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The Construction of the Lemarié-Meyer Smooth Wavelets with Respect to a Riesz Basis

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Abstract

Auscher, Weiss, and Wickerhauser showed that the local orthonormal sine and cosine bases of Coifman and Meyer generate the Lemarié-Meyer smooth wavelets [1]. In this paper we show that more general Riesz basis of Coifman and Meyer type[3] also generate the same wavelets.

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1 Introduction

In [1], Auscher, Weiss, and Wickerhauser showed that the Lemarié-Meyer smooth wavelets is generated by the local sine and cosine orthonormal bases of Coifman and Meyer [2]. In this context, a natural question is whether one can obtain the same result with Riesz basis instead. We use the Riesz basis of Coifman and Meyer type [3], to provide conditions for a positive answer to this question(Theorem 3.4).

2 Riesz basis of Coifman and Meyer type

Throughout this paper, we will consider the usual inner product of functions f, g of $L^2[\alpha, \beta]$ given by

$$< f, g >= \int_{\alpha}^{\beta} f\bar{g}.$$

Choose an even non-negative function ψ with Supp $\psi = [\text{-}\varepsilon,\ \varepsilon]$, normalized so that $\|\psi\|_{L^1} = \frac{\pi}{2}$, and let

$$\theta(x) = \int_{-\infty}^{x} \psi(t)\, dt.$$

We put $c_\varepsilon(x) = cos\theta(x)$ and $s_\varepsilon(x) = sin\theta(x)$ and define the bell function b_I by $b_I = s_\varepsilon(x-\alpha)c_{\varepsilon'}(x-\beta)$. By taking the 'smoother' orthonormal projection P_I associated with the bell function, Auscher, Weiss, and Wickerhauser [1] show that

$$L^2(\mathbb{R}) = \bigoplus_{k\epsilon\mathbb{Z}} P_{I_k} L^2(\mathbb{R})$$

where $\mathbb{R} = \bigcup_{k\epsilon\mathbb{Z}} I_k$ and provide four orthonormal bases. In our previous work [3], we showed how to obtain a Riesz basis from one of such orthonormal bases. More precisely,

Theorem 2.1 *([3], Theorem 3.2) For an interval $I = [0,1]$ and for an orthonormal basis $\{\sqrt{2} b_I(x) \sin \frac{(2n+1)}{2}\pi x\}$ of $P_I L^2(\mathbb{R})$,*

$$\left\| \sum c_n \sqrt{2} b_I(x) \left(\sin\left(\frac{2n+1}{2}\pi x \right) - \sin(\lambda_n x) \right) \right\| \leq 1 + \phi(L).$$

Thus $\{\sqrt{2} b_I(x) \sin \lambda_n x\}$ forms a Riesz basis for $P_I L^2(\mathbb{R})$, if $|\frac{2n+1}{2}\pi - \lambda_n| \leq L$ for some L for which the function $\phi(L) < 0$, where

$$\phi = \phi_A + \phi_B + \phi_C + \phi_D + \phi_E + \phi_F$$

and

$$\begin{aligned}
\phi_A(L) &\equiv -\frac{\sin 2L}{3L} - \frac{\sin L}{3L}, \\
\phi_B(L) &\equiv \frac{1}{2}\left(\frac{\sin 2L \sin \frac{3L}{2}}{\cos \frac{3L}{2}} - \frac{\sin L \sin \frac{3L}{2}}{\cos \frac{3L}{2}} \right), \\
\phi_C(L) &\equiv \frac{1}{2}\left(\frac{2\sin 2L}{3L} + \frac{2\sin L}{3L} - \frac{\sin 2L \cos \frac{3L}{2}}{\sin \frac{3L}{2}} - \frac{\sin L \cos \frac{3L}{2}}{\sin \frac{3L}{2}} \right), \\
\phi_D(L) &\equiv \frac{\cos L - \cos 2L}{3L}, \\
\phi_E(L) &\equiv \frac{1}{2}\left(\frac{\cos 2L \sin \frac{3L}{2}}{\cos \frac{3L}{2}} + \frac{\cos L \sin \frac{3L}{2}}{\cos \frac{3L}{2}} \right), \\
\phi_F(L) &\equiv \frac{1}{2}\left(\frac{2\cos 2L}{3L} - \frac{2\cos L}{3L} - \frac{\cos 2L \cos \frac{3L}{2}}{\sin \frac{3L}{2}} + \frac{\cos L \cos \frac{3L}{2}}{\sin \frac{3L}{2}} \right).
\end{aligned}$$

