Int. Journal of Math. Analysis, Vol. 4, 2010, no. 40, 1995 - 2004

Generalized Fixed Point Theorem in Three Metric Spaces

Kristaq Kikina and Luljeta Kikina

Department of Mathematics and Computer Sciences
Faculty of Natural Sciences, University of Gjirokastra, Albania
kristaqkikina@yahoo.com, gjonileta@yahoo.com

Abstract

We prove a related fixed point theorem for three mappings in three metric spaces using an implicit relation. This result generalizes and unifies several of well-known fixed point theorems in complete metric spaces.

Keywords: Cauchy sequence, complete metric space, fixed point, implicit relation.

Mathematics Subject Classification: 47H10, 54H25

1. Introduction

In [8] and [4] the following theorems are proved:

Theorem 1.1 $(Nung)$ [8] *Let* $(X,d),(Y,\rho)$ *and* (Z,σ) *be complete metric spaces and suppose* T *is a continuous mapping of* X *into* Y, S *is a continuous mapping of* Y *into* Z *and* R *is a continuous mapping of* Z *into* X *satisfying the inequalities*

$$d(RSTx, RSy) \le c\max\{d(x,RSy), d(x,RSTx), \rho(y,Tx), \sigma(Sy,STx)\}$$

$$\rho(TRSy, TRz) \le c\max\{\rho(y,TRz), \rho(y,TRSy), \sigma(z,Sy), d(Rz,RSy)\}$$

$$\sigma(STRz, STx) \le c\max\{\sigma(z,STx), \sigma(z,STRz), d(x,Rz), \rho(Tx,TRz)\}$$

for all x *in* X, y *in* Y *and* z *in* Z, *where* $0 \le c < 1$. *Then* RST *has a unique fixed* *point u in X, TRS has a unique fixed point v in Y and STR has a unique fixed point w in Z. Further, $Tu = v, Sv = w$ and $Rw = u$.*

Theorem 1.2 (*Jain et.al.*)[7] *Let $(X,d),(Y,\rho)$ and (Z,σ) be complete metric spaces and suppose T is a mapping of X into Y, S is a mapping of Y into Z and R is a mapping of Z into X satisfying the inequalities*

$$\begin{aligned}
d^2(RSy,RSTx) \le c\max\ &\{d(x,RSy)\rho(y,Tx),\rho(y,Tx)d(x,RSTx),\\
&d(x,RSTx)\sigma(Sy,STx),\sigma(Sy,STx)d(x,RSy)\}\\
\rho^2(TRz,TRSy) \le c\max\ &\{\rho(y,TRz)\sigma(z,Sy),\sigma(z,Sy)\rho(y,TRSy),\\
&\rho(y,TRSy)d(Rz,RSy),d(Rz,RSy)\rho(y,TRz)\}\\
\sigma^2(STx,STRz) \le c\max\ &\{\sigma(z,STx)d(x,Rz),d(x,Rz)\sigma(z,STRz),\\
&\sigma(z,STRz)\rho(Tx,TRz),\rho(Tx,TRz)\sigma(z,STx)\}
\end{aligned}$$

for all x in X, y in Y and z in Z, where $0 \le c < 1$. If one of the mappings R,S,T is continuous, then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y and STR has a unique fixed point w in Z. Further, $Tu = v, Sv = w$ and $Rw = u$.

2. Main results

This result generalizes and unifies several well-known fixed point theorems obtained in [8,4,10,9]. For this, we will use the implicit relations.

Let $\Phi_k^{(m)}$ be the set of continuous functions with k variables

$$\varphi:[0,+\infty)^k \to [0,+\infty)$$

satisfying the properties:

a. φ is non decreasing in respect with each variable $t_1,t_2,...,t_k$

b. $\varphi(t,t,...,t) \le t^m, m \in N$.

Denote $I_k = \{1,2,...,k\}$.For $k_1 < k_2$ we have $\Phi_{k_1}^{(m)} \in \Phi_{k_2}^{(m)}$.

For $k = 6$ we can give these examples:

Example 2.1 $\varphi(t_1,t_2,t_3,t_4,t_5,t_6) = \max\{t_1,t_2,t_3,t_4,t_5,t_6\}$, *with* $m = 1$.

