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On Convergence of Singular Integral Operators Dependings on Three Parameters with Radial Kernels

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Abstract

In this paper, two theorems are proved, one for existence of the operator $L(f;x,y,\lambda)$ and the others for its pointwise convergence to $f(x_0,y_0)$, as (x,y,λ) tends to (x_0,y_0,λ_0). In contrast to previous works, the kernel function is radial.

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1 Introduction

In papers [4] and [5], it is investigated that pointwise convergence of integrable functions in $L^1(P)$ by a three parameter family of convolution type singular integral operators of the form

$$U(f;x,y,\lambda) = \iint\limits_P f(s,t)\,K(s-x,t-y;\lambda)\,dsdt, \quad (x,y) \in P.$$

(In [4] and [5], the rectangles $[-a,a]\times[-b,b]$ and $[-\pi,\pi]\times[-\pi,\pi]$ denote by P, respectively.)
Similarly, in [1] and [2], it is studied that pointwise convergence of integrable functions in a given interval by a two parameter family convolution type singular integral.
Let $L_{2\pi}$ be the class of all real functions $f(s,t)$, 2π periodic in each variable, separately, Lebesgue-integrable in the square $Q = \langle-\pi,\pi\rangle\times\langle-\pi,\pi\rangle$, where $\langle-\pi,\pi\rangle\times\langle-\pi,\pi\rangle$ is an arbitrary region such as closed or semi-closed or open square region in R^2.
In the present paper, we investigate the pointwise convergence of $L(f;x,y,\lambda)$

to $f(x_0, y_0)$ in the space $L_{2\pi}$, by the three parameter family of singular operators with radial kernel of the form

$$L(f;x,y,\lambda) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(s,t)\, K\left(\sqrt{(s-x)^2+(t-y)^2};\lambda\right) dsdt \quad ,$$

where $(x,y) \in Q$ and $\lambda \in \Lambda \subset R$.

Definition 1.1

A function $\psi \in L_{2\pi}$ is said to be radial, if there exists a function $\varkappa\left(\sqrt{s^2+t^2}\right)$, defined on $0 \leq \sqrt{s^2+t^2} < \infty$, such that $\psi(s,t) = \varkappa\left(\sqrt{s^2+t^2}\right)$ a.e. [3].

Definition 1.2

$(Class\ A)$**:** We take a family $K = \left(K\left(\sqrt{s^2+t^2};\lambda\right)\right)_{\lambda\in\Lambda}$ of functions $K\left(\sqrt{s^2+t^2};\lambda\right)$: $R^2 \times \Lambda \to R$. We will say that the function $K\left(\sqrt{s^2+t^2};\lambda\right)$ belongs to class A, if the following conditions are satisfied.

a) $K\left(\sqrt{s^2+t^2};\lambda\right)$ is a 2π periodic function, defined for all (s,t) on Q and $\lambda \in \Lambda$ (where Λ is a given set of numbers with accumulation point λ_0), measurable with respect to (s,t) for each fixed $\lambda \in \Lambda$,

b) $\|K(.;\lambda)\|_{L^1} \leq M < \infty$ for every $\lambda \in \Lambda$, where M is independent on λ,

c) $\lim\limits_{\lambda\to\lambda_0} |K(\delta;\lambda)| = 0$ for every $\delta > 0$,

d) $\lim\limits_{(x,y,\lambda)\to(x_0,y,\lambda_0)} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} K\left(\sqrt{(s-x)^2+(t-y)^2};\lambda\right) dsdt = 1$,

e) $\lim\limits_{\lambda\to\lambda_0}\left[\sup\limits_{\delta\leq\sqrt{s^2+t^2}} \left|K\left(\sqrt{s^2+t^2};\lambda\right)\right|\right] = 0$ for $\delta > 0$ $(\lambda \in \Lambda)$.

2 Main Results

The proof of the theorem is based on the following Lemma.

Lemma 2.1

Let $1 \leq p < \infty$. If the kernel $K\left(\sqrt{s^2+t^2};\lambda\right)$ belongs to class A. Then $L\left(f;x,y,\lambda\right)$ defines a continuous transformation over $L^p_{2\pi}$.

