_0^\infty \frac{x^{\frac{q}{\beta}-1} G^{\frac{q}{\beta}}(y)}{y|x-y|^{\frac{2q}{\beta}}} dx\, dy \right)^{1/q}$$

$$= P^{1/p} Q^{1/q}.$$

Observe that, in view of lemma 1, we have

$$P = \int_0^\infty \left(\frac{F(x)}{x} \right)^{\frac{p}{\alpha}} dx \int_0^\infty \frac{\left(\frac{y}{x}\right)^{\frac{p}{\alpha}-1} \frac{1}{x}}{\left|1-\frac{y}{x}\right|^{\frac{2p}{\alpha}}} dy$$

$$= \int_0^\infty \left(\frac{F(x)}{x} \right)^{\frac{p}{\alpha}} dx \int_0^\infty \frac{t^{\frac{p}{\alpha}-1}}{|1-t|^{\frac{2p}{\alpha}}} dt$$

$$< 2(1+\alpha/p)^{p/\alpha+1} B\left(\frac{p}{\alpha}, 1-\frac{2p}{\alpha} \right) \int_0^\infty f^{\frac{p}{\alpha}+1}(x)\, dx.$$

Similarly, we can show that

$$Q < 2(1+\beta/q)^{q/\beta+1} B\left(\frac{q}{\beta}, 1-\frac{2q}{\beta} \right) \int_0^\infty g^{\frac{q}{\beta}+1}(y)\, dy.$$

This completes the proof of the theorem.

Theorem 2. Let $f, g, h \ge 0$, $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$, $1 < p < \alpha/4$, $1 < q < \beta/4$, $1 < r < \gamma/4$, $\alpha, \beta, \gamma > 0$. *Define*

$$F(x) = \int_0^\infty f(t)\, dt, \quad G(y) = \int_0^\infty g(t)\, dt, \quad H(z) = \int_0^z h(t)\, dt.$$

Then

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{x^{\frac{2}{\beta}+\frac{1}{\gamma}-1} y^{\frac{2}{\gamma}+\frac{1}{\alpha}-1} z^{\frac{2}{\alpha}+\frac{1}{\beta}-1} F^{\frac{1}{\alpha}+\frac{1}{p}}(x) G^{\frac{1}{\beta}+\frac{1}{q}}(y) H^{\frac{1}{\gamma}+\frac{1}{r}}(z)}{|x-y-z|^{\frac{4}{\alpha}+\frac{4}{\beta}+\frac{4}{\gamma}}} dx\, dy\, dz$$

$$< K_{p,\alpha}^{1/p} C_{q,\beta}^{1/q} K_{r,\gamma}^{1/r} \left(\int_0^\infty f^{\frac{p}{\alpha}+1}(x)\, dx \right)^{1/p} \left(\int_0^\infty g^{\frac{q}{\beta}+1}(y)\, dy \right)^{1/q} \left(\int_0^\infty h^{\frac{r}{\gamma}+1}(z)\, dz \right)^{1/r},$$

where

$$K_{p,\alpha} = 4(1+\alpha/p)^{p/\alpha+1} B(p/\alpha, 1-2p/\alpha) B(2p/\alpha, 1-4p/\alpha)$$

$$C_{q,\beta} = 2(1+\beta/q)^{q/\beta+1} B(q/\beta, q/\beta) B(2q/\beta, 1-4q/\beta).$$

Proof.

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{x^{\frac{2}{\beta}+\frac{1}{\gamma}-1} y^{\frac{2}{\gamma}+\frac{1}{\alpha}-1} z^{\frac{2}{\alpha}+\frac{1}{\beta}-1} F^{\frac{1}{\alpha}}(x) G^{\frac{1}{\beta}}(y) H^{\frac{1}{\gamma}}(z)}{|x-y-z|^{\frac{4}{\alpha}+\frac{4}{\beta}+\frac{4}{\gamma}}}\,dx\,dy\,dz$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty \frac{y^{\frac{1}{\alpha}-\frac{1}{p}} z^{\frac{2}{\alpha}-\frac{1}{p}} F^{\frac{1}{\alpha}+\frac{1}{p}}(x)}{x^{\frac{1}{p}}|x-y-z|^{\frac{4}{\alpha}}} \times \frac{z^{\frac{1}{\beta}-\frac{1}{q}} x^{\frac{2}{\beta}-\frac{1}{q}} G^{\frac{1}{\beta}+\frac{1}{q}}(y)}{y^{\frac{1}{q}}|x-y-z|^{\frac{4}{\beta}}} \times \frac{x^{\frac{1}{\gamma}-\frac{1}{r}} y^{\frac{2}{\gamma}-\frac{1}{r}} H^{\frac{1}{\gamma}+\frac{1}{r}}(z)}{z^{\frac{1}{r}}|x-y-z|^{\frac{4}{\gamma}}}\,dx\,dy\,dz$$