The largest L is about 0.3788

We can extend this result to $P_I L^2(\mathbb{R})$ for any interval $I = [\alpha, \beta]$, that is, under the same condition given in Theorem 2.1, the family $\{\sqrt{\frac{2}{|I|}} b_I(x) \sin \frac{\lambda_n}{|I|}(x - \alpha)\}$ forms a Riesz basis for $P_I L^2(\mathbb{R})$ which is 'close' to the orthonormal basis $\{\sqrt{\frac{2}{|I|}} b_I(x) \sin \frac{2n+1}{2} \frac{\pi}{|I|}(x-\alpha)\}$. Similarly, under the same condition,we also have a local cosine type Riesz basis $\{\sqrt{\frac{2}{|I|}} b_I(x) \cos \frac{\lambda_n}{|I|}(x-\alpha)\}$ regards to the orthonormal basis $\{\sqrt{\frac{2}{|I|}} b_I(x) \cos \frac{2n+1}{2} \frac{\pi}{|I|}(x-\alpha)\}$ of $P_I L^2(\mathbb{R})$ for any interval $I = [\alpha, \beta]$.

3 The construction of Lemarié-Meyer smooth wavelets by using a Riesz basis

The Lemarié and Meyer wavelet basis [4] is given by

$$\{w_{k,n}(x)\} = \{2^{-\frac{k}{2}} w(2^{-k}x - n)\}, \quad k, n \in \mathbb{Z},$$

where $w \in S(\mathbb{R})$ and Supp $\hat{w} = [-\frac{8\pi}{3}, -\frac{2\pi}{3}] \cup [\frac{8\pi}{3}, \frac{2\pi}{3}]$. Auscher, Weiss, and Wickerhauser [1] show that

Theorem 3.1 *([1], Theorem 6) The collection of functions*

$$\alpha_{k,n} \equiv \Psi_{k,n} + i\Phi_{k,n} \qquad \text{and} \qquad \beta_{k,n} \equiv \Phi_{k,n} + i\Psi_{k,n},$$

$k = 0, \pm 1, \pm 2, \cdots, n = 0, 1, 2, \cdots$, *is an orthonormal basis for* $L^2(\mathbb{R})$, *where*

$$\Psi_{k,n}(\xi) = \sqrt{\frac{1}{2\pi}} 2^{\frac{k}{2}} b(2^k|\xi|) \cos \frac{2n+1}{2}(2^k|\xi| - \pi)$$

and

$$\Phi_{k,n}(\xi) = \sqrt{\frac{1}{2\pi}} 2^{\frac{k}{2}} b(2^k|\xi|)(sgn\xi) \sin \frac{2n+1}{2}(2^k|\xi| - \pi).$$

Now we generate a Riesz basis which is close to the orthonprmal basis given in the Theorem 3.1. For $I = [\pi, 2\pi]$, $\varepsilon = \frac{\pi}{3}$, and $\varepsilon' = \frac{2\pi}{3}$, by Theorem 2.1 and with condition on it, we obtain that

$$\psi^*(n;\xi) = \sqrt{\frac{2}{\pi}} b_I(x) \cos \frac{\lambda_n}{\pi}(\xi - \pi)(\xi - \pi), \quad n = 0, 1, 2, \cdots,$$

is a Riesz basis for $P_I L^2(\mathbb{R})$, consequently, with dilation by 2^k,

$$\psi_k^*(n;\xi) \equiv 2^{\frac{k}{2}} \psi^*(n; 2^k\xi), \quad k \in \mathbb{Z}. \tag{1}$$

forms a Riesz basis of $L^2(0,\infty)$. A completely analogous construction based on the Riesz basis of local sine type

$$\varphi^*(n;\xi) = \sqrt{\frac{2}{\pi}} b_I(x) \sin \frac{\lambda_n}{\pi}(\xi - \pi), \quad n = 0, 1, 2, \cdots,$$

gives us a Riesz basis of $L^2(0,\infty)$:

$$\varphi_k^*(n;\xi) \equiv 2^{\frac{k}{2}} \varphi^*(n; 2^k\xi), \quad k \,\epsilon\, \mathbb{Z}. \tag{2}$$

Then by using even extensions of the functions (1) and odd extensions of the functions (2), we obtain our first result:

