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Some Applications of Integro-Substitution Operators in Integral Equations

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Abstract

In this paper we shall establish connection between Integro-substitution operators and integral equations.

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1.Introduction and Preliminaries

Let (X, s, μ) be a σ-finite measure space. For $1 \leq p < \infty$, let $L^p(X, \mathbb{C}) = \{f | f : X \to \mathbb{C}$ is measurable and $\int_X |f|^p d\mu < \infty\}$. Then $L^p(X, \mathbb{C})$ is a Banach space under the norm

$$||f|| = (\int_X |f|^p d\mu)^{1/p}$$

Let $T : X \to X$ be a non singular measurable transformation that is if $\mu(E) = 0$ implies that $\mu(T^{-1}(E)) = 0$. If T is non singular, then the measure μT^{-1} is absloutely continuous with respect to the measure μ. Hence by Radon-Nikodym derivative theorem, there exists a positive measureable function f_0 such that

$$\mu(T^{-1}(E)) = \int_X f_0 d\mu \quad \text{for every } E \in s.$$

The function f_0 is called the Radon Nikodym derivative of the measure μT^{-1} with respect to measure μ and we denote it by $\frac{d\mu T^{-1}}{d\mu}$.

Suppose (X, s, μ) is a measure space and $k : X \times X \to \mathbb{C}$ is a measurable function. For $1 \leq p < \infty$, if the operator

$$T_k : L^p(X, \mathbb{C}) \to L^p(X, \mathbb{C})$$

defined by

$$(T_k f)(x) = \int_X k(x, y) f(y) d\mu(y)$$

is bounded, we call it an integral operator induced by k.
If we compose an integral operator with a substitution operator C_T, we get another operator called Integro substitution operator. We define the operators $L_{Tk} : L^p(X, \mathbb{C}) \to L^q(X, \mathbb{C})$ and $R_{kT} : L^p(X, \mathbb{C}) \to (L^q X, \mathbb{C})$ respectively by

$$(L_{Tk} f)(x) = \int_X k(T(x), y) f(y) d\mu(y)$$

and

$$(R_{kT} f)(x) = \int_X k(x, y) f(T(y)) d\mu(y).$$

For

$$f_0 = \frac{d\mu T^{-1}}{d\mu}.$$

We set $k_{f_0}(x, y) = f_0(x) k(x, y)$.

The study of integral operators have been carried out by several mathematician in sevaral direction. A few of them are Halmos and Sunder (3), Whitely [4], Steponov [6], Sinnamon [2], Bloom and Kerman [5], A. Gupta and B.S. Komal [1].

In this paper we shall establish connection between Integro-substitution operators and integral equations. Some cases for which the integral equations of the type

$$g(x) = f(x) + \lambda \int_a^b k(x, y) g(T(y)) d\mu(y)$$

have unique solutions are discussed. The operator

$$R_{kT} : L^2([a, b], \mathbb{C}) \to L^2([a, b], \mathbb{C})$$

where the kernel function $k(x, y)$ is the characteristic function of the set

$$\{(x, y) : 0 \leq |y| \leq |x| \leq 1\}$$

is infact a quasinilpotent operator. In this special case R_{kT} will be denoted by R_T.

2. Applications of Integro-Substitution Operators in Integral Equations

Theorem 2.1: Suppose $k_T \in L^2([a,b] \times [a,b], \mathbb{C})$, where

$$k_T(x,y) = E(k(x,.))oT^{-1}(y)f_0(y).$$

Then the equation

$$g(x) = f(x) + \lambda \int_a^b k(x,y)g(T(y))d\mu(y) \quad ...(1)$$

has unique solution in $L^2([a,b],\mathbb{C})$ for sufficiently small value of λ.
Proof: For every $g \in L^2([a,b],\mathbb{C})$, consider the operator

$$(R_{kT}g)(x) = f(x) + \lambda \int_a^b k(x,y)g(T(y))d\mu(y).$$

We first show that this operator takes every function g in $L^2([a,b],\mathbb{C})$ to a function in $L^2([a,b],\mathbb{C})$. Since $f \in L^2([a,b],\mathbb{C})$, it sufficies to show that the operator,
$(R_{kT}g)(x) = \lambda \int_a^b k(x,y)g(T(y))d\mu(y)$ takes every function $g \in L^2([a,b],\mathbb{C})$ into a function in $L^2([a,b],\mathbb{C})$.
Consider

