Int. Journal of Math. Analysis, Vol. 4, 2010, no. 39, 1913 - 1921

Parabolic Starlike and Uniformly Convex Functions

I. B. Bapna

Dept. of Mathematics
M.L.V. Govt. P. G. College, Bhilwara
Rajasthan, India
bapnain@yahoo.com

Nidhi Jain

Tolani Institute for Excellence
Gandhidham, Gujarat, India
nidhijain2000@yahoo.com

Abstract

The main object of this paper is to derive the sufficient conditions for the function $z\left\{{}_p\psi_q\left(z\right)\right\}$ to be in the class of uniformly starlike and uniformly convex function associated with the parabolic region $\operatorname{Re}\{\omega\} > |\omega - 1|$. Further, the hadamard product of the function which are analytic in the open unit disk with negative coefficients are also investigated. Finally, similar results using an integral operator are also obtained.

Mathematics Subject Classification: 30C45

Keywords: Analytic Functions, Univalent Functions, Convex Functions, Starlike Functions, Fox Wright Function, Integral Operator, Parabolic Region

1. Introduction

Let T denote the class of function $f(z)$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad , \quad a_n \geq 0 \tag{1.1}$$

that are analytic in the open unit disk $U = \{z : |z| < 1\}$
A function $f \in T$ is said to be starlie of order $\alpha, 0 \leq \alpha < 1$, if and only if $\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha$ and convex if and only if $\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha$, with $z \in U$

Definition: 1.1 A function $f(z)$ is uniformly convex (uniformly starlike) in U if $f(z)$ is convex (starlike) and has the property that for every circular arc ξ contained in U, with center ξ also in U ,the arc $f(\xi)$is convex (starlike) with respect to $f(\xi)$.The class of uniformly convex functions is denoted by UC_p and the class of uniformly starlike functions by US_p [5] (also refer [7],[10]).
These subclasses were introduced and defined by Goodman [2] and [3]. The class UC_p describes geometrically the domain of values of the expression $\left(1+\frac{zf\prime\prime(z)}{f\prime(z)}\right)$,$z \in U f(z) \in UC_p$ as a parabolic region [8]

$$\Re = \left\{\omega \in C : (\mathrm{Im}\omega)^2 < 2\mathrm{Re}\omega - 1\right\}$$

Definition: 1.2 [12] $f \in UC_p(\lambda, \mu)$ if and only if

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) \geq \left|\frac{zf''(z)}{f'(z)}\right|, z \in U \tag{1.2}$$

Definition: 1.3 Let $f \in T, 0 \leq \lambda < \infty$ and $0 \leq \mu < 1$, then $f \in UC_p(\lambda, \mu)$ if and only if

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) \geq \lambda\left|\frac{zf''(z)}{f'(z)}\right| + \mu \tag{1.3}$$

The following sufficient condition on the coefficient of the class $UC_p(\lambda, \mu)$ was obtained in [9].
Definition: 1.4 [4] Let $0 \leq \lambda < \infty$ and $0 \leq \mu < 1$, then $f \in US_p(\lambda, \mu)$ if and only if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq \lambda\left|\frac{zf'(z)}{f(z)} - 1\right| + \mu \tag{1.4}$$

Definition: 1.5 For function $f(z)$ of the form(1.1) and $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$ we define the hadamard product (or convolution) of f and g by

$$(f * g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n, z \in U \tag{1.5}$$