Theorem 3.2 *For $k \,\epsilon\, \mathbb{Z}$ and $n = 0, 1, 2, \cdots$, let*

$$\Psi_{k,n}^*(\xi) = \sqrt{\frac{1}{2\pi}} 2^{\frac{k}{2}} b(2^k|\xi|) \cos \frac{\lambda_n}{\pi}(2^k|\xi| - \pi)$$

and

$$\Phi_{k,n}^*(\xi) = \sqrt{\frac{1}{2\pi}} 2^{\frac{k}{2}} b(2^k|\xi|)(sgn\xi) \sin \frac{\lambda_n}{\pi}(2^k|\xi| - \pi).$$

Then the collection of functions

$$\alpha_{k,n}^* \equiv \Psi_{k,n}^* + i\Phi_{k,n}^* \qquad and \qquad \beta_{k,n}^* \equiv \Phi_{k,n}^* + i\Psi_{k,n}^*$$

forms a Riesz basis on $L^2(\mathbb{R})$ if $|\frac{2k+1}{2}\pi - \lambda_k| \le L$ for some L which makes $1 - \frac{\sqrt{2}}{4} + \phi(L)$ negative (ϕ is from Theorem 2.1). The largest L is about 0.1542.

Proof : To use the Paley-Wiener Theorem [5, Chapter 1, Theorem 13], we need to find the condition for L such that $|\frac{2k+1}{2}\pi - \lambda_k| \le L$ and

$$\left\| \sum_{k\epsilon\mathbb{Z}} \sum_{n=0}^{\infty} c_{k,n}(\alpha_{k,n} - \alpha_{k,n}^*) + d_{k,n}(\beta_{k,n} - \beta_{k,n}^*) \right\|_{L^2(\mathbb{R})} < 1, \tag{3}$$

whenever $\sum_{k\epsilon\mathbb{Z}} \sum_{n=0}^{\infty} |c_{k,n}|^2 + \sum_{k\epsilon\mathbb{Z}} \sum_{n=0}^{\infty} |d_{k,n}|^2 \le 1$.

First we rewrite the left-hand side of (3) as $K_1 + K_2 + K_3 + K_4$ where

$$\begin{aligned}
K_1 &= \left\| \sum_{k\epsilon\mathbb{Z}} \sum_{n=0}^{\infty} c_{k,n}(\Psi_{k,n} - \Psi_{k,n}^*) \right\|_{L^2(\mathbb{R})}, \\
K_2 &= \left\| \sum_{k\epsilon\mathbb{Z}} \sum_{n=0}^{\infty} c_{k,n}(\Phi_{k,n} - \Phi_{k,n}^*) \right\|_{L^2(\mathbb{R})}, \\
K_3 &= \left\| \sum_{k\epsilon\mathbb{Z}} \sum_{n=0}^{\infty} d_{k,n}(\Phi_{k,n} - \Phi_{k,n}^*) \right\|_{L^2(\mathbb{R})}, \\
K_4 &= \left\| \sum_{k\epsilon\mathbb{Z}} \sum_{n=0}^{\infty} d_{k,n}(\Psi_{k,n} - \Psi_{k,n}^*) \right\|_{L^2(\mathbb{R})}.
\end{aligned}$$

Since functions $\Psi_{k,n} - \Psi^*_{k,n}$ and $\Phi_{k,n} - \Phi^*_{k,n}$ in above norms are even, $K_i \leq \frac{1}{\sqrt{2}} J_i$, for $i = 1, 2, 3$, and 4, where

$$\begin{aligned}
J_1 &= \left\| \sum_{k \epsilon \mathbb{Z}} \sum_{n=0}^{\infty} c_{k,n} (\psi_{k,n}(n;\xi) - \psi^*_{k,n}(n;\xi)) \right\|_{L^2(0,\infty)}, \\
J_2 &= \left\| \sum_{k \epsilon \mathbb{Z}} \sum_{n=0}^{\infty} c_{k,n} (\varphi_{k,n}(n;\xi) - \varphi^*_{k,n}(n;\xi)) \right\|_{L^2(0,\infty)}, \\
J_3 &= \left\| \sum_{k \epsilon \mathbb{Z}} \sum_{n=0}^{\infty} d_{k,n} (\psi_{k,n}(n;\xi) - \psi^*_{k,n}(n;\xi)) \right\|_{L^2(0,\infty)}, \\
J_4 &= \left\| \sum_{k \epsilon \mathbb{Z}} \sum_{n=0}^{\infty} d_{k,n} (\varphi_{k,n}(n;\xi) - \varphi^*_{k,n}(n;\xi)) \right\|_{L^2(0,\infty)}.
\end{aligned}$$

