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An Inventory Model with Time Dependent Linear Deteriorating Items with Partial Backlogging

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Abstract

In this study, a deterministic inventory model has been developed for which items are subject to linear deterioration. Here shortages are allowed and are partially backlogged. The unsatisfied demand is backlogged and is a function of time. We have established the optimal order quantity by minimizing the total inventory cost. To illustrate the model a numerical example has been provided and also Sensitivity analysis has been carried out to study the effect of parameters on decision variables and the total cost of an inventory of this model.

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1. Introduction

In many inventory systems, deterioration of goods is a realistic phenomenon. Many goods are subject to deterioration or decay during their normal storage period. Hence while determining the optimal inventory policy of such type of product, the loss due to deterioration has to be considered. In the study of inventory policy, deteriorating inventory model are continuously modified to accommodate more and more practical situation. Deterioration is defined as decay, spoilage, loss of utility of the product. The process of deterioration is observed in volatile liquids, beverages, medicines, blood components, food stuffs, dairy items etc. Study of deteriorating inventory model began with Ghare and Schrader (4) who established the classical no shortage inventory model for constant rate of decay. Further study in this direction can be observed from the literature survey by Raafat (6), Shah and Shah (7) and Goyal and Giri (5). Abad (1, 2) derived a pricing and ordering policy for a variable rate of deterioration and partially backlogging. The partial backlogging was assumed to be exponential function of waiting time till the next replenishment. But assumptions of exponential backlogging are not always realistic in developing countries. Abad's article also does not include backorder cost and shortage cost in the formulation of the objective function that influences service level to customers. Dye et al (3) have considered the backorder cost and lost sale in their inventory model.

In this paper we have developed a deterministic inventory model. Here shortages are allowed and are partially backlogged. The deterioration rate is assumed to be linear in this model. The unsatisfied demand is backlogged and is a function of time. We have established the optimal order quantity by minimizing the total inventory cost.

2. Assumptions and Notations

The mathematical model is based on the following notations and assumptions.

2.1. Notations

- A the ordering cost per order
- C the purchase cost per unit.
- h the inventory holding cost per unit per time unit
- π_b the backordered cost per unit short per time unit.
- π_l the cost of lost sales per unit.
- t_1 the time at which the inventory level reaches zero, $t_1 \geq 0$
- t_2 the length of period during which shortages are allowed, $t_2 \geq 0$
- $T = (t_1 + t_2)$ the length of cycle time.
- IM the maximum inventory level during $[0, T]$.
- IB the maximum backordered units during stock out period.
- $Q = IM + IB$ the order quantity during a cycle of length T.
- $I_1(t)$ the level of positive inventory at time $t, 0 \leq t \leq t_1$
- $I_2(t)$ the level of negative inventory at time t, $t_1 \leq t \leq t_1 + t_2$
- TC the total cost per time unit.

2.2. Assumptions

a. The inventory system deals with single item.
b. The demand rate ‘ a ’ is known and constant.
c. The deterioration rate ‘ r ’ is linear i.e. $r = \theta t$, where $0 < \theta << 1$
d. The replenishment rate is infinite.
e. The lead – time is zero or negligible.
f. The planning horizon is infinite.
g. During the stock out period, the backlogging rate is variable and is dependent on the length of the waiting time for the next replenishment. The proportion of the customers who would like to accept the backlogging at time ‘ t ’ is with the waiting time $(T - t)$ for the next replenishment, i.e. for the negative inventory the backlogging rate is $B(t) = \dfrac{1}{1 + \delta(T - t)}$; $\delta > 0$ denotes the backlogging parameter and $t_1 \le t \le T$

3. Mathematical Model

During the period $[0, t_1]$, the inventory depletes as a result of cumulative effects of demand and deterioration. Thus, the inventory level at any instant of time during $[0, t_1]$ is described by the differential equation

$$\frac{dI_1(t)}{dt} + r\, I_1(t) = -a \qquad 0 \le t \le t_1$$

Using the value of $r = \theta t$ in above equation, we get

$$\frac{dI_1(t)}{dt} + \theta t\, I_1(t) = -a \qquad 0 \le t \le t_1 \qquad (1)$$

As the boundary condition $I_1(t_1) = 0$, as θ is very small so using series expansion and ignoring second and higher powers of θ , solution of the differential equation (1) is

$$I_1(t) = a\left[\left(1 - \frac{\theta t^2}{2}\right)\left\{(t_1 - t) + \frac{\theta(t_1^{\,3} - t)}{6}\right\}\right] \qquad 0 \le t \le t_1 \qquad (2)$$

