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Nowhere Schur Property in Some Operator Spaces

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Abstract. A Banach space X has the Schur property if every weakly null sequence in X is norm null. Here, we obtain some classes of operator spaces that contain no infinite dimensional closed subspace with the Schur property or with the dual Schur property.

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1. Introduction

A Banach space X is a Schur space or has the Schur property if every weakly convergent sequence in X is norm convergent. Also X has the Dunford-Pettis property (abb. DPP), if for each weakly null sequences (x_n) in X and (x_n^*) in the dual X^* of X, one has $x_n^*(x_n) \to 0$ as $n \to \infty$. For example, the absolutely summable sequence space l_1 as well as finite dimensional Banach spaces have the Schur property. The classical sequence spaces c_0, l_∞, $L^1(\mu)$ for each positive measure μ, and all $C(K)$ spaces for compact Hausdorff space K, have the DPP [8]; while the space $L^1(\mu)$ has the Schur property if and only if μ is an atomic measure [4].

It is clear that every Schur space has the DPP and by a classical Theorem of [8], the dual space X^* has the Schur property if and only if X has the DPP and contains no copy of l_1. So for each closed subspace $M \subseteq c_0$, M^* has the Schur property. As another corollary, for each compact Hausdorff space K, the dual space $C(K)^*$ has the Schur property if and only if K is scattered, that is has no nonempty perfect subset[5]. Also by Rosenthal's l_1-Theorem [17], every infinite dimensional Schur space contains a copy of l_1 and there is no infinite dimensional Banach space X such that X and X^* have the Schur property.

There are many papers which have obtained several examples of Banach spaces with the Schur property and have characterized some aspects of this concept. Also there are some necessary and sufficient conditions for the Schur property of (dual) closed subspaces of several Banach spaces, such as operator ideals; see [3], [8], [15], [16] and [18].

In fact, there are Banach spaces X and some its closed subspace M of X such that M or M^* has the Schur property. For example, if a Banach space X contains a copy of l_1, then X contains a closed subspace with the Schur property and if X contains a copy of c_0, then X contains a closed subspace such that its dual has the Schur property. Here, by introducing the concept of nowhere (dual) Schur property, we obtain some classes of Banach spaces such as operator spaces that contain no infinite dimensional closed subspace with the Schur property or with the dual Schur property.

Throughout this article X and Y denote arbitrary Banach spaces. The dual of X is denoted by X^* and T^* refers to the adjoint of the operator T. $\mathcal{U}$ is an arbitrary (Banach) operator ideal and $\mathcal{U}(X,Y)$ is applied for component of $\mathcal{U}$. For arbitrary Banach spaces X and Y, $L(X,Y)$ and $K(X,Y)$ are used for the Banach spaces of all bounded linear and compact operators between X and Y, respectively, and $L_{w^*}(X^*,Y)$ and $K_{w^*}(X^*,Y)$ are the spaces of all bounded linear weak*-weak continuous and compact weak*-weak continuous operators from X^* to Y respectively. The abbreviation $K(X)$ is used for $K(X,X)$. Our notations are standard and we refer the reader to [7], [9] and [14] for undefined notations and terminologies.

Definitions 1.1. *Let X be an arbitrary Banach space. Then*

- **a):** *X has the nowhere Schur property if X contains no infinite dimensional closed subspace with the Schur property*
- **b):** *X has the nowhere dual Schur property if X contains no infinite dimensional closed subspace such that its dual has the Schur property.*

As easy consequences of the definition, we have the following two theorems:

Theorem 1.2. *A Banach space X has the nowhere Schur property if one of the following assertions hold:*

a): *X contains no copy of l_1*

b): *X has no infinite dimensional weakly sequentially complete (abb. wsc) subspace*

c): *X has no infinite dimensional subspace with the DPP.*

Proof. An easy consequence of Rosenthal's l_1-Theorem, shows that every infinite dimensional Schur space, contains a copy of l_1. This proves that (a) is a sufficient condition for the nowhere Schur property of X. Also, under the hypothesis of (b) or (c), the assertion valid, since every Banach space with the Schur property is wsc and has the DPP.

Theorem 1.3. *A Banach space X has the nowhere dual Schur property if one of the following assertions hold:*

a): *X is wsc*

b): *X is hereditarily l_1 Banach space, that is, every infinite dimensional subspace of X contains a copy of l_1*

c): *X has no infinite dimensional subspace with the DPP.*

Proof. Suppose that $M \subseteq X$ is an infinite dimensional closed subspace such that M^* has the Schur property. In particular, M contains no copy of l_1 and so M is not wsc, because of every wsc Banach space which contains no copy of l_1 is reflexive. Now since the weak sequential completeness of Banach spaces is inherited by closed subspaces, it follows that X is not wsc, so the assertion is proved by the hypothesis of (a). Statements (b) and (c) refers to the fact that the dual M^* of M has the Schur property if and only if M has the DPP and contains no copy of l_1.

