Int. Journal of Math. Analysis, Vol. 4, 2010, no. 37, 1851 - 1856

Weakly Compact Composition Operators on Hardy Spaces of the Upper Half-Plane[1]

Hong Bin Bai

School of Science
Sichuan University of Science and Engineering
Zigong, Sichuan, 643000, P.R. China
hbbai@suse.edu.cn

Zhi Jie Jiang

School of Science
Sichuan University of Science and Engineering
Zigong, Sichuan, 643000, P.R. China
matjzj@126.com

Feng Zhou

School of Science
Sichuan University of Science and Engineering
Zigong, Sichuan, 643000, P.R. China

Abstract. Let $\Pi_+ = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$ denote the upper half-plane in the complex plane $\mathbb{C}$. In this paper we obtain the necessary condition for the weak compactness of composition operator on $H^1(\Pi_+)$, and we also take an example to show that it is not sufficient.

Keywords: Upper half-plane, Hardy space, composition operator, weak compactness

Mathematics Subject Classification: Primary 47B38; Secondary 47B33, 47B37

[1]Supported by the Special Foundation for Young Scientists of Sichuan Province (No.09ZC115).

1. Introduction

Let $\Pi_+ = \{z \in \mathbb{C} : \text{Im}z > 0\}$ be the upper half-plane in complex plane $\mathbb{C}$ and $H(\Pi_+)$ the space of all analytic functions on Π_+. For $0 < p < \infty$, the Hardy space $H^p(\Pi_+)$ consists of all $f \in H(\Pi_+)$ such that

$$\|f\|^p_{H^p(\Pi_+)} = \sup_{y>0} \int_{-\infty}^{+\infty} |f(x+iy)|^p dx < \infty.$$

When $p \geq 1$ the Hardy space with the norm $\|\cdot\|_{H^p(\Pi_+)}$ becomes a Banach space(even a Hilbert space if $p = 2$), and for $0 < p < 1$

$$d(f,g) = \|f - g\|^p_{H^p(\Pi_+)}$$

defines a Fréchet space distance on $H^p(\Pi_+)$.

Let φ be an analytic self-map of Π_+. The composition operator induced by φ on $H(\Pi_+)$ is defined by

$$C_\varphi f(z) = f(\varphi(z)), \quad z \in \Pi_+.$$

During the past few decades, composition operators have been studied extensively on spaces of analytic functions on the unit disk or the unit ball. As a consequence of the Littlewood's subordination theorem it is well-known that every composition operator is bounded on Hardy spaces of the open unit disk. However, when we consider the Hardy space, or the Bergman space of the upper half-plane, we find the situation entirely different. There do exist unbounded composition operators on these spaces. Moreover, Matache[5] proved that there didn't exist compact composition operators on Hardy spaces of the upper half-plane. Shapiro and Smith[6] also proved that there were no compact composition operators on Bergman spaces of the upper half-plane.

Once boundedness and compactness have been established, a typical natural problem one can ask about any composition operator on Hardy space of the upper half-plane is: Is it weakly compact? or Is there a weakly compact composition operator?

Let X and Y be Banach spaces, $L : X \to Y$ be a bounded linear operator. We recall that $L : X \to Y$ is *weakly compact* if it maps bounded sets into relatively weakly compact sets. For some results in this topic see [2] and [3]. Since the Hardy space $H^p(\Pi_+)$ $(1 < p < \infty)$ is reflexive, the compactness of composition operator on $H^p(\Pi_+)$ is equivalent to the weak compactness. Thus, by the results in [5], for the case $1 < p < \infty$ we know that there is no weakly compact composition operator on $H^p(\Pi_+)$. Because the space $H^1(\Pi_+)$ is not reflexive, this leads us to wonder the question is: Is there a weakly compact composition operator on $H^1(\Pi_+)$? In this paper we are going to investigate this question.

