Int. Journal of Math. Analysis, Vol. 4, 2010, no. 39, 1937 - 1944

On ω-Locally Closed Sets in Bitopological Spaces

S. S. Benchalli

Department of Mathematics
Karnatak University, Dharwad-580 003
Karnataka State, India
benchalli_math@yahoo.com

P. G. Patil

Department of Mathematics
SKSVM Agadi College of Engineering and Technology
Laxmeshwar-582 116, Karnataka State, India
pgpatil01@gmail.com

T. D. Rayanagoudar

Department of Mathematics
Government First Grade College
Annigeri - 582 201, Karnataka State, India
rgoudar_td@yahoo.com

Abstract

The aim of this paper is to introduce a new class of closed sets called ω-locally closed sets, ωlc^*- sets, ωlc^{**}- sets which are weaker forms of the class of locally closed sets in bitopological spaces. Using these concepts, some of the generalizations of pairwise LC-continuous maps namely, pairwise ωLC- Continuous maps, pairwise ωLC^*- Continuous maps and pairwise ωLC^{**}-Continuous maps in bitopological spaces are introduced and studied. Several examples are provided to illustrate the behaviour of these new class of sets and maps.

Mathematics Subject Classification: 54A05, 54A10, 54C05, 54C08

Keywords: Bitopological spaces, ω-closed set, ω-open set, locally closed set, ωlc- sets, ωlc^*- sets, ωlc^{**}- set

1 Introduction

The notion of locally closed set in a bitopological space was introduced by Kuratowski and Sierpinski [8]. According to Bourbaki [3], a subset A of a topological space X is called locally closed in X if it is the intersection of an open set and a closed set in X.

In the year 1970, Levine [10] introduced the class of generalized closed (g-closed) sets in topological spaces.Using these notions, several new notions are defined in terms of g-closed sets of which two are g-continuous maps[1] and g-locally closed sets [2] by Balachandran et.al. These two notions are defined as natural generalization of the continuous maps and locally closed sets respectively. In 1963, Kelly [7] defined a bitopological space (X, τ_1, τ_2) to be a set X equipped with two topologies τ_1, τ_2 on X and he initiated a systematic study of bitopological spaces. Sheik John [11] defined and studied the concepts of ω-locally closed sets in topological spaces in 2002. In this paper, we define the new notions of ω-locally closed sets, ωlc^*- sets, ωlc^{**}-sets, ωLC- continuous maps, ωLC^*- continuous maps and ωLC^{**}-continuous maps in bitopological spaces and investigate some of their properties.

2 Preliminaries

Throughout this present paper, let X, Y and Z always represent non-empty bitopological spaces (X, τ_1, τ_2), (Y, σ_1, σ_2) and (Z, η_1, η_2) on which no separation axioms are assumed unless explicitly mentioned and the integers $i, j, k \in \{1, 2\}$.

If A is a subset of a topological space X with a topology τ, then the closure of A is denoted by τ-$cl(A)$ or $cl(A)$, the interior of A is denoted by τ-$int(A)$ or $int(A)$.

Before entering into our work we recall the following definitions from various authors.

Definition 2.1. A subset A of a topological space (X, τ) is called semi-open [9] if $A \subseteq cl(int(A))$. The complement of semi-open set is called semi-closed set. The intersection of all semi-closed sets containing A is called the semi-closure [4] of A, and is denoted by $scl(A)$.

Definition 2.2. A subset A of a topological space (X, τ) is called

1. *generalized closed (g-closed) [10] set if* $cl(A) \subseteq G$ *whenever* $A \subseteq G$ *and* G *is open in* (X, τ).
2. ω*-closed [11] set if* $cl(A) \subseteq G$ *whenever* $A \subseteq G$ *and* G *is semi-open in* (X, τ).

DEFINITION 2.3. The intersection of all ω-closed sets containing the set A is called ω-closure of A, and is denoted by $\omega\text{-cl}(A)$[11].

DEFINITION 2.4. A subset A of a bitopological space (X, τ_1, τ_2) is called a (τ_i, τ_j)-locally closed (briefly (τ_i, τ_j)-lc) set[6] if $A = G \cap F$, where G is τ_1-open and F is τ_2-closed in (X, τ_1, τ_2).

DEFINITION 2.5. A map $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called pairwise LC-continuous [5] if $f^{-1}(V) \in (\tau_1, \tau_2) - LC(X)$ for every σ_1-closed set V of (Y, σ_1, σ_2).