As easy consequences of these theorems, the sequence space c_0 and each reflexive Banach space has the nowhere Schur property. Also the sequence space l_1 and each reflexive Banach space has the nowhere dual Schur property.

The above theorem is also applied on the Azimi-Hagler's example. In fact, in [2], the authors gave a family of Banach spaces such that each element X of this family is hereditarily l_1 Banach space and fails to have the Schur property. So every infinite dimensional closed subspace Y of each element X

of this family contains a closed subspace which has the Schur property, but itself X, has nowhere dual Schur property.

2. Nowhere Schur property in operator spaces

In this section we give a good characterization of Banach spaces which have the nowhere Schur property and illustrate it in some operator spaces.

Theorem 2.1. *A Banach space X has the nowhere Schur property if and only if X contains no copy of l_1.*

Proof. The necessary condition follows from Rosenthal's l_1-Theorem and the sufficient condition follows from the facts that the sequence space l_1 has the Schur property and the Schur property is stable under isomorphism.

As an easy consequence, for any Hilbert space H, the Banach space $K(H)$ of all compact operators on H has the nowhere Schur property, since by [11] or [12], every non-reflexive subspace of $K(H)$ contains a subspace isomorphic to c_0 while c_0 does not embed to l_1. This assertion also shows that every non-reflexive subspace of $K(H)$ contains a closed subspace with dual Schur property. Here we obtain some operator ideals between some Banach spaces that contain no copy of l_1 and so have the nowhere Schur property:

Theorem 2.2. *Let X (resp. X^*) and Y contain no copy of l_1 and that either X^* (resp. X^{**}) or Y^* has the Radon- Nikodym property, then $K_{w^*}(X^*, Y)$ (resp. $K(X, Y)$) has the nowhere Schur property.*

Proof. By theorem 3.3.1 and its consequence of [6], $K_{w^*}(X^*, Y)$ (resp. $K(X, Y)$) contains no copy of l_1. Now apply theorem 2.1.

We remember from [1] that a linear subspace $\mathcal{M}$ of $\mathcal{U}(X, Y)$ has the $\mathcal{K}$- property if every sequence in $\mathcal{M}$ that converges in the weak-operator topology, converges weakly. This means that if $T_n, T \in \mathcal{M}$ and $\langle T_n x, y^* \rangle \to \langle Tx, y^* \rangle$, for all $x \in X$ and $y^* \in Y^*$; then $T_n \to T$ weakly. We know from [6] that, for every Banach spaces X and Y, the Banach space $K_{w^*}(X^*, Y)$ and so all of its closed subspaces have the $\mathcal{K}$- property. Also $K(X, Y)$ has the $\mathcal{K}$- property if and only if X is a Grothendieck space [13]. Recall that a Banach space X is a Grothendieck space if every weak* convergent sequence in X^* is weakly convergent [9]. In the following, $\mathcal{M}_1$ is the closed unit ball of $\mathcal{M}$.

Theorem 2.3. *Let $\mathcal{U}$ be a Banach operator ideal and $\mathcal{M} \subseteq \mathcal{U}(X,Y)$ be a linear subspace. If $\mathcal{M}$ has the $\mathcal{K}$- property and one of the following conditions:*

- **a):** *X is separable and all of the point evaluations $\mathcal{M}_1(x) = \{Tx : T \in \mathcal{M}_1\}$ are relatively weakly compact*
- **b):** *Y^* is separable and all of the point evaluations $\widetilde{\mathcal{M}}_1(y^*) = \{T^*y^* : T \in \mathcal{M}_1\}$ are relatively weakly compact*

holds, then $\mathcal{M}$ has the nowhere Schur property.

Proof. Let (T_n) be an arbitrary sequence in $\mathcal{M}_1$ and (a) holds. If (x_n) is a dense sequence in X, by a standard technique of diagonalization there exists a subsequence of (T_n), which we denote by (T_n) again, such that $(T_n x_i)_{n=1}^{\infty}$ converges weakly in Y, for all $i = 1, 2, \dots$. It is straightforward to check that the sequence $(T_n x)$ is weakly Cauchy in Y, for all $x \in X$ and by the $\mathcal{K}$- property of $\mathcal{M}$, the sequence (T_n) is weakly Cauchy. Now Rosenthal's l_1-Theorem completes the proof of the first part of Theorem. The proof of the second part is similar and we omit its details.

