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# A Generalization Towards the Inequalities for the Polynomial and its Derivative

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**Abstract**

In this paper, we obtain a new results concerning the maximum moduli of a complex polynomial and the derivative of a complex polynomial with restricted zeros. These results generalize and refine upon all the related results available in the literature.

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**Keywords:** Polynomials, zeros, inequalities

## 1 Introduction and Statement of the Results

The study on the relations between the location of the zeros of a complex polynomial and its derivative is a very fertile field for researchers and effort has been on in this direction, since more than sixty years. As a consequence, various inequalities on the complex polynomials and their derivatives have been obtained which can be found in the literature. We begin our discussion from two fundamental as well as important inequalities. If $P(z)$ is a polynomial of degree $n$, then,

$$\max_{|z|=1}|P'(z)| \leq n \max_{|z|=1}|P(z)|. \tag{1}$$

$$\max_{|z|=K}|P(z)| \leq K^n \max_{|z|=1}|P(z)|. \tag{2}$$

Inequality(1) is an immideate consequence of Bernstein's theorem [8] and the inequality (2) is due to Riesz [7]. Equality in (1) and (2) hold for polynomials, whose all the zeros lie at the origin. So it is natural to seek improvements under appropriate assumptions on the zeros of $P(z)$. Thus Erdös conjectured and Lax [5] verified that (1) can be replaced by a better inequality,

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{2} \max_{|z|=1}|P(z)| \tag{3}$$

if the zeros of $P(z)$ are not in the disk $|z| < 1$. Equality in (3) holds for the polynomial $P(z) = z^n - 1$. On the other hand Ankeny and Rivlin[1] used (3) to prove that if the zeros of $P(z)$ are not in the disk $|z| < 1$, then

$$\max_{|z|=K} |P(z)| \leq \left(\frac{K^n + 1}{2}\right) \max_{|z|=1} |P(z)|. \tag{4}$$

Equality in (4) holds for the polynomial $P(z) = \alpha z^n + \beta$ where $|\alpha| = |\beta|$. Further Aziz and Dawood [2] in their paper, obtained a result concerning the minimum modulus of a polynomial $P(z)$ and an inequality on its derivative analogous to (3) and (4), as stated below. If $P(z)$ is a polynomial of degree n with $P(z) \neq 0$ in $|z| < 1$, then,

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2}\left[\max_{|z|=1} |P(z)| - \min_{|z|=1} |P(z)|\right] \tag{5}$$

and

$$\max_{|z|=K} |P(z)| \leq \left(\frac{K^n + 1}{2}\right) \max_{|z|=1} |P(z)| - \left(\frac{K^n - 1}{2}\right) \min_{|z|=1} |P(z)|. \tag{6}$$

The result is best possible and equality holds in (5) and (6) for the polynomial $P(z) = \alpha z^n + \beta$ where $|\alpha| \geq |\beta|$. Further improvisation of these inequalities can be seen in the papers due to Jain V K [3,4]. Here we present an inequality relating the maximum value of a polynomial having all its zeros in $|z| \geq 1$ and the maximum value of the derivative of the polynomial with a parameter $\gamma$.

**Theorem 1.1** *If $P(z)$ is a polynomial of degree $n$ with $P(z) \neq 0$ in $|z| < 1$, then for any $\gamma$ with $|\gamma| \leq 1$,*

$$\max_{|z|=1} |zP'(z) + n\left(\frac{\gamma}{2}\right)P(z)|$$

$$\leq \frac{n}{2}\left[\left(|1 + \frac{\gamma}{2}| - |\frac{\gamma}{2}|\right)M - \left(|1 + \frac{\gamma}{2}| + |\frac{\gamma}{2}|\right)m\right] \tag{7}$$

*and*

$$\max_{|z|=K} |P(z)| \leq \frac{1}{K}\left[\left|1 - n\frac{\gamma}{2}\right| + \frac{n}{2}\left(|1 + \frac{\gamma}{2}| - |\frac{\gamma}{2}|\right)\right]\left[\frac{(K^{n+1} - 1)}{n+1}\right]M$$

$$-\frac{1}{K}\left[\frac{n}{2}\left(|1 + \frac{\gamma}{2}| + |\frac{\gamma}{2}|\right)\left[\frac{(K^{n+1} - 1)}{n+1}\right] - 1\right]m \tag{8}$$

*where*

$$M = \max_{|z|=1} |P(z)|, \quad m = \min_{|z|=1} |P(z)|.$$

**Remark**: Theorem 1.1 with the inequality (2) follows the following new inequality.

$$\max_{|z|=K} |zP'(z) + n\left(\frac{\gamma}{2}\right) P(z)|$$
$$\leq K^n \frac{n}{2}\left[\left(|1+\frac{\gamma}{2}| - |\frac{\gamma}{2}|\right) M - \left(|1+\frac{\gamma}{2}| + |\frac{\gamma}{2}|\right) m\right]. \qquad (9)$$

# 2 Lemmas

For the proof of the theorem 1.1, we require the following lemmas. Lemma 2.1 is due to Malik and Vong [6].

