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Screen Transversal Lightlike Submanifolds of Indefinite Sasakian Manifolds

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Abstract

In this paper, we introduce screen transversal, radical screen transversal and screen transversal anti-invariant lightlike submanifolds of an indefinite Sasakian manifold and give example. We prove a characterization theorem for the existence of screen transversal anti-invariant lightlike submanifolds and obtain necessary and sufficient conditions for the induced connection of screen transversal anti-invariant and radical screen transversal lightlike submanifolds to be a metric connection.

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1 Introduction

The study of Riemannian geometry of submanifolds is one of the most important topics of differential geometry. It is well known that Semi-Riemannian submanifolds[1], have many similarities with their Riemannian case. However, the lightlike submanifolds are different since their normal vector bundle intersect with the tangent bundle making it more interesting to study these submanifolds. The lightlike submanifolds were introduced and studied by Duggal and Bejancu[8]. In [2], B. Sahin initiated the study of transversal lightlike submanifolds of an indefinite Kaehler manifold which are different from CR-lightlike[8], Screen CR[9] and generalized CR-lightlike[12] submanifolds. Later on, B.Sahin[3] gave the notion of screen transversal lightlike submanifolds of indefinite Kaehler manifolds and obtained many interesting results. On the other hand, lightlike submanifolds of indefinite Sasakian manifold were introduced by Duggal and Sahin in [10] and studied the integrability of distributions, geometry of leaves of the distributions involved as well as other properties of this submanifold. Recently, Yildirim and Sahin[4] defined transversal lightlike submanifolds of indefinite Sasakian manifolds and investigated both radical transversal and transversal lightlike submanifolds. However, a general notion of screen transversal lightlike submanifolds of indefinite Sasakian manifolds has not been introduced as yet.

The paper is arranged as follows. In section 2, we recall definitions for indefinite Sasakian manifolds and give basic information on the lightlike geometry needed for this paper. Section 3 is devoted to the study of the geometry of screen transversal lightlike submanifolds. In section 4, we give example of a screen transversal anti-invariant lightlike submanifold, obtain a characterization of screen transversal anti-invariant lightlike submanifolds and give geometric conditions for the induced connection to be a metric connection. In section 5, we study radical transversal lightlike submanifolds and find the integrability of distributions.

2 Preliminaries

We follow [8] for the notation and fundamental equations for lightlike submanifolds used in this paper. A submanifold M^m immersed in a semi-Riemannian manifold $(\overline{M}^{m+n}, \overline{g})$ is called a lightlike submanifold if it is a lightlike manifold with respect to the metric g induced from $\overline{g}$ and the radical distribution $Rad\,TM$ is of rank r, where $1 \leq r \leq m$. Let $S(TM)$ be a screen distribution which is a semi-Reimannian complementary distribution of $Rad\,TM$ in TM, i.e.,

$$TM = (Rad\,TM) \perp S(TM)$$

Consider a screen transversal vector bundle $S(TM^{\perp})$, which is a semi-Riemannian complementry vector bundle of $Rad\ \ TM$ in $TM^{\perp}$. Since for any local basis $\{\xi_i\}$ of $Rad\,TM$, there exist a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\overline{g}(\xi_i, N_j) = \delta_{ij}$, it follows that there exist a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$ [[8] ,pg-144]. Let $tr(TM)$ be complementary (but not orthogonal)vector bundle to TM in $T\overline{M}|_M$.Then

$$tr(TM) = ltr(TM)\perp S(TM^{\perp}),$$

$$T\overline{M}|_M = S(TM)\perp[(Rad\ \ TM)\oplus ltr(TM)]\perp S(TM^{\perp}).$$

Following are four subcases of a lightlike submanifold $(M, g, S(TM), S(TM^{\perp})$.
Case 1: r-lightlike if $r < min\{m, n\}$.
Case 2: Co-isotropic if $r = n < m$; $S(TM^{\perp}) = \{0\}$.
Case 3: Isotropic if $r = m < n$; $S(TM) = \{0\}$
Csae 4: Totally lightlike if $r = m = n$; $S(TM) = \{0\} = S(TM^{\perp})$.
The Gauss and Weingarten equations are