Definition: 1.6 [6] The Fox-Wright function appearing in the present paper is defined by

$${}_p\psi_q(z) = {}_p\psi_q\left[\begin{array}{cc} (a_j, \alpha_j)_{1,p}; & z \\ (b_j, \beta_j)_{1,q}; & \end{array}\right] = \sum_{n=0}^{\infty} \frac{\prod\limits_{j=1}^{p} \Gamma(a_j + \alpha_j n) z^n}{\prod\limits_{j=1}^{q} \Gamma(b_j + \beta_j n) n!} \tag{1.6}$$

where $\alpha_j\,(j = 1, \ldots, p)$ and $\beta_j\,(j = 1, \ldots, q)$ are real and positive and

$$1 + \sum_{j=1}^{q} \beta_j > \sum_{j=1}^{p} \alpha_j$$

Definition: 1.7 For ${}_p\Psi_q$ given by (1.6) we have

$$z_p\psi_q(z) = \sum_{n=1}^{\infty} \frac{\prod_{j=1}^{p} \Gamma\left(a_j + \alpha_j\left(n-1\right)\right) z^n}{\prod_{j=1}^{q} \Gamma\left(b_j + \beta_j\left(n-1\right)\right)\left(n-1\right)!}$$

$$= \frac{\prod_{j=1}^{p} \lceil a_j}{\prod_{j=1}^{q} \lceil b_j} z + \sum_{n=2}^{\infty} \frac{\prod_{j=1}^{p} \lceil (a_j + a_j(n-1))\, z^n}{\prod_{j=1}^{q} \lceil (b_j + \beta_j(n-1))\, (n-1)!}$$

$$(1.7) \qquad z\frac{\prod_{j=1}^{q} \Gamma b_j}{\prod_{j=1}^{p} \Gamma a_j} {}_p\psi_q = z_p\Omega_q = z + \sum_{n=2}^{\infty} \frac{\prod_{j=1}^{q} \Gamma b_j}{\prod_{j=1}^{p} \Gamma a_j} \frac{\prod_{j=1}^{p} \Gamma\left(a_j + \alpha_j\left(n-1\right)\right) z^n}{\prod_{j=1}^{q} \Gamma\left(b_j + \beta_j\left(n-1\right)\right)\left(n-1\right)!}$$

where

$${}_p\Omega_q = \frac{\prod_{j=1}^{q} \Gamma b_j}{\prod_{j=1}^{p} \Gamma a_j} {}_p\psi_q$$

Definition: 1.8 Let $G_q^p : T \to T$ be a linear operator defined by

$$G_q^p f\left(z\right) = z_p\Omega_q * f\left(z\right)$$

$$= z - \sum_{n=2}^{\infty} \frac{\prod_{j=1}^{q} \Gamma b_j}{\prod_{j=1}^{p} \Gamma a_j} \frac{\prod_{j=1}^{p} \Gamma\left(a_j + \alpha_j\left(n-1\right)\right)}{\prod_{j=1}^{q} \Gamma\left(b_j + \beta_j\left(n-1\right)\right)} a_n \frac{z^n}{(n-1)!}$$

$$(1.8) \qquad G_q^p f\left(z\right) = z - \sum_{n=2}^{\infty} \mathrm{B}\left(a_j, n\right) a_n z^n$$

where

$$(1.9) \qquad \mathrm{B}\left(a_j, n\right) = \frac{\prod_{j=1}^{q} \Gamma b_j}{\prod_{j=1}^{p} \Gamma a_j} \frac{\prod_{j=1}^{p} \Gamma\left(a_j + \alpha_j\left(n-1\right)\right)}{\prod_{j=1}^{q} \Gamma\left(b_j + \beta_j\left(n-1\right)\right)}$$

Definition: 1.9 For $-1 \le \gamma < 1$, we let $S_q^p(\gamma)$ denote the subclass of starlike functions corresponding to the family UC_p for $f(z)$ s.t.

$$(1.10) \qquad \mathrm{Re}\left\{\frac{z\left(G_q^p f\left(z\right)\right)'}{G_q^p f\left(z\right)} - \gamma\right\} \ge \left|\frac{z\left(G_q^p f\left(z\right)\right)'}{G_q^p f\left(z\right)} - 1\right|$$

Lemma (1.1) A function $f(z)$ of the form (1.1) is in $US_p(\lambda,\mu)$ if and only if

$$\sum_{n=2}^{\infty}[n\,(1+\lambda)-(\lambda+\mu)]a_n \le (1-\mu)\,M_1 \tag{1.11}$$

where $M_1>0$ is a suitable constant

Lemma (1.2) [1] A function $f(z)$ of the form (1.1) is in $f\in UC_p(\lambda,\mu)$ if it satisfies the condition

$$\sum_{n=2}^{\infty}n\,[n\,(1+\lambda)-(\lambda+\mu)]a_n \le (1-\mu)\,M_2 \tag{1.12}$$

where $M_2>0$is a suitable constant.