Example 2.2 $\varphi(t_1,t_2,t_3,t_4,t_5,t_6) = \max\{t_it_j : i, j \in I_6\}$, *with* $m = 2$.

Example 2.3 $\varphi(t_1,t_2,t_3,t_4,t_5,t_6) = \max\{t_1^p,t_2^p,t_3^p,t_4^p,t_5^p,t_6^p\}$, *with* $m = p$.

Example 2.4 $\varphi(t_1,t_2,t_3,t_4,t_5,t_6) = \max\{t_1^p,t_2^p,t_3^p,t_4^p\}$, *with* $m = p$, *etc.*

Example 2.5 $\varphi(t_1,t_2,t_3,t_4,t_5,t_6) = \dfrac{t_1+t_2+t_3}{3}$ *or* $\dfrac{t_1+t_2}{2}$ *with* $m=1$, *etc.*

Theorem 2.6. *Let* $(X,d),(Y,\rho)$ *and* (Z,σ) *be complete metric spaces and suppose* T *is a mapping of* X *into* Y, S *is a mapping of* Y *into* Z *and* R *is a mapping of* Z *into* X, *such that at least one of them is a continuous mapping. Let* $\varphi_i \in \Phi_6^{(m)}$ *for* $i=1,2,3$. *If there exists* $q\in[0,1)$ *and the following inequalities hold*

$$(1)\qquad d^m(RSy,RSTx) \le q\varphi_1(d(x,RSy),d(x,RSTx),\rho(y,Tx),\rho(y,TRSy),\rho(Tx,TRSy,\sigma(Sy,STx))$$

$$(2)\qquad \rho^m(TRz,TRSy) \le q\varphi_2(\rho(y,TRz),\rho(y,TRSy),\sigma(z,Sy),\sigma(z,STRz),\sigma(Sy,STRz),d(Rz,RSy))$$

$$(3)\qquad \sigma^m(STx,STRz) \le q\varphi_3(\sigma(z,STx),\sigma(z,STRz),d(x,Rz),d(x,RSTx),d(Rz,RSTx),\rho(Tx,TRz))$$

for all $x\in X, y\in Y$ *and* $z\in Z$, *then* RST *has a unique fixed point* $\alpha\in X$, TRS *has a unique fixed point* $\beta\in Y$ *and* STR *has a unique fixed point* $\gamma\in Z$. *Further,* $T\alpha=\beta, S\beta=\gamma$ *and* $R\gamma=\alpha$.

Proof. Let $x_0\in X$ be an arbitrary point. We define the sequences $(x_n),(y_n)$ and (z_n) in X,Y and Z respectively as follows:

$$x_n=(RST)^n x_0,\, y_n = Tx_{n-1},\, z_n = Sy_n,\, n=1,2,...$$

Denote

$$d_n = d(x_n,x_{n+1}),\rho_n=\rho(y_n,y_{n+1}),\sigma_n=\sigma(z_n,z_{n+1}),n=1,2,...$$

We will assume that $x_n\ne x_{n+1}$, $y_n\ne y_{n+1}$ *and* $z_n \ne z_{n+1}$ for all *n*, otherwise if $x_n = x_{n+1}$ for some n, then $y_{n+1}=y_{n+2}, z_{n+1}=z_{n+2}$ we can take $\alpha = x_{n+1},\beta=y_{n+1},\gamma=z_{n+1}$.

By the inequality (2), for $y=y_n$ and $z=z_{n-1}$ we get:

$$\rho^m(y_n,y_{n+1}) \le q\varphi_2(\rho(y_n,y_n),\rho(y_n,y_{n+1}),\sigma(z_{n-1},z_n),\sigma(z_{n-1},z_n),\sigma(z_n,z_n),\ d(x_{n-1},x_n))$$

or

$$\rho_n^m \le q\varphi_2(0,\rho_n,\sigma_{n-1},\sigma_{n-1},0,d_{n-1}) \qquad (4)$$

For the coordinates of the point $(0,\rho_n,\sigma_{n-1},\sigma_{n-1,}0,d_{n-1})$ we have:

$$\rho_n \le \max\{d_{n-1},\sigma_{n-1}\} = \lambda, \forall n\in N \qquad (5)$$

because, in case that $\rho_n > \max\{d_{n-1},\sigma_{n-1}\}$ for some n, if we replace the coordinates with ρ_n and apply the property (b) of φ_2 we get:

$$\rho_n^m \le q\varphi_2(\rho_n,\rho_n,\rho_n,\rho_n,\rho_n,\rho_n) \le q\rho_n^m$$

This is impossible since $0 \le q < 1$.