2. Main results

In order to deal with the weak compactness of composition operator, we need introduce the *Carleson set*. For $t \in \mathbb{R}$, $h > 0$, the Carleson set is defined by

$$S_{t,h} = (t, t+h) \times (0, h).$$

Let φ be an analytic self-map of Π_+. For each fixed $y > 0$ the measurable mapping $\varphi_y(x) = \varphi(x + iy)$, $x \in \mathbb{R}$ naturally induces a Borel measure on Π_+, $m\varphi_y^{-1}$ called the pull-back measure induced by φ_y

$$m\varphi_y^{-1}(E) = |\{x \in \mathbb{R} : \varphi(x+iy) \in E\}| \tag{1}$$

for each Borel subset $E \subseteq \Pi_+$. In (1) $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}$. For each $f \in H^p(\Pi_+)$, it follows that

$$\int_{-\infty}^{+\infty} |f(\varphi(x+iy))|^p dx = \int_{-\infty}^{+\infty} |f \circ \varphi_y(x)|^p dx = \int_{\Pi_+} |f|^p dm\varphi_y^{-1}. \tag{2}$$

Before obtaining the main result, we need quote the following lemma, which was proved in [2].

Lemma 2.1 *Let X, Y, Z be Banach spaces and $T : X \to Y$, $S : X \to Z$ be bounded operators such that $\|Sx\| \le \|Tx\|$. Suppose that there are two linear topologies τ_1 on X and τ_2 on Y such that T is $\tau_1 - \tau_2$ continuous, (B_X, τ_1) is metrizable and compact and the weak topology of Y is finer than τ_2. If T is weakly compact, then so is S.*

We now formulate and prove the main result of this paper.

Theorem 2.2 *Suppose the operator C_φ is bounded on $H^1(\Pi_+)$, then C_φ is weakly compact on $H^1(\Pi_+)$ only if for any $y > 0$*

$$\lim_{h\to 0} \frac{m\varphi_y^{-1}(S_{t,h})}{h} = 0. \tag{3}$$

Proof. Let τ_1 the topology of uniform convergence on compact subsets of Π_+, τ_2 the topology of the pointwise convergence, $X = Y = H^1(\Pi_+)$, $Z = L^1(\Pi_+, m\varphi_y^{-1})$ and $S : H^1(\Pi_+) \to L^1(\Pi_+, m\varphi_y^{-1})$ given by $f \mapsto f$. Then applying Lemma 2.1, it follows that $S : H^1(\Pi_+) \to L^1(\Pi_+, m\varphi_y^{-1})$ is weakly compact.

Suppose the condition in (2) is false, which means that

$$\lim_{h\to 0} \frac{m\varphi_y^{-1}(S_{t,h})}{h} \ne 0,$$

for some $y > 0$. This implies that we can find t_n in $\mathbb{R}$, $h_n \to 0$ as $n \to \infty$ and $\varepsilon_0 > 0$ such that

$$m\varphi_y^{-1}(S_{t_n,h_n}) \geq \varepsilon_0 h_n.$$

For each fixed $n \in \mathbb{N}$, set

$$z_n = t_n + ih_n,$$

and

$$f_n(z) = \frac{h_n^4}{4\pi^2(z - \overline{z}_n)^2}.$$

We have $f_n \in H^1(\Pi_+)$ and $\|f_n\|_{H^1(\Pi_+)} = h_n^3/4\pi$. Taking $g_n = f_n/\|f_n\|_{H^1(\Pi_+)}$, $n \in \mathbb{N}$, it will be enough to prove that for each subsequence $(g_{n_k})_{k\in\mathbb{N}}$ of $(g_n)_{n\in\mathbb{N}}$, the sequence $(Sg_{n_k})_{k\in\mathbb{N}}$ is not weakly convergent in $L^1(\Pi_+, m\varphi_y^{-1})$. Then by [1, p.137], we only prove that $(Sg_{n_k})_{k\in\mathbb{N}}$ is not uniformly integrable, i.e., there exists $\varepsilon > 0$ such that for every $\eta > 0$ there is a measurable subset E of Π_+ such that $m\varphi_y^{-1}(E) \leq \eta$ and $\int_E |g_{n_k}| dm\varphi_y^{-1} \geq \varepsilon$.

Take $\varepsilon = \varepsilon_0$ and fix an arbitrary η. Since $m\varphi_y^{-1}$ is a Carleson measure, there is a constant $C > 0$ such that

$$m\varphi_y^{-1}(S_{t_n,h_n}) \leq Ch_n. \tag{4}$$

Since $h_n \to 0$ as $n \to \infty$, we choose $k \in \mathbb{N}$ such that $m\varphi_y^{-1}(S_{t_{n_k},h_{n_k}}) \leq \eta$.