Corollary 2.4. *Suppose that Y^* is separable and $\mathcal{M}$ is either a closed subspace of $K(X,Y)$ or $K_{w^*}(X^*,Y)$. If all of the point evaluations $\widetilde{\mathcal{M}}_1(y^*)$ are relatively weakly compact, then $\mathcal{M}$ has the nowhere Schur property.*

Proof. For the case $\mathcal{M} \subseteq K_{w^*}(X^*,Y)$, this is a direct consequence of Theorem 2.3(b). If $\mathcal{M} \subseteq K(X,Y)$ and (T_n) is a sequence in $\mathcal{M}_1$, the proof of Theorem 2.3 shows that the sequence $(T_n^* y^*)$ is weakly Cauchy in X^*, for all $y^* \in Y^*$. So by Corollary 3 of [13], (T_n) is weakly Cauchy. Therefore an appeal to Rosenthal's l_1-Theorem completes the proof.

3. Nowhere dual Schur property in operator spaces

Finally, we obtain some conditions that guarantee some classes of operators, have the nowhere dual Schur property.

Theorem 3.1. *If H is an arbitrary Hilbert space, then the space $\mathcal{N}(H)$ of all nuclear operators (or trace class operators) on H, has the nowhere dual Schur property.*

Proof. We know that by Theorem 3 of [12] (see also [11]), every infinite dimensional closed subspace of $\mathcal{N}(H)$ is either isomorphic to H or contains a subspace isomorphic to l_1.
Now suppose that $\mathcal{M}$ is an infinite dimensional subspace of $\mathcal{N}(H)$. if $\mathcal{M}$ is isomorphic to H, then the reflexive space $\mathcal{M}^*$ does not have the Schur property. If $\mathcal{M}$ contains a copy of l_1, then also $\mathcal{M}^*$ does not have the Schur property.

Recall that a bounded subset A of X is a Dunford-Pettis set (abb. DPS) if for all weakly null sequence $(x_n^*) \subseteq X^*$, one has

$$\lim_{n\to\infty} \sup_{x\in A} |\langle x_n^*, x\rangle| = 0.$$

The Banach space X has DP$_{rc}$P if every DPS in X is relatively norm compact. For example, if Y contains no copy of l_1, then Y^* has the DP$_{rc}$P [10].

Theorem 3.2. *Let X (resp. X^*) has the Schur property and Y has the DP$_{rc}$P. Then $K_{w^*}(X^*, Y)$ (resp. $K(X, Y)$) has the nowhere dual Schur property.*

Proof. By Theorem 7 of [10], $K_{w^*}(X^*, Y)$ (resp. $K(X, Y)$) has the DP$_{rc}$P. If $\mathcal{M} \subseteq K_{w^*}(X^*, Y)$ (resp. $K(X, Y)$) is a linear subspace such that $\mathcal{M}^*$ has the Schur property, then by the definition of norm in $\mathcal{M}^*$, the closed unit ball $\mathcal{M}_1$ of $\mathcal{M}$ is a DPS in $\mathcal{M}$ and so in $K_{w^*}(X^*, Y)$ (resp. $K(X, Y)$. Thus $\mathcal{M}_1$ is relatively compact and $\mathcal{M}$ is finite dimensional.

Theorem 3.3. *If X and Y are wsc and $K_{w^*}(X^*, Y)$ is weak-operator topology sequentially closed in $L_{w^*}(X^*, Y)$, then $K_{w^*}(X^*, Y)$ has the nowhere dual Schur property.*

Proof. By Proposition 3.1 of [6], $K_{w^*}(X^*, Y)$ is wsc and so has the nowhere dual Schur property, thanks to Theorem 1.3.

We conclude this section by two another corollaries of Theorem 1.3.

Corollary 3.4. *Let X be a wsc Banach space and $L(X,Y)$ has the $\mathcal{K}$- property. Then $L(X,Y)$ has the nowhere dual Schur property.*

Proof. By the method of Proposition 6 of [18], every weakly Cauchy sequence in $L(X,Y)$ is weakly convergent and so $L(X,Y)$ is wsc. Now apply Theorem 1.3.

Corollary 3.5. *If X^* has the Schur property and Y is wsc, then $K(X,Y)$ has the nowhere dual Schur property.*

Proof. If X^* has the Schur property, it is wsc. Also by Theorem 2.1 of [15], $K(X,Y) = L(X,Y)$. So $K(X,Y)$ is wsc.

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