By the inequalities (4), (5) and properties of φ_2 we get:

$$\rho_n^m \le q\varphi_2(\lambda,\lambda,\lambda,\lambda,\lambda,\lambda\} \le q\lambda^m = q\left[\max\left\{d_{n-1},\sigma_{n-1}\right\}\right]^m .$$

Thus

$$\rho_n \le \sqrt[m]{q}\lambda = \sqrt[m]{q}\max\left\{d_{n-1},\sigma_{n-1}\right\} \qquad (6)$$

By the inequality (3), for $x = x_{n-1}$ and $z = z_n$ we get:

$$\sigma^m(z_n, z_{n+1}) \le q\varphi_3(\sigma(z_n, z_n), \sigma(z_n, z_{n+1}), d(x_{n-1}, x_n),$$
$$d(x_{n-1}, x_n), d(x_n, x_n, \rho(y_n, y_{n+1}))$$

or

$$\sigma_n^m \le q\varphi_3(0,\sigma_n, d_{n-1}, d_{n-1}, 0, \rho_n) \qquad (7)$$

In similar way, we get:

$$\sigma_n \le \sqrt[m]{q}\max\{d_{n-1},\rho_n\}, \forall n \in N .$$

By this inequality and (6) we get:

$$\sigma_n \le \sqrt[m]{q}\max\{d_{n-1},\sigma_{n-1}\}, \forall n \in N \qquad (8)$$

By (1) for $x = x_n$ and $y = y_n$ we get:

$$d^m(x_n, x_{n+1}) \le q\varphi_1(d(x_n, x_n), d(x_n, x_{n+1}), \rho(y_n, y_{n+1}),$$
$$\rho(y_n, y_{n+1}, \rho(y_{n+1}, y_{n+1}, \sigma(z_n, z_{n+1}))$$

or

$$d_n^m \le q\varphi_1(0, d_n, \rho_n, \rho_n, 0, \sigma_n) \qquad (9)$$

In similar way, we get:

$$d_n \le \sqrt[m]{q}\max\{\rho_n,\sigma_n\}, \forall n \in N .$$

By this inequality and the inequalities (6), (8) we get:

$$d_n \le \sqrt[m]{q}\max\{\rho_n,\sigma_n\} \le \sqrt[m]{q}(\sqrt[m]{q}\max\{d_{n-1},\sigma_{n-1}\}) =$$
$$= \sqrt[m]{q}(\sqrt[m]{q})\max\{d_{n-1},\sigma_{n-1}\} \le \sqrt[m]{q}\max\{d_{n-1},\sigma_{n-1}\}$$

or

$$d_n \le \sqrt[m]{q}\max\{d_{n-1},\sigma_{n-1}\} \qquad (10)$$

By the inequalities (6), (8) and (10), using the mathematical induction, we get:

$$d(x_n, x_{n+1}) \le r^{n-1}\max\{d(x_1, x_2), \sigma(z_1, z_2)\}$$
$$\rho(y_n, y_{n+1}) \le r^{n-1}\max\{d(x_1, x_2), \sigma(z_1, z_2)\}$$
$$\sigma(z_n, z_{n+1}) \le r^{n-1}\max\{d(x_1, x_2), \sigma(z_1, z_2)\}$$

where $\sqrt[m]{q} = r < 1$.