**Lemma 2.1** *Let $Q(z)$ be a polynomial of degree $n$ having all its zeros in $|z| \leq 1$ and $P(z)$ be a polynomial of degree not exceeding that of $Q(z)$. If $|P(z)| \leq |Q(z)|$ for $|z| = 1$, then for any $\gamma$ with $|\gamma| \leq 1$ ,*

$$|zP'(z) + n\left(\frac{\gamma}{2}\right) P(z)| \leq |zQ'(z) + n\left(\frac{\gamma}{2}\right) Q(z)|$$

*for $|z| = 1$.*

**Lemma 2.2** *: If $P(z)$ is a polynomial of degree $n$ with $M = 1$, then for any $\gamma$ with $|\gamma| \leq 1$ and $|z| = 1$,*

$$|zP'(z) + n\left(\frac{\gamma}{2}\right) P(z)| + |zQ'(z) + n\left(\frac{\gamma}{2}\right) Q(z)| \leq \left[|1+\frac{\gamma}{2}| - |\frac{\gamma}{2}|\right]$$

*where $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$.*

Proof. It is obvious that for any $\alpha$ and $\beta$ with $|\alpha| = |\beta| > 1$, the polynomial $P_1(z) = \frac{P(z)}{\alpha} - \beta$ does not vanish in $|z| \leq 1$. The polynomial $Q_1(z) = z^n\overline{P_1(\frac{1}{\bar{z}})} = \frac{Q(z)}{\overline{\alpha}} - \overline{\beta}z^n$ has all its zeros in $|z| \leq 1$ and hence the polynomial $Q_2(z) = \frac{Q(z)}{\overline{\beta}} - \overline{\alpha}z^n$ has all its zeros in $|z| \leq 1$. Also $|P_1(z)| = |Q_2(z)|$ for $|z| = 1$ . Hence by lemma 2.1 we have, for $\gamma$ with $|\gamma| \leq 1$ ,

$$|zP_1'(z) + n\left(\frac{\gamma}{2}\right) P_1(z)| \leq |zQ_2'(z) + n\left(\frac{\gamma}{2}\right) Q_2(z)|.$$

$\Rightarrow$

$$\left|\frac{zP'(z) + n\left(\frac{\gamma}{2}\right) P(z)}{\alpha} - n\beta\left(\frac{\gamma}{2}\right)\right| \leq \left|\frac{zQ'(z) + n\left(\frac{\gamma}{2}\right) Q(z)}{\overline{\beta}} - n\overline{\alpha}z^n\left(1+\frac{\gamma}{2}\right)\right|.$$

$\Rightarrow \left|zP'(z) + n\left(\frac{\gamma}{2}\right) P(z) - n\alpha\beta\left(\frac{\gamma}{2}\right)\right|$

$$\leq \left|\frac{\alpha(zQ'(z) + n\left(\frac{\gamma}{2}\right) Q(z))}{\overline{\beta}} - n|\alpha|^2 z^n\left(1+\frac{\gamma}{2}\right)\right|.$$

Without loss of generality, we can choose the arguments of $\alpha$, $\beta$, and $\frac{\alpha}{\beta}$, such that,

$$\left|zP'(z)+n\left(\tfrac{\gamma}{2}\right)P(z)\right|+n\left|\alpha\beta\left(\tfrac{\gamma}{2}\right)\right| \leq \left|\left|\frac{\alpha(zQ'(z)+n\left(\frac{\gamma}{2}\right)Q(z))}{\overline{\beta}}\right|-n|\alpha|^2|z|^n\left|1+\frac{\gamma}{2}\right|\right|. \quad (10)$$

Applying lemma 2.1 to the polynomials $\frac{\alpha}{\beta}Q(z)$ and $|\alpha|^2z^n$ , we get,

$$\left|\frac{\alpha(zQ'(z)+n\left(\frac{\gamma}{2}\right)Q(z))}{\beta}\right| \leq n|\alpha|^2\left|1+\frac{\gamma}{2}\right|$$

for $|z|=1$. Hence the inequality (10) becomes,

$$\left|zP'(z)+n\left(\frac{\gamma}{2}\right)P(z)\right|+n\left|\alpha\beta\left(\frac{\gamma}{2}\right)\right| \leq n|\alpha|^2\left|1+\frac{\gamma}{2}\right|-\left|\frac{\alpha(zQ'(z)+n\left(\frac{\gamma}{2}\right)Q(z))}{\beta}\right|.$$

As $|\alpha|\to 1$ and $|\beta|\to 1$, we get the desired result.