3 ω-Locally Closed Sets in Bitopological Spaces

In this section, we introduce $(\tau_i, \tau_j) - \omega lc$-sets, $(\tau_i, \tau_j) - \omega lc^*$- sets and $(\tau_i, \tau_j) - \omega lc^{**}$- sets and obtain relationships among them.

DEFINITION 3.1. Let $i, j \in \{1, 2\}$ be fixed integers. A subset A of a bitopological space (X, τ_1, τ_2) is called $(\tau_i, \tau_j) - \omega$- locally closed (briefly $(\tau_i, \tau_j) - \omega lc$) set if $A = S \cap F$, where S is τ_1- ω-open and F is τ_2- ω-closed in (X, τ_1, τ_2).

DEFINITION 3.2. A subset A of a bitopological space (X, τ_1, τ_2) is called $(\tau_i, \tau_j) - \omega lc^*$ if there exists a τ_1- ω-open set S and τ_2- closed set F of (X, τ_1, τ_2) such that $A = S \cap F$.

DEFINITION 3.3. A subset A of a bitopological space (X, τ_1, τ_2) is called $(\tau_i, \tau_j) - \omega lc^{**}$ if there exists a τ_1-open set S and τ_2- ω-closed set F of (X, τ_1, τ_2) such that $A = S \cap F$.

REMARK 3.4. If $\tau_1 = \tau_2 = \tau$ in Definitions 3.1, 3.2 and 3.3, then $(\tau_1, \tau_2) - \omega lc$ (resp. $(\tau_1, \tau_2) - \omega lc^*$, $(\tau_1, \tau_2) - \omega lc^{**}$) is a ωlc (resp. ωlc^*, ωlc^{**}) set in a topological space [11].

The collection of all $(\tau_1, \tau_2) - \omega lc$ (resp. $(\tau_1, \tau_2) - \omega lc^*$, $(\tau_1, \tau_2) - \omega lc^{**}$) sets of (X, τ_1, τ_2) will be denoted by $(\tau_1, \tau_2) - \omega LC(X)$ (resp. $(\tau_1, \tau_2) - \omega LC^*(X)$, $(\tau_1, \tau_2) - \omega LC^{**}(X)$).

THEOREM 3.5. *Let A be a subset of a space (X, τ_1, τ_2).*

1. *If $A \in (\tau_1, \tau_2) - LC(X)$, then $A \in (\tau_1, \tau_2) - \omega LC(X)$, $(\tau_1, \tau_2) - \omega LC^*(X)$ and $(\tau_1, \tau_2) - \omega LC^{**}(X)$.*

2. *If $A \in (\tau_1, \tau_2) - \omega LC^*(X)$, then $A \in (\tau_1, \tau_2) - \omega LC(X)$.*

3. *If $A \in (\tau_1, \tau_2) - \omega LC^{**}(X)$, then $A \in (\tau_1, \tau_2) - \omega LC(X)$.*

PROOF. Since every closed set is ω-closed set, the proof follows. □

Remark 3.6. The following examples show that the reverse implications of the Theorem 3.5 are not true.

Example 3.7. Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a, b\}\}$ and $\tau_2 = \{X, \phi, \{a\}\}$. Then the subset $\{a\} \in (\tau_1, \tau_2) - \omega LC(X)$ and $(\tau_1, \tau_2) - \omega LC^*(X)$ but $\{a\} \notin (\tau_1, \tau_2) - LC(X)$ and also $\{a\} \notin (\tau_1, \tau_2) - \omega LC^{**}(X)$.

Example 3.8. Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a, b\}\}$ and $\tau_2 = \{X, \phi, \{a\}, \{a, b\}\}$. Then $\{c\} \in (\tau_1, \tau_2) - \omega LC^{**}(X)$ but $\{c\} \notin (\tau_1, \tau_2) - LC(X)$.

Example 3.9. Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}, \{a, b\}\}$ and $\tau_2 = \{X, \phi, \{a, b\}\}$. Then $\{b\} \in (\tau_1, \tau_2) - \omega LC(X)$ but $\{b\} \notin (\tau_1, \tau_2) - \omega LC^*(X)$.

Remark 3.10. The concept of $(\tau_1, \tau_2) - \omega LC^*(X)$ and $(\tau_1, \tau_2) - \omega LC^{**}(X)$ are independent of each other as seen from the following examples.

Example 3.11. In Example 3.8, the subset $\{a\} \in (\tau_1, \tau_2) - \omega LC^*(X)$ but $\{a\} \notin (\tau_1, \tau_2) - \omega LC^{**}(X)$.

Example 3.12. In Example 3.9, the subset $\{b\} \in (\tau_1, \tau_2) - \omega LC^{**}(X)$ but $\{b\} \notin (\tau_1, \tau_2) - \omega LC^*(X)$.