Thus the sequences $(x_n),(y_n)$ and (z_n) are Cauchy sequences. Since the metric spaces $(X,d),(Y,\rho)$ and (Z,σ) are complete metric spaces, we have:

$$\lim_{n\to\infty} x_n = \alpha \in X, \lim_{n\to\infty} y_n = \beta \in Y, \lim_{n\to\infty} z_n = \gamma \in Z .$$

Assume that T is a continuous mapping. Then by

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} y_n .$$

it follows

$$T\alpha = \beta. \qquad (11)$$

By (1), for $y = T\alpha$ and $x = x_n$ we get:

$$\begin{aligned} d^m(RST\alpha, x_{n+1}) \le q\varphi_1(&d(x_n, RST\alpha), d(x_n, x_{n+1}), \rho(T\alpha, y_{n+1}), \\ &\rho(T\alpha, TRST\alpha), \rho(y_{n+1}, TRST\alpha), \sigma(ST\alpha, z_{n+1})) \end{aligned}$$

By this inequality and (11), letting n tend to infinity, we get:

$$\begin{aligned} d^m(RST\alpha,\alpha) &\le q\varphi_1(d(\alpha,RST\alpha),0,0,\rho(\beta,TRS\beta),\rho(\beta,TRS\beta),\sigma(S\beta,\gamma)) \\ &\le q[\max\{\rho(\beta,TRS\beta),\sigma(S\beta,\gamma)\}]^m \end{aligned} \qquad (12)$$

By (2), for $z = z_n$ and $y = \beta$ we get:

$$\begin{aligned} \rho^m(y_{n+1}, TRS\beta) \le q\varphi_2(&\rho(\beta, y_{n+1}), \rho(\beta, TRS\beta), \sigma(z_n, S\beta), \\ &\sigma(z_n, z_{n+1}), \sigma(S\beta, z_{n+1}), d(x_n, RS\beta)) \end{aligned}$$

By this inequality letting n tend to infinity and using (11), we get:

$$\begin{aligned} \rho^m(\beta,TRS\beta) &\le q\varphi_2(0,\rho(\beta,TRS\beta),\sigma(\gamma,S\beta),0,\sigma(S\beta,\gamma),d(\alpha,RST\alpha)) \\ &\le q[\max\{d(\alpha,RST\alpha),\sigma(\gamma,S\beta)\}]^m \end{aligned} . \qquad (13)$$

By (3) for $x = \alpha$, $z = z_n$ we get:

$$\begin{aligned} \sigma^m(ST\alpha, z_{n+1}) \le q\varphi_3(&\sigma(z_n, ST\alpha), \sigma(z_n, z_{n+1}), d(\alpha, x_n), \\ &d(\alpha, RST\alpha), d(x_n, RST\alpha), \rho(T\alpha, y_{n+1})) \end{aligned}$$

By this inequality letting n tend to infinity and using (11), we have:

$$\begin{aligned} \sigma^m(S\beta,\gamma) &\le q\varphi_3(\sigma(\gamma,S\beta),0,0,d(\alpha,RST\alpha),\ d(\alpha,RST\alpha),0,) \\ &\le qd^m(\alpha,RST\alpha) \end{aligned} \qquad (14)$$

By the inequalities (12),(13) and (14) we get:

$$d^m(RST\alpha,\alpha) \le q^2 d^m(RST\alpha,\alpha)$$

Thus

$$d(RST\alpha,\alpha) = 0 \ \ or\ RST\alpha = \alpha \qquad (15)$$

By (14) and (13) we obtain

$$S\beta = \gamma$$
$$TRS\beta = \beta$$
$$STR\gamma = STR(S\beta) = S(TRS\beta) = S\beta = \gamma$$

Thus, we proved that the points α, β *and* γ are fixed points of RST, TRS and STR respectively.

In the same conclusion we would arrive if one of the mappings R or T would be continuous.

We now prove the uniqueness of the fixed points α, β and γ. Let us prove for α. Assume that there is α' a fixed point of RST different from α.