Proof. First we assume that $p = 1$.

$$L\left(f;x,y,\lambda\right) = \iint\limits_{Q} f\left(s,t\right) K\left(\sqrt{\left(s-x\right)^2+\left(t-y\right)^2};\lambda\right) dsdt, \;\; (x,y) \in Q.$$

By the linearity of the operator $L\left(f;x,y,\lambda\right)$, it is sufficient to show that

$$\left\|L\left(\lambda\right)\right\| = \sup_{f\neq 0} \frac{\left\|L\left(f;x,y,\lambda\right)\right\|_{L^1_{2\pi}}}{\left\|f\right\|_{L^1_{2\pi}}}$$

is bounded. Hence, by generalized Minkowski inequality, we get that

$$\left\|L\left(f;x,y,\lambda\right)\right\|_{L^1_{2\pi}} = \iint\limits_{Q} \left|\iint\limits_{Q} f\left(s,t\right) K\left(\sqrt{\left(s-x\right)^2+\left(t-y\right)^2};\lambda\right) dsdt\right| dxdy$$

$$\begin{aligned}
&\leq \iint\limits_{Q} \left|f\left(s,t\right)\right| \left(\iint\limits_{Q} \left|K\left(\sqrt{\left(s-x\right)^2+\left(t-y\right)^2};\lambda\right)\right| dxdy\right) dsdt \\
&= \left\|f\right\|_{L^1_{2\pi}} \left\|K\right\|_{L^1_{2\pi}} .
\end{aligned}$$

Consequently

$$\left\|L\left(\lambda\right)\right\| = \sup_{f\neq 0} \frac{\left\|L\left(f;x,y,\lambda\right)\right\|_{L^1_{2\pi}}}{\left\|f\right\|_{L^1_{2\pi}}} \leq M < \infty.$$

Thus $L\left(f;x,y,\lambda\right)$ defines a continuous transformation over $L_{2\pi}$.
Secondly we assume that $1 < p < \infty$.
By the linearity of the operator $L\left(f;x,y,\lambda\right)$, it is sufficient to show that

$$\left\|L\left(\lambda\right)\right\| = \sup_{f\neq 0} \frac{\left\|L\left(f;x,y,\lambda\right)\right\|_{L^p_{2\pi}}}{\left\|f\right\|_{L^p_{2\pi}}}$$

is bounded.

Hence, by generalized Minkowski inequality, we get following inequalities.

$$\begin{aligned}
&\left\| L\left(f;x,y,\lambda\right)\right\|_{L_{2\pi}^{p}} \\
&= \left(\iint_{Q} \left| \iint_{Q} f\left(s,t\right) K\left(\sqrt{\left(s-x\right)^{2}+\left(t-y\right)^{2}};\lambda\right) dsdt\right|^{p} dxdy\right)^{\frac{1}{p}} \\
&\leq \iint_{Q} \left| K\left(\sqrt{s^{2}+t^{2}};\lambda\right)\right| \left(\iint_{Q} \left|f\left(s+x,t+y\right)\right|^{p} dxdy\right)^{\frac{1}{p}} dsdt \\
&\leq \iint_{Q} \left| K\left(\sqrt{s^{2}+t^{2}};\lambda\right)\right| \left(\iint_{Q} \left|f\left(u,v\right)\right|^{p} dudv\right)^{\frac{1}{p}} dsdt \\
&\leq \left\|K\right\|_{L_{2\pi}^{1}} \left\|f\right\|_{L_{2\pi}^{p}}
\end{aligned}$$

Consequently

$$\left\|L\left(\lambda\right)\right\| = \sup_{f\neq 0} \frac{\left\| L\left(f;x,y,\lambda\right)\right\|_{L_{2\pi}^{p}}}{\left\|f\right\|_{L_{2\pi}^{p}}} \leq M < \infty.$$

Thus $L\left(f;x,y,\lambda\right)$ defines a continuous transformation over $L_{2\pi}^{p}$.

Theorem 2.2

Assume that the kernel $K\left(\sqrt{s^{2}+t^{2}};\lambda\right)$ belongs to class A. Let (x_0, y_0) be a continuous point of function $f\left(s,t\right) \in L_{2\pi}^{1}$ then

$$\lim_{(x,y,\lambda)\to(x_0,y_0,\lambda_0)} L\left(f;x,y,\lambda\right) = f\left(x_0,y_0\right).$$

Proof. Suppose that $(x_0, y_0) \in Q$, $|x_0 - x| < \frac{\delta}{2}$ and $|y_0 - y| < \frac{\delta}{2}$ for any $0 < \delta < \pi$.
Set $I := L\left(f;x,y,\lambda\right) - f\left(x_0,y_0\right)$. According to condition (d), we shall write

$$\begin{aligned}
I\left(x,y,\lambda\right) &= \iint_{Q} f\left(s,t\right) K\left(\sqrt{\left(s-x\right)^{2}+\left(t-y\right)^{2}};\lambda\right) dsdt - f\left(x_0,y_0\right) \\
&\leq \iint_{Q} \left|f\left(s,t\right)-f\left(x_0,y_0\right)\right| \left| K\left(\sqrt{\left(s-x\right)^{2}+\left(t-y\right)^{2}};\lambda\right)\right| dsdt \\
&\quad + \left|f\left(x_0,y_0\right)\right| \left| \iint_{Q} K\left(\sqrt{\left(s-x\right)^{2}+\left(t-y\right)^{2}};\lambda\right) dsdt - 1\right| \\
&= I_1\left(x,y,\lambda\right) + I_2\left(x,y,\lambda\right)
\end{aligned}$$

