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A Survey on Hardy-Hilbert's Integral Inequality

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Abstract. New kind of Hardy-Hilbert's integral inequality through a generalization have been obtained. Other results also have been deduced as an application.

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1. Introduction

Let $f, g \geq 0$ satisfy

$$0 < \int_0^\infty f^2(t)\,dt < \infty \;\; and \;\; 0 < \int_0^\infty g^2(t)\,dt < \infty ,$$

then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \pi\left(\int_0^\infty f^2(t)\,dt\int_0^\infty g^2(t)\,dt\right)^{1/2} , \qquad (1)$$

where the constant factor π is the best possible (cf. Hardy et al. [2]). Inequality (1) is well known as Hilbert's integral inequality. This inequality had been extended by Hardy [1] as follows

If p>1, $\frac{1}{p}+\frac{1}{q}=1$, $f, g \geq 0$ satisfy

$$0 < \int_0^\infty f^p(t)\,dt < \infty \;\; and \;\; \int_0^\infty g^q(t)\,dt < \infty ,$$

then

(2) $$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(t)\,dt\right)^{1/p}\left(\int_0^\infty g^q(t)\,dt\right)^{1/q},$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequality (2) is called Hardy-Hilbert's integral inequality and is important in analysis and application (cf. Mitrinovic et al. [3]).

Gradually, B. Yang gave the following extensions of (2) as follows :

Theorem A[4]. *If* $\lambda > 2 - \min\{p,q\}$, $f, g \ge 0$, satisfy

$$0 < \int_0^\infty t^{1-\lambda} f^p(t)\,dt < \infty \quad and \quad \int_0^\infty t^{1-\lambda} g^q(t)\,dt < \infty,$$

then

(3) $$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\,dx\,dy < k_\lambda(p)\left(\int_0^\infty t^{1-\lambda} f^p(t)\,dt\right)^{1/p}\left(\int_0^\infty t^{1-\lambda} g^q(t)\,dt\right)^{1/q},$$

where the constant factor $k_\lambda(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$ is the best possible, B is the beta function.

Theorem B[5]. *If* $n \in N - \{1\}$, $p_i > 1$, $\sum_{i=1}^n \frac{1}{p_i} = 1$, $\lambda > 0$, $f_i \ge 0$ satisfy

$$0 < \int_0^\infty t^{p_i-1-\lambda} f_i^{p_i}(t)\,dt < \infty \qquad (i = 1,2,...,n),$$

then

(4) $$\int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n x_j\right)^\lambda} \prod_{i=1}^n f_i(x_i)\,dx_1 \ldots dx_n$$

$$< \frac{1}{\Gamma\lambda}\prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i}\right)\left(\int_0^\infty t^{p_i-1-\lambda} f_i^{p_i}(t)\,dt\right)^{1/p_i},$$

where the constant factor $\frac{1}{\Gamma\lambda}\prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i}\right)$ is the best possible.

2. Lemma

The following Lemma is needed for our project

Lemma 2.1 . *Let* $a_i \geq 0$, $p_i > 1$, $i = 1,...,n$, $\sum_{i=1}^{n} 1/p_i = 1$. *Then*

$$\prod_{i=1}^{n} a_i \leq \sum_{i=1}^{n} \frac{a_i^{p_i}}{p_i} .$$

Proof . For $n = 2$, the inequality

$$a_1 a_2 \leq \frac{a_1^{p_1}}{p_1} + \frac{a_2^{p_2}}{p_2} , \quad \tfrac{1}{p_1} + \tfrac{1}{p_2} = 1$$

is well known . For $n = 3$, as

$$\frac{1}{\frac{p_1 p_2}{p_1 + p_2}} + \frac{1}{p_3} = 1, \text{ and } \frac{1}{\frac{p_1 + p_2}{p_1}} + \frac{1}{\frac{p_1 + p_2}{p_2}} = 1 ,$$