By (1) for $x = \alpha'$ and $y = T\alpha$ we get:

$$\begin{aligned} d^m(\alpha,\alpha') &= d^m(RST\alpha, RST\alpha') \le \\ &\le q\varphi_1(d(\alpha', RST\alpha), d(\alpha', RST\alpha'), \rho(T\alpha, T\alpha'), \\ &\quad \rho(T\alpha, TRST\alpha), \rho(T\alpha', TRST\alpha, \sigma(ST\alpha, ST\alpha')) = \\ &= q\varphi_1(d(\alpha',\alpha), 0, \rho(T\alpha, T\alpha'), 0, \rho(T\alpha', T\alpha), \sigma(ST\alpha, ST\alpha') \\ &\le q[\max\{d(\alpha',\alpha), \rho(T\alpha', T\alpha), \sigma(ST\alpha, ST\alpha')\}]^m \end{aligned}$$

or

$$d^m(\alpha,\alpha') = q(\max A)^m \tag{14}$$

where $A = \{d(\alpha',\alpha); \rho(T\alpha, T\alpha'); \sigma(ST\alpha, ST\alpha')\}$.

We distinguish the following three cases:

Case I: If $\max A = d(\alpha',\alpha)$, then the inequality (14) implies

$$d^m(\alpha,\alpha') \le q d^m(\alpha',\alpha) \Leftrightarrow \alpha' = \alpha .$$

Case II: If $\max A = \rho(T\alpha, T\alpha')$, then the inequality (14) implies

$$d^m(\alpha,\alpha') \le q\rho^m(T\alpha, T\alpha') \quad (15)$$

Continuing our argumentation for the Case 2, by (2) for $z = ST\alpha$ and $y = T\alpha'$ we have:

$$\begin{aligned} \rho^m(T\alpha, T\alpha') &= \rho^m(TRST\alpha, TRST\alpha') \le \\ &\le q\varphi_2(\rho(T\alpha', TRST\alpha), \rho(T\alpha', TRST\alpha'), \sigma(ST\alpha, ST\alpha'), \\ &\quad \sigma(ST\alpha, STRST\alpha), \sigma(ST\alpha', STRST\alpha), d(RST\alpha', RST\alpha)) = \\ &= q\varphi_2(\rho(T\alpha', T\alpha), 0, \sigma(ST\alpha, ST\alpha'), 0, \sigma(ST\alpha', ST\alpha), d(\alpha,\alpha')) = \\ &\le q(\max A)^m \end{aligned} \tag{16}$$

Since in Case II, $\max A = \rho(T\alpha, T\alpha')$, by (16) it follows

$$\rho^m(T\alpha, T\alpha') \le q\rho^m(T\alpha, T\alpha')$$

or

$$\rho(T\alpha, T\alpha') = 0 .$$

By (15), it follows $d(\alpha,\alpha')=0$.

Case III: If $\max A=\sigma(ST\alpha,ST\alpha')$, then by (14) it follows

$$d^m(\alpha,\alpha')\le q\sigma^m(ST\alpha,ST\alpha') \quad (17)$$

By the inequality (3), for $x=RST\alpha, z=ST\alpha'$, in similar way we obtain:

$$\sigma^m(ST\alpha,ST\alpha')\le q(\max A)^m=q\sigma^m(ST\alpha,ST\alpha')$$

It follows

$$\sigma(ST\alpha,ST\alpha')=0$$

and by (17) it follows

$$d(\alpha,\alpha')=0.$$

Thus, we have again $\alpha=\alpha'$.

In the same way, it is proved the uniqueness of β and γ.

Example 2.7. Let $X=[0,1], Y=[1,2], Z=[1,2]$ *and* $d=\rho=\sigma$ is the usual metric for the real numbers. Define*:*

$$Tx=\begin{cases}\frac{5}{4} & if \ \ 0\le x<\frac{1}{2}\\ \frac{3}{2} & if \ \ \frac{1}{2}\le x\le 1\end{cases} \qquad Sy=\frac{3}{2} \qquad Rz=\begin{cases}\frac{3}{4} & if \ \ 1\le z<\frac{5}{4}\\ 1 & if \ \ \frac{5}{4}\le z\le 2\end{cases}$$

Then S is continuous but T and R are not continuous.

We have

$$STx=\frac{3}{2}, RSy=1, TRz=\frac{3}{2}$$

$$RSTx=1, TRSy=\frac{3}{2}, STRz=\frac{3}{2}$$

and

$$\text{RST1=1, TRS}\frac{3}{2}=\frac{3}{2}\text{, STR}\frac{3}{2}=\frac{3}{2}\text{ and T1=}\frac{3}{2}\text{, S}\frac{3}{2}=\frac{3}{2}\text{, R}\frac{3}{2}=1$$

These inequalities (1), (2) and (3) are satisfies since the value of left hand side of each inequality is 0.