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y),\ \ \forall\ X, Y \in \Gamma(TM) \tag{2.1}$$

$$\overline{\nabla}_X U = -A_U X + \nabla^t_X U,\ \ \forall\ X \in \Gamma(TM), U \in \Gamma(tr(TM)) \tag{2.2}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^t_X U\}$ belongs to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively, ∇ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$, respectively. Moreover, we have

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \tag{2.3}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \tag{2.4}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \tag{2.5}$$

$\forall\ X, Y \in \Gamma(TM), N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Then, by using (2.1), (2.3)-(2.5) and the fact that $\overline{\nabla}$ is a metric connection, we get

$$\overline{g}(h^s(X, Y), W) + \overline{g}(Y, D^l(X, W)) = g(A_W X, Y) \tag{2.6}$$

In general, the induced connection ∇ on M is not a metric connection. Since $\overline{\nabla}$ is a metric connection, by using (2.3) we get

$$(\nabla_X g)(Y, Z) = \overline{g}(h^l(X, Y), Z) + \overline{g}(h^l(X, Z), Y) \tag{2.7}$$

An odd dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ is called a contact metric manifold [[7],[10]] if there exists a (1,1) tensor field ϕ, a vector field V, called the characteristic vector field, and its 1-form satisfying

$$\left.\begin{array}{l} \overline{g}(\phi X, \phi Y) = \overline{g}(X, Y) - \epsilon\eta(X)\eta(Y), \overline{g}(V, V) = \epsilon \\ \phi^2 X = -X + \eta(X)V, \overline{g}(X, V) = \epsilon\eta(X), \\ d\eta(X, Y) = \overline{g}(X, \phi Y), \forall X, Y \in \Gamma(TM), \end{array}\right\} \tag{2.8}$$

where $\epsilon = \pm 1$. It follows that $\phi V = 0$, $\eta o \phi = 0$, $\eta(V) = \epsilon$.
Then $(\phi, V, \eta, \overline{g})$ is called a contact metric structure of $\overline{M}$. We say that $\overline{M}$ has a normal contact structure if $N_\phi + d\eta \otimes \xi = 0$, where N_ϕ is the Nijenhuis tensor field of ϕ[7]. A normal contact metric manifold is called a Sasakian manifold [14] for which we have

$$\overline{\nabla}_X V = \phi X, \tag{2.9}$$

$$(\overline{\nabla}_X \phi)Y = -\overline{g}(X, Y)V + \epsilon\eta(Y)X. \tag{2.10}$$

$\overline{M}$ is called an indefinite Sasakian form, denoted by $\overline{M}(c)$, if it has the constant ϕ-sectional curvature c[13]. The curvature tensor $\overline{R}$ of a Sasakian space form $\overline{M}(c)$ is given by

$$\overline{R}(X, Y)Z = \frac{(c+3)}{4}\{\overline{g}(Y, Z)X - \overline{g}(X, Z)Y\} + \frac{(c-1)}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \tag{2.11}$$

$+\overline{g}(X, Z)\eta(Y)V - \overline{g}(Y, Z)\eta(X)V + \overline{g}(\phi Y, Z)\phi X + \overline{g}(\phi Z, X)\phi Y - 2\overline{g}(\phi X, Y)\phi Z\}$
for any $X, Y\, and\, Z \in \Gamma(T\overline{M})$.

3 Screen transversal lightlike submanifolds

In this section, we introduce and study screen transversal(ST), radical screen transversal and screen transversal anti-invariant lightlike submanifolds of indefinite Sasakian manifolds. We first prove the following lemma.