Inventory level reaches to zero at time t_1 there after shortages occur. During the interval $[t_1 , t_1 + t_2]$, the inventory level depends on demand and a fraction of the demand is backlogged. The state of inventory during $[t_1 , t_1 + t_2]$ can be represented by the differential equation,

$$\frac{dI_2(t)}{dt} = -\frac{a}{1 + \delta(t_1 + t_2 - t)} \qquad t_1 \le t \le t_1 + t_2 \qquad (3)$$

Using the boundary condition $I_2(t_1) = 0$, the solution of differential equation is

$$I_2(t) = \frac{a}{\delta}\left(\ln(1 + \delta(t_1 + t_2 - t)) - \ln(1 + \delta t_2)\right) \qquad t_1 \le t \le t_1 + t_2 \qquad (4)$$

The maximum positive inventory is $IM = I_1(0) = a\left[t_1 + \frac{\theta t_1^3}{6}\right]$ (5)

The maximum backordered units are $IB = -I_2(t_1 + t_2) = \frac{a}{\delta}\ln(1+\delta t_2)$ (6)

Hence, the order size during $[0, T]$ is $Q = IM + IB$

$$Q = a\left[t_1 + \frac{\theta t_1^3}{6} + \frac{1}{\delta}\ln(1+\delta t_2)\right] \tag{7}$$

The total cost per cycle consists of following cost components.

i. Ordering cost per cycle; $OC = A$

ii. Inventory holding cost per cycle is $IHC = h\int_0^{t_1} I_1(t)\,dt$

Solving and neglecting second and higher powers of θ as θ is very small we get;

$$IHC = ha\left[\frac{t_1^2}{2} + \frac{\theta}{6}\left(t_1^4 - \frac{t_1^2}{2}\right) - \frac{\theta t_1^4}{24}\right]$$

iii. Backordered cost per cycle is $BC = \pi_b \int_{t_1}^{t_1+t_2} -I_2(t)\,dt = \frac{\pi_b a}{\delta^2}[\delta t_2 - \ln(1+\delta t_2)]$

iv. Cost due to lost sales per cycle;

$$LS = \pi_l a \int_{t_1}^{t_1+t_2}\left(1 - \frac{1}{1+\delta(t_1+t_2-t)}\right)dt = \frac{\pi_l a}{\delta}[\delta t_2 - \ln(1+\delta t_2)]$$

v. Purchase cost per cycle is $PC = C \times Q = Ca\left[t_1 + \frac{\theta t_1^3}{6} + \frac{1}{\delta}\ln(1+\delta t_2)\right]$

Therefore, the total cost per time unit is

$$TC = \frac{1}{t_1+t_2}[OC + IHC + BC + LS + PC]$$

$$= \frac{1}{t_1+t_2}\left[\begin{array}{l} A + ha\left[\frac{t_1^2}{2} + \frac{\theta}{6}\left(t_1^4 - \frac{t_1^2}{2}\right) - \frac{\theta t_1^4}{24}\right] + \frac{\pi_b a}{\delta^2}[\delta t_2 - \ln(1+\delta t_2)] \\ + \frac{\pi_l a}{\delta}[\delta t_2 - \ln(1+\delta t_2)] + Ca\left[t_1 + \frac{\theta t_1^3}{6} + \frac{1}{\delta}\ln(1+\delta t_2)\right] \end{array}\right] \tag{8}$$

The necessary condition for the total cost per time unit, to be minimize is

$$\frac{\partial TC}{\partial t_1}=\frac{1}{(t_1+t_2)^2}\left[\begin{array}{l} -A+ha\left[\begin{array}{l}\frac{\theta}{6}\left(3t_1^{\,4}+4t_1^{\,3}t_2-\frac{t_1^{\,2}+2t_1t_2}{2}\right)\\ -\frac{\theta(3t_1^{\,4}+4t_1^{\,3}t_2)}{24}+\frac{t_1^{\,2}+2t_1t_2}{2}\end{array}\right]+\frac{\pi_b a}{\delta^2}\left[\ln(1+\delta t_2)-\delta t_2\right]\\ +\frac{\pi_l a}{\delta}\left[\ln(1+\delta t_2)-\delta t_2\right]+Ca\left[\frac{3t_2+\theta t_1^{\,3}-3\ln(1+\delta t_2)}{3}\right]\end{array}\right]=0 \tag{9}$$