we have

$$\begin{aligned} a_1 a_2 a_3 &= (a_1 a_2) a_3 \\ &\leq \frac{p_1 + p_2}{p_1 p_2} (a_1 a_2)^{\frac{p_1 p_2}{p_1 + p_2}} + \frac{1}{p_3} a_3^{p_3} \\ &\leq \frac{p_1 + p_2}{p_1 p_2} \left(\frac{p_2}{p_1 + p_2} \left(a_1^{\frac{p_1 p_2}{p_1 + p_2}} \right)^{\frac{p_1 + p_2}{p_2}} + \frac{p_1}{p_1 + p_2} \left(a_2^{\frac{p_1 p_2}{p_1 + p_2}} \right)^{\frac{p_1 + p_2}{p_1}} \right) + \frac{1}{p_3} a_3^{p_3} \\ &= \frac{a_1^{p_1}}{p_1} + \frac{a_2^{p_2}}{p_2} + \frac{a_3^{p_3}}{p_3}. \end{aligned}$$

We aim to prove the inequality by induction. Let $\sum_{i=1}^{n+1} \frac{1}{p_i} = 1$, and assume that the inequality is true for m = n. That is

$$\prod_{i=1}^{n} a_i \leq \sum_{i=1}^{n} \frac{a_i^{p_i}}{p_i} \qquad \textit{whenever} \qquad \sum_{i=1}^{n} \frac{1}{p_i} = 1 .$$

Since

$$\frac{1}{p_{n+1}} + \frac{1}{\frac{p_{n+1}}{p_{n+1} - 1}} = 1, \ \textit{and} \ \left(\frac{1}{\frac{p_{n+1} - 1}{p_{n+1}}} \right) \left(\frac{1}{p_1} + ... + \frac{1}{p_n} \right) = 1,$$

We have

$$\prod_{i=1}^{n+1} a_i = \left(\prod_{i=1}^{n} a_i \right) a_{n+1} \leq \frac{\left(\prod_{i=1}^{n} a_i \right)^{\frac{p_{n+1}}{p_{n+1} - 1}}}{\frac{p_{n+1}}{p_{n+1} - 1}} + \frac{a_{n+1}^{p_{n+1}}}{p_{n+1}} = \left(\frac{p_{n+1} - 1}{p_{n+1}} \right) \prod_{i=1}^{n} a_i^{\frac{p_{n+1}}{p_{n+1} - 1}} + \frac{a_{n+1}^{p_{n+1}}}{p_{n+1}}$$

$$\leq \left(\frac{p_{n+1}-1}{p_{n+1}}\right)\sum_{i=1}^{n}\frac{\left(a_i^{\frac{p_{n+1}}{p_{n+1}-1}}\right)^{\left(\frac{p_{n+1}-1}{p_{n+1}}\right)p_i}}{\left(\frac{p_{n+1}-1}{p_{n+1}}\right)p_i}+\frac{a_{n+1}^{p_{n+1}}}{p_{n+1}}=\sum_{i=1}^{n+1}\frac{a_i^{p_i}}{p_i}.$$

Therefore , the inequality is true for any n .

3. Main Result

We state and prove the following

Theorem 3.1. *Let* $F_i, G_i, f_i, g_i > 0$, , $G(0)=0$, $G(\infty)=\infty$, $p_i > 1$, $\lambda > p_i - 1$, $i = 1,...,n$, $\sum_{i=1}^{n} 1/p_i = 1$. *Define*

$$G_i\left(f(t)\right)=\int_0^{f(t)} g_i(u)\,du \quad .$$

Then

$$\int_0^\infty \ldots \int_0^\infty \frac{F_1(t_1)\,g_1\left(f_1(t_1)\right)\ldots F_n(t_n)\,g_n\left(f_n(t_n)\right)}{\left(G_1\left(f_1(t_1)\right)+\ldots+G_n\left(f_n(t_n)\right)\right)^\lambda}\,dt_1\ldots dt_n \tag{5}$$

$$\leq \frac{1}{\Gamma(\lambda)}\sum_{i=1}^{n}\frac{\Gamma(1+\lambda-p_i)}{p_i}\int_0^\infty F_i^{p_i}(t)\,G_i^{p_i-\lambda-1}\left(f_i(t)\right)g_i\left(f_i(t)\right)dt\,,$$