On the other hand, if $z = x + iy \in S_{t_n,h_n}$, then $t_n < x < t_n + h_n$, $0 < y < h_n$ and

$$|f_n(z)| = \frac{h_n^4}{4\pi^2|z - \overline{z}_n|^2} = \frac{h_n^4}{4\pi^2[(x - t_n)^2 + (y + t_n)^2]} \geq \frac{h_n^2}{4\pi^2}.$$

From this, and applying (2) and (4), we obtain

$$\begin{aligned}\int_{S_{t_{n_k},h_{n_k}}} |g_{n_k}(z)| dm\varphi_y^{-1}(z) &\geq \frac{h_{n_k}^2}{4\pi^2\|f_{n_k}\|_{H^1(\Pi_+)}} m\varphi_y^{-1}(S_{t_{n_k},h_{n_k}}) \\ &\geq \frac{4}{\pi}\varepsilon_0,\end{aligned}$$

from which it follows that our hypothesis is false, and we deduce that the condition in (3) is a necessary condition for the weak compactness of C_φ on $H^1(\Pi_+)$.

The following example shows that the condition in (3) is not sufficient for the weak compactness of C_φ on $H^1(\Pi_+)$.

Example 2.3 Suppose $\varphi : \Pi_+ \to \Pi_+$ is defined as $\varphi(z) = z + z_0$, $z_0 = x_0 + iy_0 \in \Pi_+$. Then by Example 2.3 in [5], C_φ is a bounded operator on $H^1(\Pi_+)$. Since φ is invertible, from the Theorem 3.1 of [7], C_φ is invertible.

Then by [4], it can not be weakly compact. However, we will prove that φ satisfies the condition in (3). For $t \in \mathbb{R}$, $h > 0$ and each $y > 0$,

$$\varphi_y^{-1}(\{(x, y + y_0) : t < x < t + h\}) = (-x_0 + t, -x_0 + t + h).$$

So we have

$$\varphi_y^{-1}(S_{t,h}) = (-x_0 + t, -x_0 + t + h), \text{ if } h > y + y_0;$$

and

$$\varphi_y^{-1}(S_{t,h}) = \emptyset, \text{ if } h \leq y + y_0.$$

This implies that for each $y > 0$,

$$m\varphi_y^{-1}(S_{t,h}) = h, \text{ if } h > y + y_0;$$

and

$$m\varphi_y^{-1}(S_{t,h}) = 0, \text{ if } h \leq y + y_0.$$

Thus we have

$$\lim_{h \to 0} \frac{m\varphi_y^{-1}(S_{t,h})}{h} = 0.$$

We conclude that φ satisfies the condition in (3).

Remark. Since $H^p(\Pi_+)$ $(1 < p < \infty)$ is a reflexive Banach space, the compactness of composition operator on $H^p(\Pi_+)$ is equivalent to the weak compactness. Then by [6], we know that there are no weakly compact composition operators on $H^p(\Pi_+)$ for all p $(1 < p < \infty)$. For the case $H^1(\Pi_+)$, which is not reflexive, we only obtain the necessary conditions for weak compactness of composition operators. Therefore we conjecture that there don't exit weakly compact composition operators on $H^1(\Pi_+)$.

References

[1] S. Banach, Linear operator, Chelsea, New York, 1932.

[2] M. D. Contress, H. Diaz, Weighted composition operators on Hardy spaces, *J. Math. Anal. Appl.* **263** (2001), 224-233.

[3] Z. J. Jiang, Weighted composition operator on Hardy space $H^p(B_N)$, *Advances in Mathematics.* **37** (6) (2008), 749-754 (In Chinese).

[4] R. E. Megginson, An introduction to Banach space theory, Springer-Verlag, New York, 1998.

[5] V. Matache, Composition operators on Hardy spaces of a half-plane, *Proc. Amer. Math. Soc.* **127** (5) (1999), 1483-1491.

[6] J. H. Shapiro, W. Smith, Hardy spaces that support no compact composition operators, *J. Functional Analysis.* **205** (2003), 62-89.

[7] R. K. Singh, S. D. Sharma, Composition operators on a functional Hilbert space, *Bull. Austral. Math. Soc.* **20** (2007), 3775-384.

Received: April, 2010