and

$$\frac{\partial TC}{\partial t_2}=\frac{1}{(t_1+t_2)^2}\left[\begin{array}{l} -A+ha\left[\frac{\theta}{6}\left(\frac{t_1^{\,2}}{2}-t_1^{\,4}\right)-t_1^{\,2}+\frac{\theta t_1^{\,4}}{24}\right]\\ +\frac{\pi_b a}{\delta^2}\left[\delta t_2-\frac{(t_1+t_2)\delta-\ln(1+\delta t_2)(1+\delta t_2)}{(1+\delta t_2)}\right]\\ +\frac{\pi_l a}{\delta}\left[\delta t_2-\frac{(t_1+t_2)\delta-\ln(1+\delta t_2)(1+\delta t_2)}{(1+\delta t_2)}\right]\\ +Ca\left[\frac{(t_1+t_2)\delta-\ln(1+\delta t_2)(1+\delta t_2)}{\delta(1+\delta t_2)}-\left(t_1+\frac{\theta t_1^{\,3}}{6}\right)\right]\end{array}\right]=0 \tag{10}$$

Provided

$$\frac{\partial^2 TC}{\partial t_1^{\,2}}\times\frac{\partial^2 TC}{\partial t_2^{\,2}}-\frac{\partial^2 TC}{\partial t_1\partial t_2}>0 \tag{11}$$

for obtained pair of (t_1, t_2). The equations (9) and (10) are highly non-linear. Using mathematica-5.1 software, for given set of parametric equations (9) and (10) can be solved. The obtained values of t_1 and t_2 must satisfy equation (11) to minimize the total cost per time unit of an inventory system. To illustrate and validate the proposed model, let us consider a numerical data in the following section and carry out sensitivity analysis with respect to backlogging parameter, deterioration rate and demand.

4. Numerical example

Example-1: Consider an inventory system with following parametric values in proper units.
$[A, C, h, \pi_b\ \pi_l, a, \delta, \theta]$ = [200, 10, 0.4, 15, 20, 40, 4, 0.04], we get optimum value of $t_1^* = 6.42009$, $t_2^* = 4.08238$, putting the optimum values of t_1^* and t_2^* in equation (7) and (8) we get the optimum values of $Q^* = 355.893$ and minimum total cost $TC^* = 679.713$ respectively.

5. Sensitivity Analysis

We have performed sensitivity analysis by changing parameters θ, δ and a (increasing or decreasing) the parameters by 25% and 50%, and keeping the remaining parameters at their original values using the values of the above numerical example. The results are given in the following tables namely Table-1 to Table-3.

Table-1 (Variation in deterioration rate 'θ')

Parameter	% Change	Change in t_1^*	Change in t_2^*	Change in Q^*	Change in TC^*
	+25	5.96452	3.83925	337.258	681.999
θ	+50	5.61143	3.65143	322.650	684.111
	-25	7.04682	4.41796	381.129	677.243
	-50	8.00508	4.93421	418.919	674.641

In table-1, it is observed that increase in deterioration rate increase total cost per time unit of an inventory system and decreases inventory time period and ordering quantity.

Table-2(Variation in backlogging parameter 'δ')

Parameter	% Change	Change in t_1^*	Change in t_2^*	Change in Q^*	Change in TC^*
	+25	6.55396	4.33311	362.198	686.512
δ	+50	6.65361	4.52866	366.949	691.374
	-25	6.22658	3.75024	346.846	669.413
	-50	5.91875	3.30470	332.629	652.359

In table-2, it is observed that increase in backlogging parameter increase total cost per time unit of an inventory system and increase inventory time period and ordering quantity.

Table-3(Variation in demand 'a')

Parameter	% Change	Change in t_1^*	Change in t_2^*	Change in Q^*	Change in TC^*
	+25	6.35510	3.98384	438.678	840.408
a	+50	6.31038	3.91734	521.340	1000.90
	-25	6.52328	4.24368	272.883	518.638
	-50	6.71319	4.55641	189.384	356.740

In table-3, it is observed that increase in demand parameter increase total cost per time unit of an inventory system and decrease inventory time period and increase ordering quantity.

6. Conclusion

In this paper, we have developed a deterministic inventory model for linearly deteriorating items with shortages which are partially backlogged. The optimal order quantity has been calculated by minimizing the total inventory cost.

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