$$\begin{aligned}
&\int_a^b (|\int_a^b k(x,y)g(T(y))d\mu(y)|^2)d\mu(x) \\
&\quad = \int_a^b |\int_a^b E(k_x)oT^{-1}(y)f_0(y)g(y)d\mu(y)|^2 d\mu(x) \\
&\quad \leq \int_a^b (\int_a^b |f_0(y)E(k_x)oT^{-1}(y)|^2 d\mu(y) \int_a^b |g(y)|^2 d\mu(y)d\mu(x) \\
&\quad = ||g||^2 \int_a^b \int_a^b |f_0(y)E(k_x)oT^{-1}(y)|^2 d\mu(y)d\mu(x) \\
&\quad = ||g||^2.M
\end{aligned}$$

Now

$||R_{kT}g - R_{kT}h||$

$$\begin{aligned}
&= (\int_a^b |(R_{kT}g)(x) - (R_{kT}h)(x)|^2 d\mu(y))^{1/2} \\
&= (|\lambda|^2 \int_a^b | \int_a^b k(x,y)(g(T(y)) - h(T(y))|^2 d\mu(y) d\mu(x))^{1/2} \\
&\leq |\lambda|(\int_a^b (\int_a^b |k(x,y)||g(T(y) - h(T(y))|)^2 d\mu(y) d\mu(x))^{1/2} \\
&= |\lambda| \int_a^b (\int_a^b E|k(x,.)|^2 oT^{-1}(y) f_0^2(y)(\int_a^b |g(y) - h(y)|^2 d\mu(y) d\mu(x))^{1/2} \\
&= |\lambda| ||g-h||^2 \int_a^b \int_a^b f_0^2(y) E(|k(x,.)|) oT^{-1}(y) d\mu(y) \\
&< ||g-h|| \quad \text{if} \quad |\lambda| \int_a^b \int_a^b |k(x,.)| oT^{-1}(y) f_0^2(y) d\mu(y) d\mu(x) < 1.
\end{aligned}$$

By Banach contraction principle there exists unique $g_0 \in L^2([a,b],\mathbb{C})$ such that $R_{kT}g_0 = g_0$, that is $g_0(x) = f(x) + \lambda \int_a^b k(x,y) g(T(y)) d\mu(y)$. Hence the given equation (1) has a unique solution.

Theorem 2.2 : Let $k_T \in L^2([0,1] \times [0,1], \mathbb{C})$ and $T^{-1}[0,x] \supset [0,x]$ for each $x \in [0,1]$. Then the equation

$$g(s) = f(s) + \lambda \int_0^s k(s,t) g(T(t)) dt$$

has unique solution in $L^2([0,1],\mathbb{C})$ for arbitrary $\lambda \in \mathbb{C}$.

Proof : Let $p(s) = \int_0^s |k(s,t)|^2 dt, q(t) = \int_t^1 |k(s,t)|^2 dt$ and $r(s) = \int_0^1 f_0(z) p(z) dz$. Define the mapping $A : L^2([0,1],\mathbb{C}) \to L^2([0,1],\mathbb{C})$ by

$$(Ag)(s) = f(s) + \lambda \int_0^s k(s,t) g(T(t)) dt$$

we first prove that A^n is a contraction for some $n \in N$.
Now

$$\begin{aligned}
Ag &= f + \lambda R_{kT} g \text{ so that} \\
A^n g &= f + \lambda R_{kT} f + \lambda^2 R_{k_T}^2 f + - - - + \lambda^n R_{k_T}^n f.
\end{aligned}$$

and $(R^n_{k_T}g)(s) = \int_0^s k_{T_n}(s,t)g(T(t))dt$, where $k_{T_1}(s,t) = k$.
$k_{T_2}(s,t) = \int_0^s k(s,z)k(T(z),t)dz$
....
....
$k_{T_m}(s,t) = \int k(s,z)k_{T_{m-1}}(T(z),t)dz$
To calculate $||R^n_{kT}||$, consider

$$\begin{aligned}
|k_{T_2}(s,t)|^2 &= |\int_0^s k(s,z)k(T(z),t)|^2 dz \\
&\leq \int_0^s |k(s,z)|^2 dz \int_0^s |k(T(z),t)|^2 dz \\
&\leq p(s) \int_{T^{-1}([0,s])} |k(T(z),t)|^2 dz \\
&\leq p(s) \int_0^s f_0(z)|k(z,t)|^2 dz \\
&\leq p(s)q(t)
\end{aligned}$$