Lemma 3.1. *Let M be a r-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ and let $\phi(Rad\ TM)$ be a vector subbundle of $S(TM^\perp)$. Then $\phi(ltr(TM))$ is also a vector subbundle of the screen transversal bundle $S(TM^\perp)$ and $\phi(Rad\,TM) \cap \phi(ltr(TM)) = \{0\}$.*

Proof. Let us assume that $ltr(TM)$ is invariant with respect to ϕ, i.e, $\phi(ltr(TM)) = ltr(TM)$. By the definition of a lightlike submanifold, there exist vector field $\xi \in \Gamma(Rad\,TM)$ and $N \in \Gamma(ltr(TM))$ such that $g(\xi, N) = 1$. Also from (2.8), we get

$$1 = \overline{g}(\xi, N) = \overline{g}(\phi\xi, \phi N)$$

However, if $\phi N \in \Gamma(ltr(TM))$, then by hypothesis, we get $\overline{g}(\phi N, \phi\xi) = 0$, which contradicts the fact that $\overline{g}(\phi N, \phi\xi) = 1$. Hence ϕN does not belong to $ltr(TM)$. Now suppose that $\phi N \in \Gamma(S(TM))$. Then, using (2.8)and since $\phi\xi \in \Gamma(S(TM^{\perp}))$, we have

$$1 = \overline{g}(\xi, N) = \overline{g}(\phi N, \phi\xi) = 0,$$

which shows that ϕN does not belong to $(S(TM))$. In a similar way, we can easily obtain that ϕN does not belong to $Rad\ TM$. Then, from the decomposition of a lightlike submanifold, we have $\phi N \in \Gamma(S(TM^{\perp}))$.
On contrary, suppose there exists a vector field $X \in \phi(Rad\,TM) \cap \phi(ltr(TM))$. Then, we have $X \in \Gamma(\phi(Rad\,TM))$. Hence $\overline{g}(X, \phi N) = 0$, because $X \in \Gamma(\phi(ltr(TM)))$. But for a r-lightlike submanifold, there exists some vector fields $\phi X \in \Gamma(Rad\ TM)$ such that $\overline{g}(\phi X, N) \neq 0$. Then, from (2.8), we have $0 \neq \overline{g}(\phi X, N) = -\overline{g}(X, \phi N) = 0$, which is a contradiction. Thus, proof is complete.

Definition 3.2. *Let M be an r-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then, we say that M is a screen transversal lightlike submanifold of $\overline{M}$ if there exists a screen transversal bundle $S(TM^{\perp})$ such that*

$$\phi(Rad\,TM) \subset S(TM^{\perp}) \tag{3.1}$$

From the above definition and lemma 3.1, it follows that $\phi(ltr(TM)) \subset S(TM^{\perp})$. In addition to this, if $S(TM^{\perp}) = 0$, then obviously there is no co-isotropic and totally lightlike ST-lightlike submanifold of indefinite sasakian manifold.
It is important to note that $\phi(Rad\,TM)$ and $\phi(ltr(TM))$ are not orthogonal otherwise $S(TM^{\perp})$ will be degenerate.

We now give a new definition by putting some conditions on the screen distribution.

Definition 3.3. *Let M be a ST-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then*
(i) We say that M is a radical ST-lightlike submanifold if $S(TM)$ is invariant with respect to ϕ.
(ii) We say that M is a ST-anti-invariant lightlike submanifold of M if $S(TM)$ is screen transversal with respect to ϕ i.e.,

$$\phi(S(TM)) \subset S(TM^{\perp}). \tag{3.2}$$

In view of the above definition, if M is a screen transversal anti-invariant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$, then we have

$$S(TM^{\perp}) = (\phi(Rad\,TM) \oplus \phi(ltr(TM)))\perp\phi(S(TM))\perp\mu, \tag{3.3}$$

where μ is a non-degenerate orthogonal complementary distribution to

$$(\phi(Rad\,TM) \oplus \phi(ltr(TM)))\perp\phi(S(TM))$$

in $S(TM^{\perp})$.