In particular, *for* $1+\lambda = 2p_i$,

$$\int_0^\infty \ldots \int_0^\infty \frac{F_1(t_1)\,g_1\left(f_1(t_1)\right)\ldots F_n(t_n)\,g_n\left(f_n(t_n)\right)}{\left(G_1\left(f_1(t_1)\right)+\ldots+G_n\left(f_n(t_n)\right)\right)^\lambda}\,dt_1\ldots dt_n \tag{6}$$

$$\leq \frac{1}{\Gamma(\lambda)}\sum_{i=1}^{n}\frac{\Gamma(p_i)}{p_i}\int_0^\infty \left(\frac{F_i(t)}{G_i\left(f_i(t)\right)}\right)^{p_i} g_i\left(f_i(t)\right)\,dt\ ,$$

Provided the integrals on the right do exist.

Proof . Define, for $i=1,...,n$,

$$L_i(s) = s^{\frac{\lambda-1}{p_i}}\int_0^\infty F_i(t_i)\,e^{-sG_i\left(f_i(t_i)\right)}\ g_i\left(f_i(t_i)\right)dt_i\ .$$

Then we have, by virtue of the Lemma

$$\int_0^\infty L_1(s)\ldots L_n(s)\,ds \le \int_0^\infty \sum_{i=1}^n \frac{1}{p_i} L_i^{p_i} ds = \sum_{i=1}^n \frac{1}{p_i}\int_0^\infty L_i^{p_i} ds \;.$$

Now, since by the hypothesis, $G_i'\big(f(t)\big)\, f'(t) = g_i\big(f(t)\big)$, we have

$$L_i^{p_i}(s) = \left(s^{\frac{\lambda-1}{p_i}} \int_0^\infty F_i(t_i)\, e^{-sG_i(f_i(t_i))}\, g_i\big(f_i(t_i)\big)\, dt_i \right)^{p_i}$$

$$\le s^{\lambda-1}\int_0^\infty F_i^{p_i}(t_i)\, e^{-sG_i(f_i(t_i))}\, g_i\big(f_i(t_i)\big) dt_i \left(\int_0^\infty e^{-sG_i(f_i(t_i))} g_i\big(f_i(t_i)\big)\, dt_i\right)^{p_i-1}$$

$$= s^{\lambda-p_i}\int_0^\infty F_i^{p_i}(t_i)\, e^{-sG_i(f_i(t_i))}\, g_i\big(f_i(t_i)\big)\, dt_i \;.$$

Therefore

$$\int_0^\infty L_1(s)\ldots. L_n(s)\,ds \le \sum_{i=1}^n \frac{1}{p_i}\int_0^\infty s^{\lambda-p_i}\int_0^\infty F_i^{p_i}(t_i)\, e^{-sG_i(f_i(t_i))} g_i\big(f_i(t_i)\big)\, dt_i\, \mathrm{ds}$$

$$= \sum_{i=1}^n \frac{1}{p_i}\int_0^\infty F_i^{p_i}(t_i)\, g_i\big(f_i(t_i)\big) dt_i \int_0^\infty s^{\lambda-p_i} e^{-sG_i(f_i(t_i))}\, ds$$

$$= \sum_{i=1}^n \frac{1}{p_i}\int_0^\infty F_i^{p_i}(t_i)\, G_i^{p_i-\lambda-1}\big(f_i(t_i)\big) g_i\big(f_i(t_i)\big) dt_i \int_0^\infty z^{\lambda-p_i} e^{-z}\, dz$$

$$= \sum_{i=1}^n \frac{\Gamma\big(1+\lambda-p_i\big)}{p_i}\int_0^\infty F_i^{p_i}(t_i)\, G_i^{p_i-\lambda-1}\big(f_i(t_i)\big) g_i\big(f_i(t_i)\big) dt_i \;.$$