2. Main Results

Theorem2.1 If $\sum\limits_{j=1}^{q} b_j > \sum\limits_{j=1}^{p} a_j{+}1, a_j>0$ and $1+\sum\limits_{j=1}^{q}\beta_j > \sum\limits_{j=1}^{p}\alpha_j$ then a sufficient condition for the function $z\left\{{}_p\psi_q\,(z)\right\}$ to be in the class $US_p\,(\lambda,\mu)$, $0\le\lambda<\infty$ and $0\le\lambda<\infty$ is

$$\frac{(1+\lambda)}{(1-\mu)}{}_p\psi_q\left[\begin{array}{cc}(a_j+\alpha_j,\alpha_j)_{1,p}; & 1\\ (b_j+\beta_j,\beta_j)_{1,q}; & \end{array}\right]+{}_p\psi_q\left[\begin{array}{cc}(a_j,\alpha_j)_{1,p}; & 1\\ (b_j,\beta_j)_{1,q}; & \end{array}\right]\le M_1+\frac{\prod\limits_{j=1}^{p}\Gamma a_j}{\prod\limits_{j=1}^{q}\Gamma b_j} \tag{2.1}$$

Proof: Since

$$z_p\psi_q(z)=z\sum_{n=0}^{\infty}\frac{\prod\limits_{j=1}^{p}\Gamma\left(a_j+\alpha_j n\right)z^n}{\prod\limits_{j=1}^{q}\Gamma\left(b_j+\beta_j n\right)n!}$$

$$=\frac{\prod\limits_{j=1}^{p}\Gamma a_j}{\prod\limits_{j=1}^{q}\Gamma b_j}z+\sum_{n=2}^{\infty}\frac{\prod\limits_{j=1}^{p}\Gamma\left(a_j+\alpha_j(n-1)\right)z^n}{\prod\limits_{j=1}^{q}\Gamma\left(b_j+\beta_j(n-1)\right)(n-1)!}$$

So by the virtue of Lemma 1.1, we need only to show that

$$\sum_{n=2}^{\infty}[n\,(1+\lambda)-(\lambda+\mu)]\left[\frac{\prod\limits_{j=1}^{p}\Gamma\left(a_j+\alpha_j(n-1)\right)}{\prod\limits_{j=1}^{q}\Gamma\left(b_j+\beta_j(n-1)\right)(n-1)!}\right]\le(1-\mu)\,M_1$$

Now,

$$\sum_{n=2}^{\infty}[n\,(1+\lambda)-(\lambda+\mu)]\left[\frac{\prod\limits_{j=1}^{p}\Gamma\left(a_j+\alpha_j(n-1)\right)}{\prod\limits_{j=1}^{q}\Gamma\left(b_j+\beta_j(n-1)\right)(n-1)!}\right]$$

$$= \sum_{n=0}^{\infty} \left[(n+2)(1+\lambda) - (\lambda+\mu) \right] \left[\frac{\prod_{j=1}^{p} \Gamma\left(a_j + \alpha_j(n+1)\right)}{\prod_{j=1}^{q} \Gamma\left(b_j + \beta_j(n+1)\right)(n+1)!} \right]$$

$$= \sum_{n=0}^{\infty} (n+1)(1+\lambda) \left[\frac{\prod_{j=1}^{p} \Gamma\left(a_j + \alpha_j(n+1)\right)}{\prod_{j=1}^{q} \Gamma\left(b_j + \beta_j(n+1)\right)(n+1)!} \right] + \sum_{n=0}^{\infty} (1-\mu) \left[\frac{\prod_{j=1}^{p} \Gamma\left(a_j + \alpha_j(n+1)\right)}{\prod_{j=1}^{q} \Gamma\left(b_j + \beta_j(n+1)\right)(n+1)!} \right]$$