Hence all the conditions of Theorem 2.6 are satisfies

We emphasize the fact that it is necessary the continuity of at least one of the mappings T,S and R. The following example shows this.

Example 2.8. Let $X=Y=Z=[0,1]; d=\rho=\sigma$ such that $d(x,y)=|x-y|, \forall x,y\in[0,1]$. We consider the mappings $T,S,R:[0,1]\to[0,1]$ such that

$$Tx = Rx = Sx = \begin{cases} 1 & for \quad x = 0 \\ \dfrac{x}{2} & for \quad x \in (0,1] \end{cases}$$

We have

$$STx = RSx = TRx = \begin{cases} \dfrac{1}{2} & for \; x = 0 \\ \dfrac{x}{4} & for \; x \in (0,1] \end{cases}$$

and

$$RSTx = TRSx = STRx = \begin{cases} \dfrac{1}{4} & for \; x = 0 \\ \dfrac{x}{8} & for \quad x \in (0,1] \end{cases}$$

We observe that the inequalities (1), (2) and (3) are satisfied for $\varphi_1 = \varphi_2 = \varphi_3 = \varphi \in \Phi_6^{(1)}$ with $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = \max\{t_1, t_2, t_3, t_4, t_5, t_6\}$ and $q = \dfrac{1}{2}$. It can be seen that none of the mappings RST, TRS, STR has a fixed point. This is because none of the mappings T, R, S is a continuous mapping.

3. Corollaries

Corollary 3.1 *Let $(X, d), (Y, \rho)$ and (Z, σ) be complete metric spaces and suppose T is a mapping of X into Y, S is a mapping of Y into Z and R is a mapping of Z into X, such that at least one of them is a continuous mapping. If there exists $q \in [0,1)$ and $m \in N$ such that the following inequalities hold*

$$d^m(RSy, RSTx) \le q \max\{(d^m(x, RSy), d^m(x, RSTx), \rho^m(y, Tx),$$
$$\rho^m(y, TRSy), \rho^m(Tx, TRSy, \sigma^m(Sy, STx)\}$$
$$\rho^m(TRz, TRSy) \le q \max(\rho^m(y, TRz), \rho^m(y, TRSy), \sigma^m(z, Sy),$$
$$\sigma^m(z, STRz), \sigma^m(Sy, STRz), d^m(Rz, RSy)\}$$
$$\sigma^m(STx, STRz) \le q \max(\sigma^m(z, STx), \sigma^m(z, STRz), d^m(x, Rz),$$
$$d^m(x, RSTx), d^m(Rz, RSTx), \rho^m(Tx, TRz)\}$$

for all $x \in X, y \in Y$ and $z \in Z$, then RST has a unique fixed point $\alpha \in X$, TRS has a unique fixed point $\beta \in Y$ and STR has a unique fixed point $\gamma \in Z$. Further, $T\alpha = \beta, S\beta = \gamma$ and $R\gamma = \alpha$.

Proof. The proof follows by Theorem 2.6 in the case $\varphi_1 = \varphi_2 = \varphi_3 = \varphi \in \Phi_6^{(m)}$ such that $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = \max\{t_1^m, t_2^m, t_3^m, t_4^m, t_5^m, t_6^m\}$.

Corollary 3.2 *Theorem 1.1 (Nung [8]) is taken by theorem 2.6 for* $m = 1$ *and* $\varphi_1 = \varphi_2 = \varphi_3 = \varphi$ *such* that $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = \max\{t_1, t_2, t_3, t_6\}$

Corollary 3.3 *Theorem 1.2 (Jain et. al. [4]) is taken by Theorem 2.6 in case* $\varphi_1 = \varphi_2 = \varphi_3 = \varphi \in \Phi_6^{(2)}$ *such that* $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = \max\{t_1t_3, t_2t_3, t_2t_6, t_6t_1\}$.