Theorem 3.13. *Let A be a subset of a space (X, τ_1, τ_2).*

(i) If $A \in (\tau_1, \tau_2) - \omega LC^*(X)$.

(ii) $A = G \cap \tau_2$-cl(A) for some $\tau_1 - \omega$-open set G.

(iii) τ_2-cl(A)-A is $\tau_1 - \omega$-closed.

(iv) $A \cup (X - \tau_2$-cl(A)) is $\tau_1 - \omega$-open.

Proof. $(i) \Rightarrow (ii)$: Let $A \in (\tau_1, \tau_2) - \omega LC^*(X)$. Then there exist $\tau_1 - \omega$-open set G and a τ_2-closed set F in (X, τ_1, τ_2) such that $A = G \cap F$. Since $A \subseteq G$ and $A \subseteq \tau_2$-cl(A), we have $A \subseteq G \cap \tau_2$-cl$(A)$. Also, since $\tau_2 - cl(A) \subseteq F$, $G \cap \tau_2$-cl$(A) \subseteq G \cap F = A$. Therefore $A = G \cap \tau_2$-cl(A).

$(ii) \Rightarrow (i)$: Since G is $\tau_1 - \omega$-open and τ_2-cl(A) is a τ_2-closed set, we have $A = G \cap \tau_2$-cl$(A) \in (\tau_1, \tau_2) - \omega LC^*(X)$.

$(iii) \Rightarrow (iv)$: Let $F = \tau_2$-cl(A)-A. Then by (iii), F is $\tau_1 - \omega$-closed. Now $X - F = A \cup (X - \tau_2$-cl$(A))$. Since $(X - F)$ is $\tau_1 - \omega$-open, we get $A \cup (X - \tau_2$-cl$(A))$ is $\tau_1 - \omega$-open.

$(iv) \Rightarrow (iii)$: Let $G = A \cup (X - \tau_2$-cl$(A))$. This implies that $X - G$ is $\tau_1 - \omega$-closed and $X - G = \tau_2$-cl(A)-A. Hence τ_2-cl(A)-A is $\tau_1 - \omega$-closed.

(iv) $\Rightarrow$ *(ii)*: Let $G = A \cup (X - \tau_2\text{-cl}(A))$. So $G \cap \tau_2\text{-cl}(A) = A$. Hence $A = G \cap \tau_2\text{-cl}(A)$ for some $\tau_1 - \omega$-open set G.

(ii) $\Rightarrow$ *(iv)*: Let $A = G \cap \tau_2\text{-cl}(A)$ for some $\tau_1 - \omega$-open set G. Then $A \cup (X - \tau_2\text{-cl}(A)) = G$ which is $\tau_1 - \omega$-open. □

THEOREM 3.14. *Let A and B be any two subsets of a space (X, τ_1, τ_2). If $A \in (\tau_1, \tau_2) - \omega LC(X)$ and B is $\tau_1 - \omega$-open or $\tau_2 - \omega$-closed, then $A \cap B \in (\tau_1, \tau_2) - \omega LC(X)$.*

PROOF. Let $A \in (\tau_1, \tau_2) - \omega LC(X)$. This implies $A = G \cap F$, where G is $\tau_1 - \omega$-open and F is $\tau_2 - \omega$-closed in (X, τ_1, τ_2). Now $A \cap B = (G \cap B) \cap F$.
case (i): If B is $\tau_1 - \omega$-open, then $G \cap B$ is also $\tau_1 - \omega$-open and F is $\tau_2 - \omega$-closed in (X, τ_1, τ_2). Hence $A \cap B \in (\tau_1, \tau_2) - \omega LC(X)$.
case (ii): If B is $\tau_2 - \omega$-closed, then $A \cap B = G \cap (B \cap F)$, where G is $\tau_1 - \omega$-open and $B \cap F$ is $\tau_2 - \omega$-closed in (X, τ_1, τ_2). Hence $A \cap B \in (\tau_1, \tau_2) - \omega LC(X)$.
It is clear that, the intersection of any two $(\tau_1, \tau_2) - \omega lc$-sets is again a $(\tau_1, \tau_2) - \omega lc$-set. □

THEOREM 3.15. *If $A, B \in (\tau_1, \tau_2) - \omega LC^*(X)$, then $A \cap B \in (\tau_1, \tau_2) - \omega LC^*(X)$.*

PROOF. Let $A, B \in (\tau_1, \tau_2) - \omega LC^*(X)$. Then there exist $\tau_1 - \omega$-open sets G and H such that $A = G \cap \tau_2\text{-cl}(A)$ and $B = H \cap \tau_2\text{-cl}(A)$ by Theorem 3.13. Since $G \cap H$ is $\tau_1 - \omega$-open and $A \cap B = (G \cap H) \cap (\tau_2 - cl(A) \cap \tau_2 - cl(B))$, then $A \cap B \in (\tau_1, \tau_2) - \omega LC^*(X)$. □

REMARK 3.16. The union of any two $(\tau_1, \tau_2) - \omega lc^*$-sets need not be a $(\tau_1, \tau_2) - \omega lc^*$-set as seen from the following example.