On the other hand

$$\int_0^\infty L_1(s)\ldots L_n(s)\,ds$$

$$= \int_0^\infty s^{\lambda-1}\left(\int_0^\infty F_1(t_1)\, e^{-sG_1(f_1(t_1))} g_1\big(f_1(t_1)\big) dt_1\right)\ldots\left(\int_0^\infty F_n(t_n)\, e^{-sG_n(f_n(t_n))}\, g_n\big(f_n(t_n)\big) dt_n\right) ds$$

$$= \int_0^\infty\ldots\int_0^\infty F_1(t_1)\, g_1\big(f_1(t_1)\big)\ldots F_n(t_n)\, g_n\big(f_n(t_n)\big) dt_1\ldots dt_n \times$$

$$\int_0^\infty s^{\lambda-1} e^{-s(G_1(f_1(t_1)))+\ldots+G_n(f_n(t_n))}\, ds$$

$$= \int_0^\infty\ldots\int_0^\infty \frac{F_1(t_1)\, g_1\big(f_1(t_1)\big)\ldots F_n(t_n)\, g_n\big(f_n(t_n)\big)}{\big(G_1\big(f_1(t_1)+\ldots+G_n(f_n(t_n)\big)\big)^{\lambda}}\, dt_1\ldots dt_n \int_0^\infty z^{\lambda-1} e^{-z\,dz}$$

$$= \Gamma\big(\lambda\big)\int_0^\infty\ldots\int_0^\infty \frac{F_1(t_1)\, g_1\big(f_1(t_1)\big)\ldots F_n(t_n)\, g_n\big(f_n(t_n)\big)}{\big(G_1\big(f_1(t_1)\big)+\ldots+G_n\big(f_n(t_n)\big)\big)^{\lambda}}\, dt_1\ldots dt_n \;.$$

Summarizing, we obtain

$$\int_0^\infty \cdots \int_0^\infty \frac{F_1(t_1)\, g_1\big(f_1(t_1)\big) \ldots F_n(t_n)\, G_n\big(f_n(t_n)\big)}{\big(G_1\big(f_1(t_1)\big)+\ldots+G_n\big(f_n(t_n)\big)\big)^{\lambda}}\, dt_1 \ldots dt_n$$

$$\leq \frac{1}{\Gamma(\lambda)} \sum_{i=1}^{n} \frac{\Gamma(1+\lambda-p_i)}{p_i} \int_0^\infty F_i^{p_i}(t)\, G_i^{p_i-\lambda-1}\big(f_i(t)\big)\, g_i\big(f_i(t)\big)\, dt.$$

4. Applications

Corolary 4.1. On putting $f_i(t)=t$ in Theorem 3.1, we obtain

(7) $$\int_0^\infty \cdots \int_0^\infty \frac{F_1(t_1)\, g_1\big(f_1(t_1)\big) \ldots F_n(t_n)\, g_n\big(f_n(t_n)\big)}{\big(G_1\big(f_1(t_1)\big)+\ldots+G_n\big(f_n(t_n)\big)\big)^{\lambda}}\, dt_1 \ldots dt_n$$

$$\leq \frac{1}{\Gamma(\lambda)} \sum_{i=1}^{n} \frac{\Gamma(1+\lambda-p_i)}{p_i} \int_0^\infty F_i^{p_i}(t)\, G_i^{p_1-\lambda-1}\big(f_i(t)\big)\, g_i\big(f_i(t)\big)\, dt\ .$$

Corollary 4.2. On putting $g_i(u)=1$, and hence $G(t)=t$ in Theorem 3.1, we obtain

(8) $$\int_0^\infty \cdots \int_0^\infty \frac{F_1(t_1)\, g(t_1) \ldots F_n(t_n)\, g_n(t_n)}{\big(G_1(t_1)+\ldots+.G_n(t_n)\big)^{\lambda}}\, dt_1 \ldots dt_n$$

$$\leq \frac{1}{\Gamma(\lambda)} \sum_{i=1}^{n} \frac{\Gamma(1+\lambda-p_i)}{p_i} \int_0^\infty F_i^{p_i}(t)\, G_i^{p_i-\lambda-1}(t)\, g_i(t)\, dt\ .$$