$$\begin{aligned}
|k_{T_3}(s,t)|^2 &= |\int_0^s k(s,z)k_{T_2}(T(z),t)dz|^2 \\
&\leq \int_0^s |k(s,z)|^2 dz \int_0^s |k_{T_2}(T(z),t)|^2 dz \\
&\leq \int_0^s |k(s,z)|^2 dz \int_{T^{-1}([0,s])} k_{T_2}(T(z),t)|^2 dz \\
&\leq p(s) \int_0^s f_0(z)|k_{T_2}(z,t)|^2 dz \\
&\leq p(s) \int_t^s f_0(z)p(z)q(t) dz \\
&= p(s)q(t) \int_t^s f_0(z)p(z) dz \\
&= p(s)q(t)[r(s) - r(t)]
\end{aligned}$$

$$\begin{aligned} |k_{T_4}(s,t)|^2 &= |\int_0^s k(s,z)k_{T_3}(T(z),t)|^2 dz \\ &\leq \int_0^s |k(s,z)|^2 dz \int_0^s |(T(z),t)|^2 dz \\ &= p(s)\int_t^s f_0(y)|k_{T_3}(z,t)|^2 dz \\ &\leq p(s)\int_t^s f_0(z)p(z)q(t)(r(s)-r(t))dz \\ &= p(s)q(t)\frac{(r(s)-r(t))^2}{2} \end{aligned}$$

Similarly we can prove that

$$|k_{T_n}(s,t)|^2 \leq p(s)q(t)\frac{[r(s)-r(t)]^{n-2}}{(n-2)!}, \text{ for } n \geq 2$$

Now

$$\begin{aligned} |(A^n g)(s)-(A^n h)(s)|^2 &= |\lambda^n \int_0^s k_T^n(s,t)[g(T(t)-h(T(t))]|^2 dt \\ &\leq |\lambda|^{2n}\int_0^s |k_T^n(s,t)|^2 dt \int_0^s |g(T(t))-h(T(t))|^2 dt \\ &= |\lambda|^{2n}\int_0^s p(s)q(t)\frac{[r(s)-r(t)]^{n-2}}{(n-2)!}dt \int_0^s f_0(t)|g(t)-h(t)|^2 dt \\ &\leq |\lambda|^{2n}p(s)\frac{[r(s)-r(t)]^{n-2}}{(n-2)!}M.M_1||g-h||^2 \end{aligned}$$

Integrating with respect to s, we get

$$\begin{aligned} ||A^n g - A^n h||^2 &\leq |\lambda|^{2n}\frac{||g-h||^2}{(n-2)!}MM_1\int_0^1 (r(s))^{n-2}p(s)ds \\ &\leq |\lambda|^{2n}\frac{||g-h||^2}{(n-2)!}M^{n-1}M_1\int_0^1 p(s)ds \\ &\leq |\lambda|^{2n}\frac{||g-h||^2}{(n-2)!}M^n M_1, \text{ for } n \geq 2 \end{aligned}$$

Therefore, we can choose n_0 such that $\frac{|\lambda|^{2n_0}M^{n_0}M_1}{(n_0-2)!} < 1$, so that A^{n_0} is a contraction. Hence A has a unique fixed point and that fixed point is a unique solution.

Theorem 2.3 : The integro substitution operator R_T is a quasinilpotent operator i.e $\sigma(R_T) = \{0\}$.

Proof : For $f \in L^2([a, b], \mathbb{C})$, consider

$$\begin{aligned}
(R_T f)(x) &= \int_x^{-x} (foT)(x)dx \text{ and} \\
R_T(R_T f)(x) &= \int_{-x}^{x} (R_T f)(y)dy \\
&= \int_{-x}^{0} (R_T f)(y)dy + \int_0^x (R_T f)(y)dy \\
&= -\int_0^{-x} (R_T f)(y)dy + \int_0^x (R_T f)(y)dy \\
&= -\int_0^{x} (R_T f)(t)dt + \int_0^x (R_T f)(y)dy \\
&= 0
\end{aligned}$$

Thus R_T is a quasinilpotent operator of index 2. Hence $\sigma(R_T) = \{0\}$.

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