EXAMPLE 3.17. In Example 3.9, the subsets $\{a\}$, $\{c\} \in (\tau_1, \tau_2) - \omega LC^*(X)$ but their union $\{a, c\} \notin (\tau_1, \tau_2) - \omega LC^*(X)$.

THEOREM 3.18. *If $A, B \in (\tau_1, \tau_2) - \omega LC^{**}(X)$, then $A \cap B \in (\tau_1, \tau_2) - \omega LC^{**}(X)$.*

PROOF. The proof is similar to Theorem 3.15. □

THEOREM 3.19. *If $A \in (\tau_1, \tau_2) - \omega LC^{**}(X)$ and B is either τ_2-closed or τ_1-open subset of (X, τ_1, τ_2), then $A \cap B \in (\tau_1, \tau_2) - \omega LC^{**}(X)$.*

PROOF. Let $A \in (\tau_1, \tau_2) - \omega LC^{**}(X)$. This implies that $A = G \cap F$, where G is $\tau_1 - \omega$-open and F is $\tau_2 - \omega$-closed. Now $A \cap B = (G \cap F) \cap B$. If B is τ_1-open, then $B \cap G$ is τ_1-open. Hence $A \cap B \in (\tau_1, \tau_2) - \omega LC^{**}(X)$. If B is τ_2-closed, then $B \cap F$ is $\tau_2 - \omega$-closed. Therefore $A \cap B \in (\tau_1, \tau_2) - \omega LC^{**}(X)$. □

4 Pairwise ωLC-Continuous and Other related Maps

In this section we introduce new class of LC-continuous maps namely, pairwise ωLC-continuous maps, ωLC^{*}-continuous maps, ωLC^{**}-continuous maps, ωLC-irresolute maps, ωLC^{*}-irresolute maps and pairwise ωLC^{**}-irresolute maps and investigate some of their relationship.

Definition 4.1. A map $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called pairwise ωLC-continuous (resp. pairwise ωLC^{*}-continuous, pairwise ωLC^{**}-continuous) if $f^{-1}(V) \in (\tau_1, \tau_2) - \omega LC(X)$ (resp. $f^{-1}(V) \in (\tau_1, \tau_2) - \omega LC^{*}(X)$, $f^{-1}(V) \in (\tau_1, \tau_2) - \omega LC^{**}(X)$) for every σ_1-closed set V in (Y, σ_1, σ_2).

Definition 4.2. A map $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called pairwise ωLC-irresolute (resp. pairwise ωLC^{*}-irresolute, pairwise ωLC^{**}-irresolute) if $f^{-1}(V) \in (\tau_1, \tau_2) - \omega LC(X)$ (resp. $f^{-1}(V) \in (\tau_1, \tau_2) - \omega LC^{*}(X)$, $f^{-1}(V) \in (\tau_1, \tau_2) - \omega LC^{**}(X)$) for every $V \in (\sigma_1, \sigma_2) - \omega LC(Y)$ (resp. $V \in (\sigma_1, \sigma_2) - \omega LC^{*}(Y)$, $V \in (\sigma_1, \sigma_2) - \omega LC^{**}(Y)$).

Theorem 4.3. *Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a map. Then the following are satisfied.*

1. *If f is pairwise LC-continuous, then it is pairwise ωLC-continuous (resp. pairwise ωLC^{*}-continuous, pairwise ωLC^{**}-continuous).*

2. *If f is pairwise ωLC^{*}-continuous, then it is pairwise ωLC-continuous.*

3. *If f is pairwise ωLC^{**}-continuous, then it is pairwise ωLC-continuous.*

4. *If f is pairwise ωLC-irresolute, then it is pairwise ωLC-continuous.*

5. *If f is pairwise ωLC^{*}-irresolute, then it is pairwise ωLC^{*}-continuous.*

Proof. Omitted. □

Remark 4.4. The converses of the Theorem 4.3 need not be true in general as seen from the following examples.