$$= \sum_{n=0}^{\infty} (1+\lambda) \left[\frac{\prod_{j=1}^{p} \Gamma\left(a_j + \alpha_j(n+1)\right)}{\prod_{j=1}^{q} \Gamma\left(b_j + \beta_j(n+1)\right) n!} \right] + (1-\mu) \left[\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma\left(a_j + \alpha_j n\right)}{\prod_{j=1}^{q} \Gamma\left(b_j + \beta_j n\right) n!} - \frac{\prod_{j=1}^{p} \Gamma a_j}{\prod_{j=1}^{q} \Gamma b_j} \right]$$

$$(1+\lambda)\, {}_p\psi_q \left[\begin{array}{cc} (a_j+\alpha_j, \alpha_j)_{1,p}; & \\ (b_j+\beta_j, \beta_j)_{1,q}; & 1 \end{array} \right] + (1-\mu)\, {}_p\psi_q \left[\begin{array}{cc} (a_j, \alpha_j)_{1,p}; & \\ (b_j, \beta_j)_{1,q}; & 1 \end{array} \right] - (1-\mu) \frac{\prod_{j=1}^{p} \Gamma a_j}{\prod_{j=1}^{q} \Gamma b_j}$$

$$= (1-\mu) \left\{ \frac{(1+\lambda)}{(1-\mu)} {}_p\psi_q \left[\begin{array}{cc} (a_j+\alpha_j, \alpha_j)_{1,p}; & \\ (b_j+\beta_j, \beta_j)_{1,q}; & 1 \end{array} \right] + {}_p\psi_q \left[\begin{array}{cc} (a_j, \alpha_j)_{1,p}; & \\ (b_j, \beta_j)_{1,q}; & 1 \end{array} \right] - \frac{\prod_{j=1}^{p} \Gamma a_j}{\prod_{j=1}^{q} \Gamma b_j} \right\}$$

This expression is bounded above by $(1-\mu) M_1$, if and only if

$$= \frac{(1+\lambda)}{(1-\mu)} {}_p\psi_q \left[\begin{array}{cc} (a_j+\alpha_j, \alpha_j)_{1,p}; & \\ (b_j+\beta_j, \beta_j)_{1,q}; & 1 \end{array} \right] + {}_p\psi_q \left[\begin{array}{cc} (a_j, \alpha_j)_{1,p}; & \\ (b_j, \beta_j)_{1,q}; & 1 \end{array} \right] \leq M_1 + \frac{\prod_{j=1}^{p} \Gamma a_j}{\prod_{j=1}^{q} \Gamma b_j}$$

Remark 2.1 In Theorem 2.1 $\lambda = 2$ and $\mu = 0$ gives the sufficient condition for the function $z\left\{{}_p\psi_q(z)\right\}$ to be in the class $US_p(2,0), 0 \leq \lambda < \infty$ and $0 \leq \mu < 1$ obtained in [11].

We get this similar result for $\alpha = 0$ in [11, Th 2.2] i.e.

$$= 3\,{}_p\psi_q \left[\begin{array}{cc} (a_j+\alpha_j, \alpha_j)_{1,p}; & \\ (b_j+\beta_j, \beta_j)_{1,q}; & 1 \end{array} \right] + {}_p\psi_q \left[\begin{array}{cc} (a_j, \alpha_j)_{1,p}; & \\ (b_j, \beta_j)_{1,q}; & 1 \end{array} \right] \leq M_1 + \frac{\prod_{j=1}^{p} \Gamma a_j}{\prod_{j=1}^{q} \Gamma b_j}$$