Corollary 4.3. On putting $f(t)=t$ in Corollary 4.2, we obtain

(9) $$\int_0^\infty \cdots \int_0^\infty \frac{F_1(t_1) \ldots F_n(t_n)}{\big(t_1.+\ldots+t_n\big)}\, dt_1 \ldots dt_n$$

$$\leq \frac{1}{\Gamma(\lambda)} \sum_{i=1}^{n} \frac{\Gamma(1+\lambda-p_i)}{p_i} \int_0^\infty t^{p_i-\lambda-1} F_i^{p_i}(t)\, dt\ .$$

Corollary 4.4. On putting $f(t)=\sinh t$ in theorem 3.1, we obtain

(10) $$\int_0^\infty \cdots \int_0^\infty \frac{F_1(t_1)\, g_1(\sinh t_1) \ldots F_n(t_n)\, g_n(\sinh t_n)}{\big(G_1(\sinh t_1)+\ldots.+G_n(\sinh t_n)\big)^{\lambda}}\, dt_1 \ldots dt_n$$

$$\le \frac{1}{\Gamma(\lambda)}\sum_{i=1}^{n}\frac{\Gamma(1+\lambda-p_i)}{p_i}\int_0^\infty F_i^{\,p_i}(t)\,G_i^{\,p_i-\lambda-1}(\sinh t)\,g_i(\sinh t)\,dt\ .$$

Corollary 4.5. On putting $f(t)=\sinh^{-1} t$ in Theorem 3.1, we obtain

(11)
$$\int_0^\infty\!\!\dots\!\int_0^\infty \frac{F_1(t_1)\,g_1(\sinh^{-1} t_1)\dots F_n(t_n)\,g_n(\sinh^{-1} t_n)}{\left(G_1(\sinh^{-1} t_1)+\dots+G_n(\sinh^{-1} t_n)\right)^{\lambda}}\,dt_1\dots dt_n$$
$$\le \frac{1}{\Gamma(\lambda)}\sum_{i=1}^{n}\frac{\Gamma(1+\lambda-p_i)}{p_i}\int_0^\infty F_i^{\,p_i}(t)\,G_i^{\,p_i-\lambda-1}(\sinh^{-1} t)\,g(\sinh^{-1} t)\,dt\ .$$

Corollary 4.6. On putting $f(t)=1/t$ in Theorem 3.1 , we obtain

(12)
$$\int_0^\infty\!\!\dots\!\int_0^\infty \frac{F_1(t_1)\,g_1(t_1^{-1})\dots F_n(t_n)\,g_n(t_n^{-1})}{\left(G_1(t_1^{-1})+\dots+G_n(t_n^{-1})\right)^{\lambda}}\,dt_1\dots dt_n$$
$$\le \frac{1}{\Gamma(\lambda)}\sum_{i=1}^{n}\frac{\Gamma(1+\lambda-p_i)}{p_i}\int_0^\infty F_i^{\,p_i}(t)\,G_i^{\,p_i-\lambda-1}(t^{-1})\,g_i(t^{-1})\,dt\ .$$

Corollary 4.7. On putting $g_i(u)=u$ in Theorem 3.1, we obtain

(13)
$$\int_0^\infty\!\!\dots\!\int_0^\infty \frac{F_1(t_1)\,f_1(t_1)\dots F_n(t_n)\,f_n(t_n)}{\left(f_1^2(t_1)+\dots+f_n^2(t_n)\right)^{\lambda}}\,dt_1\dots dt_n$$
$$\le \frac{1}{\Gamma(\lambda)}\sum_{i=1}^{n}\frac{\Gamma(1+\lambda-p_i)}{2^{p_i-1}\,p_i}\int_0^\infty F_i^{\,p_1}(t)\,f_i^{\,2p_i-2\lambda-1}(t)\,dt\ .$$

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