**Lemma 2.3** *: If $P(z)$ is a polynomial of degree $n$ with all its zeros in $|z|\leq 1$, then for any $\gamma$ with $|\gamma|\leq 1$ and $|z|=1$,*

$$|zP'(z)+n\left(\frac{\gamma}{2}\right)P(z)| \geq mn\left|1+\frac{\gamma}{2}\right|.$$

Proof: Applying lemma 2.1 to the polynomials $P(z)$ and $mz^n$, we can easily get the above inequality.

# 3 Proof of the Theorem 1.1

First we prove the inequality (7). If $P(z)$ has a zero on $|z|=1$, then the theorem can be proved directly by lemma 2.1 and lemma 2.2. Therefore assume that $P(z)$ has all its zeros in $|z|>1$. Now for any $\alpha$ and $\beta$ with $0<|\alpha|=|\beta|<1$, we have, $|\alpha\beta|m<m\leq|P(z)|$ for $|z|=1$. Hence by Rouche's theorem, the polynomial $P(z)-\alpha\beta m$ has no zeros in $|z|<1$. But then $P_1(z)=\frac{P(z)}{\alpha}-\beta m$ does not vanish in $|z|<1$. Hence the polynomial $Q_1(z)=z^n\overline{P_1(\frac{1}{\overline{z}})}=\frac{Q(z)}{\overline{\alpha}}-\overline{\beta}z^n m$ has all its zeros in $|z|\leq 1$, by which we can conclude that the polynomial $Q_2(z)=\frac{Q(z)}{\overline{\beta}}-\overline{\alpha}mz^n$ has all its zeros in $|z|\leq 1$. Also $|P_1(z)|=|Q_2(z)|$ for $|z|=1$. Hence by lemma 2.1, we get,

$$|zP_1'(z)+n\left(\frac{\gamma}{2}\right)P_1(z)| \leq |zQ_2'(z)+n\left(\frac{\gamma}{2}\right)Q_2(z)|$$

for $|z| = 1$, by which it follows that,

$$\left|\frac{zP'(z) + n\left(\frac{\gamma}{2}\right)P(z)}{\alpha} - mn\beta\left(\frac{\gamma}{2}\right)\right| \leq \left|\frac{zQ'(z) + n\left(\frac{\gamma}{2}\right)Q(z)}{\overline{\beta}} - mn\overline{\alpha}z^n\left(1 + \frac{\gamma}{2}\right)\right|$$

$$\Rightarrow \left|zP'(z) + n\left(\tfrac{\gamma}{2}\right)P(z) - mn\alpha\beta\left(\tfrac{\gamma}{2}\right)\right|$$

$$\leq \left|\frac{\alpha(zQ'(z) + n\left(\frac{\gamma}{2}\right)Q(z))}{\overline{\beta}} - mn|\alpha|^2 z^n\left(1 + \frac{\gamma}{2}\right)\right|.$$

Now, we can choose the arguments of $\alpha$, $\beta$, and $\frac{\alpha}{\beta}$, such that

$$\left|zP'(z) + n\left(\tfrac{\gamma}{2}\right)P(z)\right| + mn\left|\alpha\beta\left(\tfrac{\gamma}{2}\right)\right|$$

$$\leq \left|\left|\frac{\alpha(zQ'(z) + n\left(\frac{\gamma}{2}\right)Q(z))}{\overline{\beta}}\right| - mn|\alpha|^2|z|^n\left|1 + \frac{\gamma}{2}\right|\right|. \tag{11}$$

Since the polynomial $Q(z)$ has all its zeros in $|z| \leq 1$ and

$$\min_{|z|=1}|P(z)| = m = \min_{|z|=1}|Q(z)|,$$

by lemma 2.3 we have,

$$mn\left|1 + \frac{\gamma}{2}\right| \leq \left|zQ'(z) + n\left(\frac{\gamma}{2}\right)Q(z)\right|, \quad |\gamma| \leq 1, \quad |z| = 1.$$

Hence the inequality (11) becomes,

$$\left|zP'(z) + n\left(\tfrac{\gamma}{2}\right)P(z)\right| + mn\left|\alpha\beta\left(\tfrac{\gamma}{2}\right)\right|$$

$$\leq \left|\frac{\alpha(zQ'(z) + n\left(\frac{\gamma}{2}\right)Q(z))}{\beta}\right| - mn|\alpha|^2\left|1 + \frac{\gamma}{2}\right|.$$