Theorem 2.2 Let $f(z)$ be given by (1.1). if $-1 \leq \gamma < 1$ and

$$(2.2) \sum_{n=2}^{\infty} (2n-1-\gamma)\, B(a_j, n)\, |a_n| \leq 1-\gamma$$

then $f(z) \in S_q^p(\gamma)$ where $B(a_j, n)$ is given by (1.9).
Proof: By Def (1.9) of the class $S_q^p(\gamma)$, it suffices to show that

$$\text{Re}\left\{\frac{z\left(G_q^p f(z)\right)'}{G_q^p f(z)} - \gamma\right\} \geq \left|\frac{z\left(G_q^p f(z)\right)'}{G_q^p f(z)} - 1\right| \tag{2.3}$$

$$\text{or} \qquad 2\left|\frac{z\left(G_q^p f(z)\right)'}{G_q^p f(z)} - 1\right| \leq 1-\gamma$$

$$\text{or} \qquad 2\left|\frac{z\left\{1-\sum\limits_{n=2}^{\infty} B(a_j,n)\, a_n n z^{n-1}\right\}}{z-\sum\limits_{n=2}^{\infty} B(a_j,n)\, a_n z^n} - 1\right| \leq 1-\gamma$$

$$\text{or} \qquad 2\sum_{n=2}^{\infty} B(a_j,n)(n-1)|a_n| \leq (1-\gamma)\left\{1-\sum_{n=2}^{\infty} B(a_j,n)|a_n|\right\}$$

$$\text{or} \qquad \sum_{n=2}^{\infty}(2n-1-\gamma)\, B(a_j,n)|a_n| \leq 1-\gamma$$

Hence, the theorem is proved.

3. An Integral Operator

In this section we illustrate some results obtained by a particular integral operator defined by

$${}_p\phi_q\left[\begin{matrix}(a_j,\alpha_j)_{1,p}; \\ (b_j,\beta_j)_{1,q};\end{matrix}\quad z\right] = \int_0^z {}_p\psi_q(x)dx$$

Theorem 3.1 If $\sum\limits_{j=1}^{q} b_j > \sum\limits_{j=1}^{p} a_j, a_j > 0$ and $1+\sum\limits_{j=1}^{q}\beta_j > \sum\limits_{j=1}^{p}\alpha_j$ then a sufficient condition for the function ${}_p\phi_q(z) = \int\limits_0^z {}_p\psi_q(x)dx$ to be in the class $US_p(\lambda,\mu)$ $0 \leq \lambda < \infty$ and $0 \leq \mu < 1$ is

$$\frac{(1+\lambda)}{(1-\mu)}{}_p\psi_q\left[\begin{matrix}(a_j,\alpha_j)_{1,p}; \\ (b_j,\beta_j)_{1,q};\end{matrix}\quad 1\right] - \frac{(\lambda+\mu)}{(1-\mu)}{}_p\psi_q\left[\begin{matrix}(a_j-\alpha_j,\alpha_j)_{1,p}; \\ (b_j-\beta_j,\beta_j)_{1,q};\end{matrix}\quad 1\right] \tag{3.1}$$

$$+\frac{(\lambda+\mu)}{(1-\mu)}\frac{\prod\limits_{j=1}^{p}\Gamma(a_j-\alpha_j)}{\prod\limits_{j=1}^{q}\Gamma(b_j-\beta_j)} \leq M_1 + \frac{\prod\limits_{j=1}^{p}\Gamma a_j}{\prod\limits_{j=1}^{q}\Gamma b_j}$$

Proof: Since ${}_p\phi_q\left(z\right)=\int\limits_0^z {}_p\psi_q\left(x\right)dx$

$$=\frac{\prod\limits_{j=1}^{p}\Gamma a_j}{\prod\limits_{j=1}^{q}\Gamma b_j}z+\sum_{n=2}^{\infty}\frac{\prod\limits_{j=1}^{p}\Gamma\left[\left(a_j-\alpha_j\right)+\alpha_j n\right]z^n}{\prod\limits_{j=1}^{q}\Gamma\left[\left(b_j-\beta_j\right)+\beta_j n\right]n!} \tag{3.2}$$