Now, we have:

Proposition 3.4. *Let M be a ST-anti-invariant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then the distribution μ is invariant with respect to ϕ.*

Proof. For $X \in \Gamma(\mu)$, $\xi \in \Gamma(Rad\,TM)$ and $N \in \Gamma(ltr(TM))$, we have

$$\overline{g}(\phi X, \xi) = -\overline{g}(X, \phi\xi) = 0 \;\; \textit{and} \;\; \overline{g}(\phi X, N) = -\overline{g}(X, \phi N) = 0$$

which shows that $\phi(\mu) \cap Rad\,TM = 0$ and $\phi(\mu) \cap ltr(TM) = 0$. From (3.1), we have

$$\overline{g}(\phi X, \phi\xi) = \overline{g}(X, \xi) = 0 \;\; \textit{and} \;\; \overline{g}(\phi X, \phi N) = \overline{g}(X, N) = 0$$

which implies that $\phi(\mu) \cap \phi(Rad\;TM) = 0$ and $\phi(\mu) \cap \phi(ltr(TM)) = 0$. Moreover, for $Z \in \Gamma(S(TM))$, since $\phi Z \in \Gamma(\phi(S(TM)))$, $\phi(S(TM))$ and μ are orthogonal, we obtain $\overline{g}(\phi X, Z) = -\overline{g}(X, \phi Z) = 0$, which shows that $\phi(\mu) \cap S(TM) = 0$. Hence, we also have $\phi(\mu) \cap \phi(S(TM)) = 0$. Thus , we arrive at

$$\phi(\mu) \cap TM = 0, \phi(\mu) \cap ltr(TM) = 0$$

and

$$\phi(\mu) \cap \{\phi(S(TM))\perp(\phi(ltr(TM)) \oplus \phi(Rad\,TM))\} = 0,$$

which shows that μ is invariant.

4 ST-anti-invariant lightlike submanifolds

In the present section, we study the geometry of ST-anti-invariant lightlike submanifolds of an indefinite Sasakian manifold and give an example. We first recall the Sasakian structure defined on R_q^{2m+1}.

Hereafter, $(R_q^{2m+1}, \phi_0, V, \eta, g)$ will denote the manifold R_q^{2m+1} with its usual Sasakian structure given by

$$\eta = \frac{1}{2}(dz - \sum_{i=1}^{m} y^i dx^i), V = 2\partial z$$

$$\overline{g} = \eta \otimes \eta + \frac{1}{4}(-\sum_{i=1}^{\frac{q}{2}} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=q+1}^{m} dx^i \otimes dx^i + dy^i \otimes dy^i)$$

$$\phi_0(\sum_{i=1}^{m}(X_i\partial x^i + Y_i\partial y^i) + Z\partial z) = \sum_{i=1}^{m}(Y_i\partial x^i - X_i\partial y^i) + \sum_{i=1}^{m} Y_i\, y^i\partial z$$

where (x^i, y^i, z) are the cartesian co-ordinates.

Example 4.1. *Let M be a submanifold in R_2^9 given by the following equations:*
$x^1 = u_1,\ x^2 = u_2,\ x^3 = u_3,\ x^4 = 0$
$y^1 = 0,\ y^2 = u_3,\ y^3 = u_2,\ y^4 = u_1.$
Then, the tangent bundle of M is spanned by

$$Z_1 = 2(\partial x_1 + \partial y_4 + y^1\partial z)$$
$$Z_2 = 2(\partial x_2 + \partial y_3 + y^2\partial z)$$
$$Z_3 = 2(\partial x_3 + \partial y_2 + y^3\partial z)$$
$$Z_4 = 2\partial z$$

Thus, M is a 1-lightlike submanifold with Rad TM = span$\{Z_1\}$. Choose a screen distribution $S(TM)$ = span$\{Z_2, Z_3\}$. It is easy to see that $S(TM)$ is not invariant with respect to ϕ. Since $\{\phi Z_2, \phi Z_3\}$ is non-degenerate, it follows that $\phi(S(TM)) \subset S(TM^\perp)$. The lightlike transversal bundle ltr(TM) is spanned by

$$N = \{-\partial x_1 + \partial y_4 - y^1\partial z\}.$$

Also, screen transversal bundle is spanned by

$$W_1 = 2(\partial x_4 - \partial y_1 + y^4\partial z)$$
$$W_2 = 2(\partial x_3 - \partial y_2 + y^3\partial z)$$
$$W_3 = 2(\partial x_2 - \partial y_3 + y^2\partial z)$$
$$W_4 = 2(\partial x_4 + \partial y_1 + y^4\partial z).$$