As $|\alpha| \to 1$, $|\beta| \to 1$, we get,

$$\left|zP'(z) + n\left(\tfrac{\gamma}{2}\right)P(z)\right| - \left|zQ'(z) + n\left(\tfrac{\gamma}{2}\right)Q(z)\right|$$

$$\leq -mn\left(\left|1 + \frac{\gamma}{2}\right| + \left|\frac{\gamma}{2}\right|\right), \quad |z| = 1. \tag{12}$$

But by lemma 2.2 we have,

$$\left|zP'(z) + n\left(\tfrac{\gamma}{2}\right)P(z)\right| + \left|zQ'(z) + n\left(\tfrac{\gamma}{2}\right)Q(z)\right|$$

$$\leq Mn\left(\left|1 + \frac{\gamma}{2}\right| - \left|\frac{\gamma}{2}\right|\right), \quad |z| = 1. \tag{13}$$

Addition of (12) and (13) gives the desired inequality (7).

Now we prove the second part. By the first part we have,

$$|zP'(z)+n\left(\frac{\gamma}{2}\right)P(z)|\le\frac{n}{2}\left[\left(|1+\frac{\gamma}{2}|-|\frac{\gamma}{2}|\right)M-\left(|1+\frac{\gamma}{2}|+|\frac{\gamma}{2}|\right)m\right]$$

for $|z|=1$. Since $P'(z)$ is a polynomial of degree $n-1$ it follows that for all $r\ge 1$ and $0\le\theta<2\pi$,

$$|re^{i\theta}P'(re^{i\theta})+n\left(\frac{\gamma}{2}\right)P(re^{i\theta})|\le r^n\frac{n}{2}\left[\left(|1+\frac{\gamma}{2}|-|\frac{\gamma}{2}|\right)M-\left(|1+\frac{\gamma}{2}|+|\frac{\gamma}{2}|\right)m\right].$$

Also for each $\theta,\ \ 0\le\theta<2\pi$ and $K>1$, we have,

$$KP(Ke^{i\theta})-P(e^{i\theta})-(1-n\frac{\gamma}{2})\int_1^K P(te^{i\theta})dt=\int_1^R[tP'(te^{i\theta})+\frac{n\gamma}{2}P(te^{i\theta})]dt.$$

Then,

$$\left|KP(Ke^{i\theta})-P(e^{i\theta})-(1-n\frac{\gamma}{2})\int_1^R P(te^{i\theta})dt\right|\le\int_1^R\left|[tP'(te^{i\theta})+\frac{n\gamma}{2}P(te^{i\theta})]\right|dt$$

$$\le\frac{n}{2}\left[\left(|1+\frac{\gamma}{2}|-|\frac{\gamma}{2}|\right)M-\left(|1+\frac{\gamma}{2}|+|\frac{\gamma}{2}|\right)m\right]\int_1^K t^n dt$$

$$\le\frac{n}{2}\left[\left(|1+\frac{\gamma}{2}|-|\frac{\gamma}{2}|\right)M-\left(|1+\frac{\gamma}{2}|+|\frac{\gamma}{2}|\right)m\right]\left[\frac{(K^{n+1}-1)}{n+1}\right].$$

But then $|KP(Re^{i\theta})|\le|P(e^{i\theta})|+\left|(1-n\frac{\gamma}{2})\int_1^R P(te^{i\theta})dt\right|$

$$+\frac{n}{2}\left[\left(|1+\frac{\gamma}{2}|-|\frac{\gamma}{2}|\right)M-\left(|1+\frac{\gamma}{2}|+|\frac{\gamma}{2}|\right)m\right]\left[\frac{(K^{n+1}-1)}{n+1}\right]$$

which implies $|KP(Re^{i\theta})|\le m+\left|(1-n\frac{\gamma}{2})\right|\int_1^R|P(te^{i\theta})|dt$

$$+\frac{n}{2}\left[\left(|1+\frac{\gamma}{2}|-|\frac{\gamma}{2}|\right)M-\left(|1+\frac{\gamma}{2}|+|\frac{\gamma}{2}|\right)m\right]\left[\frac{(K^{n+1}-1)}{n+1}\right].$$

But from (2), it again follows that,

$$|KP(Ke^{i\theta})|\le m+\left|(1-n\frac{\gamma}{2})\right|M\left[\frac{(K^{n+1}-1)}{n+1}\right]$$

$$+\frac{n}{2}\left[\left(|1+\frac{\gamma}{2}|-|\frac{\gamma}{2}|\right)M-\left(|1+\frac{\gamma}{2}|+|\frac{\gamma}{2}|\right)m\right]\left[\frac{(K^{n+1}-1)}{n+1}\right].$$

With the simplification, we can easily arrive at the inequality (8).

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