We see that

$$\sum_{n=2}^{\infty}\left[n\left(1+\lambda\right)-\left(\lambda+\mu\right)\right]\frac{\prod\limits_{j=1}^{p}\Gamma\left[\left(a_j-\alpha_j\right)+\alpha_j n\right]}{\prod\limits_{j=1}^{q}\Gamma\left[\left(b_j-\beta_j\right)+\beta_j n\right]n!}$$

$$=\left(1+\lambda\right)\sum_{n=2}^{\infty}\frac{n\prod\limits_{j=1}^{p}\Gamma\left[\left(a_j-\alpha_j\right)+\alpha_j n\right]}{\prod\limits_{j=1}^{q}\Gamma\left[\left(b_j-\beta_j\right)+\beta_j n\right]n!}-\left(\lambda+\mu\right)\sum_{n=2}^{\infty}\frac{\prod\limits_{j=1}^{p}\Gamma\left[\left(a_j-\alpha_j\right)+\alpha_j n\right]}{\prod\limits_{j=1}^{q}\Gamma\left[\left(b_j-\beta_j\right)+\beta_j n\right]n!}$$

$$=\left(1+\lambda\right)\left[\sum_{n=0}^{\infty}\frac{\prod\limits_{j=1}^{p}\Gamma(a_j+\alpha_j n)}{\prod\limits_{j=1}^{q}\Gamma(b_j+\beta_j n)n!}-\frac{\prod\limits_{j=1}^{p}\Gamma a_j}{\prod\limits_{j=1}^{q}\Gamma b_j}\right]\left(\lambda+\mu\right)\left[\sum_{n=0}^{\infty}\frac{\prod\limits_{j=1}^{p}\Gamma[(a_j-\alpha_j)+\alpha_j n]}{\prod\limits_{j=1}^{q}\Gamma[(b_j-\beta_j)+\beta_j n]n!}-\frac{\prod\limits_{j=1}^{p}\Gamma(a_j-\alpha_j)}{\prod\limits_{j=1}^{q}\Gamma(b_j-\beta_j)}-\frac{\prod\limits_{j=1}^{p}\Gamma a_j}{\prod\limits_{j=1}^{q}\Gamma b_j}\right]$$

$$=\left(1+\lambda\right){}_p\psi_q\left[\begin{array}{cc}\left(a_j,\alpha_j\right)_{1,p}; & 1\\ \left(b_j,\beta_j\right)_{1,q}; & \end{array}\right]-\left(\lambda+\mu\right){}_p\psi_q\left[\begin{array}{cc}\left(a_j-\alpha_j,\alpha_j\right)_{1,p}; & 1\\ \left(b_j-\beta_j,\beta_j\right)_{1,q}; & \end{array}\right]$$

$$+\left(\lambda+\mu\right)\frac{\prod\limits_{j=1}^{p}\Gamma\left(a_j-\alpha_j\right)}{\prod\limits_{j=1}^{q}\Gamma\left(b_j-\beta_j\right)}-\left(1-\mu\right)\frac{\prod\limits_{j=1}^{p}\Gamma a_j}{\prod\limits_{j=1}^{q}\Gamma b_j}$$

$$=\left(1-\mu\right)\left\{\frac{\left(1+\lambda\right)}{\left(1-\mu\right)}{}_p\psi_q\left[\begin{array}{cc}\left(a_j,\alpha_j\right)_{1,p}; & 1\\ \left(b_j,\beta_j\right)_{1,q}; & \end{array}\right]-\frac{\left(\lambda+\mu\right)}{\left(1-\mu\right)}{}_p\psi_q\left[\begin{array}{cc}\left(a_j-\alpha_j,\alpha_j\right)_{1,p}; & 1\\ \left(b_j-\beta_j,\beta_j\right)_{1,q}; & \end{array}\right]\right.$$

$$\left.+\frac{\left(\lambda+\mu\right)}{\left(1-\mu\right)}\frac{\prod\limits_{j=1}^{p}\Gamma\left(a_j-\alpha_j\right)}{\prod\limits_{j=1}^{q}\Gamma\left(b_j-\beta_j\right)}-\frac{\prod\limits_{j=1}^{p}\Gamma a_j}{\prod\limits_{j=1}^{q}\Gamma b_j}\right\}$$

$$\leq\left(1-\mu\right)M_1$$

where M_1 is a constant greater than the expression in (3.2) which in view of Lemma 1.1 gives the desired result (3.1).