Corollary 3.4 Theorem *Telci (Theorem 2 [10]).Let* $(X, d), (Y, \rho)$ *be complete metric spaces and* T *is a mapping of* X *into* Y, S *is a mapping of* Y *into X.* $\varphi_i \in \Phi_3^m$ *for* $i = 1, 2$. *If there exists* $q \in [0,1)$ *such that the following inequalities hold*

$$(1') \qquad d(Sy, STx) \le q\varphi_1(d(x, Sy), d(x, STx), \rho(y, Tx)).$$

$$(2') \qquad \rho(Tx, TSy) \le q\varphi_2(\rho(y, Tx), \rho(y, TSy), d(x, Sy).$$

for all $x \in X, y \in Y$, *then* ST *has a unique fixed point* $\alpha \in X$ *and* TS *has a unique fixed point* $\beta \in Y$. *Further ,*$T\alpha = \beta, S\beta = \gamma$.

Proof. The proof follows by Theorem 2.6 in the case $Z = X, \sigma = d$, $m = 1$ and the mapping R as the identity mapping in X. Then the inequality (1) takes the form $(1')$, the inequality (2) takes the form $(2')$ and the inequality (3) is always satisfied since his left side is $\sigma(STx, STx) = 0$. Thus, the satisfying of the conditions (1), (2) and (3) is reduced in satisfying of the conditions $(1')$, $(2')$.

The mappings T and S may be not continuous, while from the mappings T, S and R for which we applied Theorem 2.6, the identity mapping R is continuous. This completes the proof.

We have the following corollary.

Corollary 3.5 *Theorem Popa (Theorem 2, [9]) is taken by Corollary 3.4 for* $\varphi_1 = \varphi_2 = \varphi$ *such that* $\varphi(t_1, t_2, t_3) = \max\{t_1t_2, t_1t_3, t_2t_3\}$ *with* $m = 2$.

We also emphasize here that the constants c_1, c_2 can be replaced by $q = \max\{c_1, c_2\}$.

Conclusions.

In this paper it has been proved a related fixed point theorem for three mappings in three metric spaces, one of mappings is continuous. This theorem generalizes and unifies several of well-known fixed point theorems for contractive-type mappings on metric spaces, for example the theorems of Nung [8], Jain et.al [4], Popa [9], Telci [10] and the theorem of Fisher [1]. As corollaries of main result we can obtain other propositions determined by the form of implicit relations.

References

[1] B. Fisher, *Fixed point in two metric spaces,* Glasnik Matem. **16(36)** (1981), 333-337.

[2] R. K. Jain, H. K. Sahu, B. Fisher, *A related fixed point theorem on three metric spaces*, Kyungpook Math. J. **36** (1996), 151-154.

[3] R. K. Jain, H. K. Sahu, B. Fisher, *Related fixed point theorems to three metric spaces*, Novi Sad J. Math., Vol. **26**, No. 1, (1996), 11-17.

[4] R. K. Jain, A. K. Shrivastava, B. Fisher, *Fixed points on three complete metric spaces*, Novi Sad J. Math. Vol. **27**, No. 1 (1997), 27-35.

[5] L. Kikina, *Fixed points theorems in three metric spaces*, Int. Journal of Math. Analysis, Vol. **3**, 2009, No. 13-16, 619-626.

[6] S. Č. Nešic′, *A fixed point theorem in two metric spaces*, Bull. Math. Soc. Sci. Math. Roumanie, Tome **44(94)** (2001), No.3, 253-257.

[7] S. Č. Nešic′, *Common fixed point theorems in metric spaces*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.), **46(94)** (200 3), No.3-4, 149-155.

[8] N. P. Nung, *A fixed point theorem in three metric spaces*, Math. Sem. Notes, Kobe Univ. **11** (1983), 77-79.

[9] V. Popa, *Fixed points on two complete metric spaces*, Zb. Rad. Prirod.-Mat. Fak. (N.S.) Ser. Mat. **21(1)** (1991), 83-93.

[10] M. Telci, *Fixed points on two complete and compact metric spaces*, Applied Mathematics and Mechanices 22(5) (2001), 564 -568.

Received: April, 2010