Then it is esay to see that $S(TM^\perp)$ = span$\{W_1 = \phi Z_1, W_2 = \phi Z_2, W_3 = \phi Z_3, W_4 = \phi N\}$. Thus M is a ST-anti-invariant lightlike submanifold.

Next, we give a characterization for ST-anti-invariant lightlike submanifolds of an indefinite Sasakian space form which is similar to the characterization given by Blair-Chen[5] for non-degenerate CR-submanifolds of a complex space form.

Theorem 4.2. *Let M be a lightlike submanifold of an indefinite complex space form $\overline{M}(c)$, $c \neq 1$ such that $\phi(Rad\ TM) \subset S(TM^{\perp})$. Then M is a ST-anti-invariant lightlike submanifold if and only if*

$$\overline{g}(\overline{R}(X,Y)\xi, \phi N) = 0. \tag{4.1}$$

for $X, Y \in \Gamma(S(TM))$, $\xi \in \Gamma(Rad\ TM)$ and $N \in \Gamma(ltr(TM))$.

Proof: Let us assume that $\phi(Rad(TM)) \subset S(TM^{\perp})$. Using lemma 3.1, we have $\phi ltr(TM) \subset S(TM^{\perp})$. From (2.8), we have $\overline{g}(\phi X, N) = -g(X, \phi N) = 0$, for $X \in \Gamma(S(TM))$ and $N \in \Gamma(ltr(TM))$. Hence $\phi(S(TM)) \cap (Rad\ TM) = \{0\}$. Also, using (2.8) we get

$$\overline{g}(\phi X, \phi\xi) = 0 \ \ and \ \ \overline{g}(\phi X, \phi N) = 0,$$

which imply that $\phi(S(TM)) \cap \phi(Rad\ TM) = \{0\}$ and $\phi(S(TM)) \cap \phi(ltr(TM)) = \{0\}$. In similar way, we can obtain that $\phi(S(TM)) \cap ltr(TM) = \{0\}$. On the other hand, using (2.11) and since $\phi\xi \in \Gamma(S(TM^{\perp}))$, we get

$$\overline{g}(\overline{R}(X,Y)\xi, \phi N) = \frac{c-1}{2}\overline{g}(\phi X, Y)\overline{g}(\xi, N).$$

Then, we can choose vector fields $\xi \in \Gamma(Rad\ TM)$ and $N \in \Gamma(ltr(TM))$ such that $\overline{g}(\xi, N) \neq 0$. Thus,

$$\overline{g}(\overline{R}(X,Y)\xi, \phi N) = 0 \Leftrightarrow \phi(S(TM)) \perp S(TM),$$

from which we conclude that $\phi(S(TM)) \cap S(TM) = \{0\}$. Since $\phi(S(TM)) \cap S(TM) = \{0\}$, therefore

$$\overline{g}(\overline{R}(X,Y)\xi, \phi N) = 0 \Leftrightarrow \phi(S(TM)) \subset S(TM^{\perp}),$$

from which our assertion follows.

Let F_1, F_2, F_3 and F_4 be the projection morphisms on $\phi(Rad\ TM)$, $\phi(S(TM))$, $\phi(ltr(TM))$ and μ, respectively. Then for $W \in \Gamma(S(TM^{\perp}))$ we have

$$W = F_1W + F_2W + F_3W + F_4W \tag{4.2}$$

On the other hand, for $W \in \Gamma(S(TM^{\perp}))$, we write

$$\phi W = BW + CW, \tag{4.3}$$

where BW and CW are tangential and transversal parts of ϕW. Then applying ϕ to (4.2), we get

$$\phi W = \phi F_1 W + \phi F_2 W + \phi F_3 W + \phi F_4 W \tag{4.4}$$

Comparing the tangential and transversal parts of the above equation, we obtain

$$\left.\begin{aligned} BW &= \phi F_1 W + \phi F_2 W \\ CW &= \phi F_3 W + \phi F_4 W \end{aligned}\right\} \tag{4.5}$$