Firstly we investigate $I_1(x,y,\lambda)$ on two regions as follows.

$$\begin{aligned} I_1(x,y,\lambda) &= \iint\limits_{Q\backslash B_\delta} |f(s,t)-f(x_0,y_0)| \left| K\left(\sqrt{(s-x)^2+(t-y)^2};\lambda\right)\right| dsdt \\ &+\iint\limits_{B_\delta} |f(s,t)-f(x_0,y_0)| \left| K\left(\sqrt{(s-x)^2+(t-y)^2};\lambda\right)\right| dsdt \\ &= I_{11}(x,y,\lambda)+I_{12}(x,y,\lambda), \end{aligned}$$

where $B_\delta := \left\{(s,t) : (s-x_0)^2+(t-y_0)^2 \le \delta^2,\ \ (x_0,y_0)\in Q\right\}$
Now we consider $I_{11}(x,y,\lambda)$. Since $K\left(\sqrt{(s-x)^2+(t-y)^2};\lambda\right)$ is non-increasing function on $Q\backslash B_\delta$, we have following inequalities.

$$\begin{aligned} I_{11}(x,y,\lambda) &\le |K(\delta;\lambda)| \iint\limits_{Q\backslash B_\delta} |f(s,t)-f(x_0,y_0)|\, dsdt \\ &\le |K(\delta;\lambda)| \left\{ \iint\limits_{Q} |f(s,t)|\, dsdt + \iint\limits_{Q} |f(x_0,y_0)|\, dsdt \right\} \\ &= |K(\delta;\lambda)| \left\{ \|f\|_{L^1_{2\pi}} + 4\pi^2 |f(x_0,y_0)| \right\}. \end{aligned}$$

Hence, by (c),

$$\lim_{\lambda\to\lambda_0} |K(\delta;\lambda)| \left\{ \|f\|_{L^1_{2\pi}} + 4\pi^2 |f(x_0,y_0)| \right\} = 0.$$

Thus

$$\lim_{(x,y,\lambda)\to(x_0,y_0,\lambda_0)} I_{11}(x,y,\lambda) = 0. \tag{1}$$

Next we can show that $I_{12}(x,y,\lambda)$ tends to zero as $(x,y,\lambda)\to(x_0,y_0,\lambda_0)$ on B_δ.
Since f is continuous at some point (x_0,y_0) ,for every $\varepsilon>0$ there exists a $\delta>0$ such that $(s-x_0)^2+(t-y_0)^2<\delta^2$ we have $|f(s,t)-f(x_0,y_0)|<\varepsilon$.
From condition (b), we get that

$$\begin{aligned} I_{12}(x,y,\lambda) &< \varepsilon \iint\limits_{B_\delta} \left| K\left(\sqrt{(s-x)^2+(t-y)^2};\lambda\right)\right| dsdt \\ &< \varepsilon \iint\limits_{B_\delta} \left| K\left(\sqrt{(s-x)^2+(t-y)^2};\lambda\right)\right| dsdt \\ &\le \varepsilon \|K\|_{L^1_{2\pi}} \le \varepsilon M. \end{aligned}$$

Thus

$$\lim_{(x,y,\lambda)\to(x_0,y_0,\lambda_0)} I_{12}\left(x,y,\lambda\right)=0. \tag{2}$$

Combining (1) and (2), we get

$$\lim_{(x,y,\lambda)\to(x_0,y_0,\lambda_0)} I_{1}\left(x,y,\lambda\right)=0.$$

Therefore we get from the condition (d)

$$\lim_{(x,y,\lambda)\to(x_0,y_0,\lambda_0)} L\left(f;x,y,\lambda\right)=f\left(x_0,y_0\right).$$

Therefore our theorem is proved.

References

[1] Gadjiev, A. D. *On the order of convergence of singular integrals which depending on two parameters*. Special Problems of Functional Analysis and its Applications to the Theory of Differential Equations and the Theory of Functions pp. 40–44 Izdat. Akad. Nauk Azerbaĭdažan, Baku, 1968.

[2] Karsli, H. and Ibikli, E. *On convergence of convolution type singular integral operators depending on two parameters*. Fasc. Math. No. 38 (2007), 25–39.

[3] Nessel, R.J. *Contributions to the theory saturation for singular integrals in several variables, III, radial kernels*. Indag. Math. 29. Ser. A, (1966), 65-73.

[4] Stanislaw, S. *A theorem of Romanovski type for double singular integrals*. Comment. Math. 29.(1989), 277-289.

[5] Taberski, R. *On double integrals and Fourier Series*. Ann. Pol. Math.(1964), 97-115.

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