Example 4.5. Let $X = Y = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a, b\}\}$, $\tau_2 = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma_1 = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\}$, $\sigma_2 = \{Y, \phi, \{a\}\}$. Define a map $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ by $f(a) = a$, $f(b) = c$ and $f(c) = b$. Then the map f is pairwise ωLC-continuous (resp. pairwise ωLC^{*}-continuous, pairwise ωLC^{**}-continuous) but not pairwise LC-continuous.

EXAMPLE 4.6. Let $X = Y = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}, \{a, b\}\}$, $\tau_2 = \{X, \phi, \{a, b\}\}$ and $\sigma_1 = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\}$, $\sigma_2 = \{Y, \phi, \{a\}\}$. Define a map $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ by $f(a) = a$, $f(b) = c$ and $f(c) = b$. Then the map f is pairwise ωLC-continuous but not pairwise ωLC^*-continuous and pairwise ωLC^{**}-continuous.

EXAMPLE 4.7. Let $X = Y = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a, b\}\}$, $\tau_2 = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma_1 = \{Y, \phi, \{a\}, \{b, c\}\}$, $\sigma_2 = \{Y, \phi, \{a\}\}$. Then the identity map $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is pairwise ωLC-continuous (resp. pairwise ωLC^*-continuous) but not pairwise ωLC-irresolute (resp.pairwise ωLC^*-irresolute) .

THEOREM 4.8. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ and $g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2)$ are functions. Then the following statements are hold.*

1. *If f is pairwise ωLC-irresolute and g is pairwise ωLC-continuous, then gof is pairwise ωLC-continuous.*
2. *If f and g are pairwise ωLC-irresolute, then gof is pairwise ωLC-irresolute.*
3. *If f is pairwise ωLC^*-irresolute and g is pairwise ωLC^*-continuous, then gof is pairwise ωLC^*-continuous.*
4. *If f is pairwise ωLC^{**}-irresolute and g is pairwise ωLC^{**}-continuous, then gof is pairwise ωLC^{**}-continuous.*

PROOF. Obvious. □

REMARK 4.9. The composition of pairwise ωLC-continuous maps need not be pairwise ωLC-continuous as seen from the following example.

EXAMPLE 4.10. Let $X = Y = Z = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}, \{a, b\}\}$, $\tau_2 = \{X, \phi, \{a\}\}$, $\sigma_1 = \{Y, \phi, \{a, b\}\}$, $\sigma_2 = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\eta_1 = \{Z, \phi, \{a\}\}$ and $\eta_2 = \{Z, \phi, \{a\}, \{b, c\}\}$. Define maps $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ by $f(a) = b$, $f(b) = c$ and $f(c) = c$ and $g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2)$ by $g(a) = c$, $g(b) = a$ and $g(c) = b$. Then the maps f and g are pairwise ωLC-continuous. However their composition fog is not a pairwise ωLC-continuous.

References

[1] K. BALACHANDRAN, P. SUNDARAM AND H. MAKI, On generalized continuous maps in topological spaces, *Mem. Fac. Sci. Kochi Univ. Ser. A. Math.*, 12 (1991), 5-13.

[2] K. Balachandran, P. Sundaram and H. Maki, Generalized locally closed sets and GLC-continuous functions, I*nd. J. Pure Appl. Math.*, 27(3), (1996), 235-244.

[3] N. Bourbaki, General Topology, Part I reading MA: Addison Wesley, 1996.

[4] S.G. Crossely and S.K.Hildebrand, Semi-topological properties, *Fund. Math.*, 74 (1972), 233-254.

[5] M. Ganster and I. L. Reilly, Locally closed sets and LC-continuous functions, *Internal J. Math. and Math. Sci.*, 12(1989), 417-424.

[6] M. Jelic, A decomposition of pairwise continuity, *J. Inst. Math. Comp. Sci.* 3 (1990), 25-29.

[7] J. C. Kelly, Bitopological Spaces, *Proc. London Math. Soc.* (3), 13 (1962), 71-89.

[8] C. Kuratowski and W. Sierpinski, Surles differences deux ensembles fermes, *Tohoku Math .J.*, 20(1921), 22-25.

[9] N. Levine, Semi-open sets and Semi-continuity in topological spaces, *Amer. Math. Monthly*, 70 (1963), 36-41.

[10] N. Levine, Generalized closed sets in topology, *Rend. Circ. Mat. Palermo*, 19 (2) (1970), 89-96.

[11] M. Sheik John, A study on generalizations of closed sets on continuous maps in topological spaces and bitopological spaces, Ph.D., thesis, Bharathiar University, Coimbatore (2002).

Received: January, 2010