If we put $\phi F_1 = B_1, \phi F_2 = B_2, \phi F_3 = C_1$ and $\phi F_4 = C_2$, then (4.4) can be rewritten as

$$\phi W = B_1 W + B_2 W + C_1 W + C_2 W, \tag{4.6}$$

where $B_1 W \in \Gamma(Rad\ TM), B_2 W \in \Gamma(S(TM)), C_1 W \in \Gamma(ltr(TM))$ and $C_2 W \in \Gamma(\mu)$.

It is known that the induced connection of a lightlike submanifold is not a metric connection. The condition under which the induced connection on a ST-anti-invariant lightlike submanifold of an indefinite Sasakian manifold to be a metric connection is given by following.

Theorem 4.3. *Let M be a ST-anti-invariant lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then the induced connection on M is a metric connection if and only if $\nabla^S_X \phi\xi$ has no components in $\phi(S(TM))$ for $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad\, TM)$.*

Proof. For $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad\, TM)$, we have

$$\overline{\nabla}_X \xi = -\phi^2(\overline{\nabla}_X \xi) + \eta(\overline{\nabla}_X \xi) V,$$

where we have used (2.8). Using (2.3),(2.5)and(4.6)in the above equation, we get

$$\nabla_X \xi + h^l(X,\xi) + h^s(X,\xi) = \phi A_{\phi\xi} X - B_1 \nabla^s_X \phi\xi - B_2 \nabla^s_X \phi\xi - C_1 \nabla^s_X \phi\xi - C_2 \nabla^s_X \phi\xi - \phi D^l(X, \phi\xi).$$

Comparing the tangential parts on both sides, we get

$$\nabla_X \xi = -B_1 \nabla^s_X \phi\xi - B_2 \nabla^s_X \phi\xi.$$

Thus our assertion follows from Theorem-2.4 in [[8],p.161].

5 Radical ST-lightlike submanifolds

In this section, we study radical ST-lightlike submanifolds. We investigate, lightlike product and give a necessary and sufficient condition for the induced connection on a radical ST-lightlike submanifold to be a metric connection. First, recall that a lightlike submanifold is called irrotational [6]if and only if $\overline{\nabla}_X\xi \in \Gamma(TM)$ for $X \in (TM)$, and $\xi \in \Gamma(Rad\ TM)$. From (2.3), M is irrotational if and only if $h^l(X, \xi) = 0$, $h^s(X, \xi) = 0$ for $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad\,TM)$.

For the integrability of the distributions involved in the definition of a radical ST-lightlike submanifold, we have:

Theorem 5.1. *Let M be a radical ST-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then screen distribution $S(TM)$ is integrable if and only if*

$$\overline{g}(h^s(X, \phi Y), \phi N) = \overline{g}(h^s(\phi Y, X), \phi N) \tag{5.1}$$

for $X, Y \in \Gamma(S(TM))$ and $N \in \Gamma(ltr(TM))$.

Proof. For $X, Y \in \Gamma(S(TM))$ and $N \in \Gamma(ltr(TM))$, we have

$$\overline{g}([X, Y], N) = \overline{g}(\overline{\nabla}_X \phi Y - (\overline{\nabla}_X \phi)Y, \phi N) - \overline{g}(\overline{\nabla}_Y \phi X - (\overline{\nabla}_Y \phi)X, \phi N),$$

where we have used (2.8). Using (2.3)and (2.10), we get

$$\overline{g}([X, Y], N) = \overline{g}(h^s(X, \phi Y), \phi N) - g(h^s(Y, \phi X), \phi N),$$

from which our assertion follows.