Remark: For $\alpha=2$ and $\mu=0$ theorem (3.1) reduce to the desired result obtained by Chaurasia and Srivastava.

Theorem (3.2) If $\sum_{j=1}^{q} b_j > \sum_{j=1}^{p} a_j, a_j > 0$ and $1+\sum_{j=1}^{q} \beta_j > \sum_{j=1}^{p} \alpha_j$ then a sufficient condition for the function ${}_p\phi_q(z) = \int_0^z {}_p\psi_q(x)dx$ to be in the class $UC_p(\lambda, \mu), 0 \le \lambda < \infty$ and $0 \le \mu < 1$ is

$$\frac{(1+\lambda)}{(1-\mu)} {}_p\psi_q \left[\begin{array}{cc} (a_j+\alpha_j, \alpha_j)_{1,p}; & 1 \\ (b_j+\beta_j, \beta_j)_{1,q}; & \end{array} \right] + {}_p\psi_q \left[\begin{array}{cc} (a_j, \alpha_j)_{1,p}; & 1 \\ (b_j, \beta_j)_{1,q}; & \end{array} \right] \le M_2 + \frac{\prod_{j=1}^{p} \Gamma a_j}{\prod_{j=1}^{q} \Gamma b_j} \tag{3.3}$$

Proof: The proof of this theorem is a direct consequence of Th.3.1

References

[1] A.Swaminathan, Hypergeometric Functions in the Parabolic Domain,*Tamsui Oxford Journal of Mathematical Sciences*, 20(1) (2004)1-16.

[2] A.W.Goodman, On Uniformly Convex Functions,*Ann. polon Math.* 56(1991), 87-92.

[3] A.W.Goodman, On Uniformly Starlike Functions,*J. Math. Anal. & Appl.*, 155(1991), 364-370.

[4] F.Ronning, Uniformly Convex Functions and a Corresponding Class of Starlike Functions,*Proc. Amer. Math. Soc.*, 118(1993), 189-196.

[5] Herb Silverman, A Class of Starlike Functions Defined by the Dziok Srivastava Operator,*Kyungpook Math. J.* 49 (2009), 95-106.

[6] H.M.Srivastava and H.L.Manocha, A Treatise on Generating Functions, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

[7] H.M.Srivastava and A.K.Mishra, Applications of Fractional Calculus to Parabolic Starlike and Uniformly Convex Functions, *Computer Math. Appl.*, 39 (2000), 57-69.

[8] Oqlah Al-Refai and Maslina Darus, An Extension to the Owa-Srivastava Fractional Operator with Applications to Parabolic Starlike and Uniformly Convex Functions, *International Journal of Differential Equations*, Volume 2009, Article ID 597292, 18 Pages.

[9] R.Bharti, R. Parvatham and A.Swaminathan, On Subclass of Uniformly Convex Functions and Corresponding Class of Starlike Functions, *Tamkang J. Math.* 28 (1997), 17-32.

[10] S.Kanas and F.Ronning, Uniformly Starlike and Convex Functions and other Related Classes of Univalent Functions, *Ann. Univ. Mariae Curie-Sklodowska Section* A, 53 (1999), 95-105.

[11] V.B.L Chaurasia and Amber Srivastava, Uniformly Starlike and Uniformly Convex Functions Pertaining to Special Functions, *J.i.p.a.m* 9 (2008), 6 Pages.

[12] W.Ma and D.Minda, Uniformly Convex Functions,*Ann. polon. Math.*, 57 (1992), 165-175.

Received: April, 2010