$$\le \left(\int_0^\infty\int_0^\infty\int_0^\infty \frac{y^{\frac{p}{\alpha}-1} z^{\frac{2p}{\alpha}-1} F^{\frac{p}{\alpha}+1}(x)}{x|x-y-z|^{\frac{4p}{\alpha}}}\,dx\,dy\,dz\right)^{1/p} \left(\int_0^\infty\int_0^\infty\int_0^\infty \frac{z^{\frac{q}{\beta}-1} x^{\frac{2q}{\beta}-1} G^{\frac{q}{\beta}+1}(y)}{y|x-y-z|^{\frac{4q}{\beta}}}\,dx\,dy\,dz\right)^{1/q}$$

$$\times \left(\int_0^\infty\int_0^\infty\int_0^\infty \frac{x^{\frac{r}{\gamma}-1} y^{\frac{2r}{\gamma}-1} H^{\frac{r}{\gamma}+1}(z)}{z|x-y-z|^{\frac{4r}{\gamma}}}\,dx\,dy\,dz\right)^{1/r}$$

$$= L^{1/p} M^{1/q} N^{1/r}.$$

We first consider

$$L \le \int_0^\infty\int_0^\infty\int_0^\infty \frac{y^{\frac{p}{\alpha}-1} z^{\frac{2p}{\alpha}-1} F^{\frac{p}{\alpha}+1}(x)}{x\big||x-y|-z\big|^{\frac{4p}{\alpha}}}\,dx\,dy\,dz$$

$$= \int_0^\infty \left(\frac{F(x)}{x}\right)^{\frac{\alpha}{p}+1} dx \int_0^\infty \frac{\left(\frac{y}{x}\right)^{\frac{p}{\alpha}-1}\frac{1}{x}}{\left|1-\frac{y}{x}\right|^{\frac{2p}{\alpha}}}\,dy \int_0^\infty \frac{\left(\frac{z}{|x-y|}\right)^{\frac{2p}{\alpha}-1}\frac{1}{|x-y|}}{\left|1-\frac{z}{|x-y|}\right|^{\frac{4p}{\alpha}}}\,dz$$

$$< (1+\alpha/p)^{p/\alpha+1}\int_0^\infty f^{\frac{p}{\alpha}+1}(x)\,dx \int_0^\infty \frac{u^{\frac{p}{\alpha}-1}}{|1-u|^{\frac{2p}{\alpha}}}\,du \int_0^\infty \frac{v^{\frac{2p}{\alpha}-1}}{|1-v|^{\frac{4p}{\alpha}}}\,dv$$

$$= 4(1+\alpha/p)^{p/\alpha+1} B\left(\frac{p}{\alpha},1-\frac{2p}{\alpha}\right) B\left(\frac{2p}{\alpha},1-\frac{4p}{\alpha}\right)\int_0^\infty f^{\frac{p}{\alpha}+1}(x)\,dx\,,$$

in view of theorem A and lemma 1.

$$M = \int_0^\infty\int_0^\infty\int_0^\infty \frac{z^{\frac{q}{\beta}-1} x^{\frac{2q}{\beta}-1} G^{\frac{2}{\beta}+1}(y)}{y|y+z-x|^{\frac{4q}{\beta}}}\,dx\,dy\,dz$$

$$= \int_0^\infty \left(\frac{G(y)}{y} \right)^{\frac{q}{\beta}+1} dy \int_0^\infty \frac{\left(\frac{z}{y}\right)^{\frac{q}{\beta}-1} \frac{1}{y}}{\left(1+\frac{z}{y}\right)^{\frac{2q}{\beta}}} dz \int_0^\infty \frac{\left(\frac{x}{y+z}\right)^{\frac{2q}{\beta}-1} \frac{1}{y+z}}{\left|1-\frac{x}{y+z}\right|^{\frac{4q}{\beta}}} dx$$

$$< (1+\beta/q)^{q/\beta+1} \int_0^\infty g^{\frac{q}{\beta}+1}(y)\,dy \int_0^\infty \frac{u^{\frac{q}{\beta}-1}}{(1+u)^{\frac{2q}{\beta}}} du \int_0^\infty \frac{v^{\frac{2q}{\beta}-1}}{|1-v|^{\frac{4q}{\beta}}} dv$$

$$= 2(1+\beta/q)^{q/\beta+1} B\left(\frac{q}{\beta}, \frac{q}{\beta}\right) B\left(\frac{2q}{\beta}, 1-\frac{4q}{\beta}\right) \int_0^\infty g^{\frac{q}{\beta}+1}(y)\,dy,$$

in view of Theorem A and Lemma 1.

Finally,

$$N \le \int_0^\infty \int_0^\infty \int_0^\infty \frac{x^{\frac{r}{\gamma}-1} y^{\frac{2r}{\gamma}-1} H^{\frac{r}{\gamma}+1}(z)}{z\,\big||x-z|-y\big|^{\frac{4r}{\gamma}}} dx\,dy\,dz$$

$$< 4(1+\gamma/r)^{r/\gamma+1} B\left(\frac{r}{\gamma}, 1-\frac{2r}{\gamma}\right) B\left(\frac{2r}{\gamma}, 1-\frac{4r}{\gamma}\right) \int_0^\infty h^{\frac{r}{\gamma}+1}(z)\,dz,$$

as in the case of M.
This completes the proof of the theorem.

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