Theorem 5.2. *Let M be a radical ST-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then, radical distribution is integrable if and only if*

$$\overline{g}(h^s(\xi_1, \phi X), \phi\xi_2) = \overline{g}(h^s(\xi_2, \phi X), \phi\xi_1)$$

for $X \in \Gamma(S(TM))$ and $\xi_1, \xi_2 \in \Gamma(Rad\,TM)$.

Proof. From (2.8), we have

$$\overline{g}([\xi_1, \xi_2], X) = \overline{g}(\overline{\nabla}_{\xi_1} \phi\xi_2 - (\overline{\nabla}\xi_1\phi)\xi_2, \phi X) - \overline{g}(\overline{\nabla}_{\xi_2} \phi\xi_1 - (\overline{\nabla}\xi_2\phi)\xi_1, \phi X)$$

for each $X \in \Gamma(S(TM))$ and $\xi_1, \xi_2 \in \Gamma(Rad\,TM)$. Using (2.5) and (2.6)in the above equation we get

$$\overline{g}([\xi_1, \xi_2], X) = \overline{g}(h^s(\xi_2, \phi X), \phi\xi_1) - \overline{g}(h^s(\xi_1, \phi X), \phi\xi_2),$$

which proves our assertion.

Theorem 5.3. *Let M be a radical ST-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then, $S(TM)$ defines a totally geodesic foliation on M if and only if $h^s(X, \phi Y)$ has no component in $\phi(Rad\ TM)$ for $X, Y \in \Gamma(S(TM))$.*

Proof. Using (2.8), for any $N \in \Gamma(Rad\, TM)$, we obtain

$$\overline{g}(\nabla_X Y, N) = \overline{g}(\overline{\nabla}_X \phi Y - (\overline{\nabla}_X \phi)Y, \phi N)$$

Now using (2.3) and (2.10), we get

$$\overline{g}(\nabla_X Y, N) = \overline{g}(h^s(X, \phi Y), \phi N),$$

from which our assertion follows.

In a similar way, we have

Theorem 5.4. *Let M be a radical ST-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then $Rad\, TM$, defines a totally geodesic foliation on M if and only if $h^s(\xi_1, \phi X)$ has no component in $\phi(ltr(TM))$ for $\xi_1 \in \Gamma(Rad\, TM)$ and $X \in \Gamma(S(TM))$.*

From Theorem 5.3 and Theorem 5.4, we have

Corollary 5.5. *Let M be a an irrotational radical ST-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then M, is a lightlike product manifold if and only if $h^s(X, \phi Y)$ has no component in $\phi(Rad\ TM)$ for $X, Y \in (S(TM))$.*

Now, we find a necessary and sufficient condition for ∇ on a radical ST-lightlike submanifold to be a metric connection.

Theorem 5.6. *Let M be a radical ST-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then, the induced connection on M is a metric connection if and only if $h^s(X, \phi Y)$ has no component in $\phi(ltr(TM))$ for $X, Y \in \Gamma(TM)$.*

Proof. From (2.8)and after calculation, we have

$$\overline{\nabla}_X \xi = -\phi(\overline{\nabla}_X \phi\xi - (\overline{\nabla}_X \phi)\xi).$$

Using(2.10), we get

$$\overline{\nabla}_X \xi = -\phi \overline{\nabla}_X \phi\xi$$

Hence, using (2.3) and (2.5), we get

$$\nabla_X \xi + h^l(X, \xi) + h^s(X, \xi) = \phi A_{\phi\xi} X - \phi \nabla^S_X \phi\xi - \phi(D^l(X, \phi\xi)). \qquad (5.2)$$

Taking inner product of (5.2) with $Y \in \Gamma(S(TM))$, we get

$$\overline{g}(\nabla_X \xi, Y) = -\overline{g}(A_{\phi\xi} X, \phi Y)$$

Using (2.6), we get

$$\overline{g}(\nabla_X \xi, Y) = -\overline{g}(h^s(X, \phi Y), \phi\xi),$$

from which our assertion follows.

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