By Theorem 2.1, we have that each $J_i \leq 1 + \phi(L)$ for $i = 1, 2, 3$, and 4. Then we obtain that

$$\frac{1}{\sqrt{2}}(J_1 + J_2 + J_3 + J_4) \leq \frac{4}{\sqrt{2}}(1 + \phi(L)).$$

Thus, by the Paley-Wiener Theorem, if $\frac{4}{\sqrt{2}}(1 + \phi(L)) < 1$, the collection of functions $\alpha^*_{k,n}$ and $\beta^*_{k,n}$ forms a Riesz basis for $L^2(\mathbb{R})$. $\square$.

Thus if λ_n is close to n with certain condition, then the collection of $\alpha^*_{k,n}$ and $\beta^*_{k,n}$ forms a Riesz basis. We now modify this Riesz basis to get another Riesz basis for $L^2(\mathbb{R})$ which is generated by the single function $\gamma^*(\xi) \equiv i\alpha^*_{0,0}(\xi)$. In this construction, it requires certain restriction to λ_n. The trick is that we apply λ_n differently by sign of n.

Theorem 3.3 *For* $0 < \delta \leq L$ *and*

$$\lambda_n = \begin{cases} \frac{2n+1}{2}\pi - \delta & if \quad n \geq 0 \\ \frac{2n+1}{2}\pi + \delta & if \quad n < 0 \end{cases}$$

The functions

$$\gamma^*_{k,n}(\xi) \equiv 2^{\frac{k}{2}} e^{-i2^k n\xi} \gamma^*(2^k \xi), \quad k, n \;\epsilon\; \mathbb{Z},$$

form a Riesz basis of $L^2(\mathbb{R})$. *(L is from the Theorem 3.2 and the largest L is about 0.1542).*

Proof : We define $\alpha^*_{k,-n}(\xi) = \overline{\alpha^*_{k,n-1}(\xi)} = -i\beta^*_{k,n-1}(\xi)$ for $k \;\epsilon\; \mathbb{Z}$ and $n > 0$. Then by Theorem 3.2, $\{\alpha^*_{k,n}\}_{k,n\epsilon\mathbb{Z}}$ forms a Riesz basis for $L^2(\mathbb{R})$. We will be done once we show that

$$\alpha^*_{0,n}(\xi) = (-1)^n(-i)e^{in\xi}\gamma^*(\xi), \quad n \;\epsilon\; \mathbb{Z}. \tag{4}$$

First if $n = 0$, then λ_0 is $\frac{1}{2}\pi - \delta$, thus we have

$$\gamma^*(\xi) = i\alpha^*_{0,0}(\xi) = \frac{b(|\xi|)}{\sqrt{2\pi}} e^{-i\frac{\delta}{\pi}\xi} e^{i\frac{\xi}{2}} (sgn\xi) e^{i(sgn\xi)\delta}.$$

Now assume n is nonzero, we define λ_n differently by sign of n. For $n \geq 0$, we let $\lambda_n = \frac{2n+1}{2}\pi - \delta$, then we have

$$\begin{aligned}
\alpha^*_{0,n}(\xi) &= i(\Psi^*_{0,n} + i\Phi^*_{0,n})(\xi) \\
&= i\frac{b(|\xi|)}{\sqrt{2\pi}} \left(\cos\frac{\lambda_n}{\pi}(|\xi| - \pi) + i(sgn\xi)\sin\frac{\lambda_n}{\pi}(|\xi| - \pi) \right) \\
&= \frac{b(|\xi|)}{\sqrt{2\pi}} e^{i\frac{2n+1}{2}\xi} e^{-i\frac{\delta}{\pi}\xi} \begin{cases} e^{-i(\frac{2n+1}{2}\pi - \delta)} & if \quad \xi \geq 0 \\ e^{i(\frac{2n+1}{2}\pi - \delta)} & if \quad \xi < 0 \end{cases} \\
&= (-1)^n (sgn\xi)(-i) e^{in\xi} \frac{b(|\xi|)}{\sqrt{2\pi}} e^{i\frac{\xi}{2}} e^{-i\frac{\delta}{\pi}\xi} e^{i(sgn\xi)\delta} \\
&= (-1)^n (-i) e^{in\xi} \gamma^*(\xi).
\end{aligned}$$

If $n < 0$, we define $\lambda_n = \frac{2n+1}{2}\pi + \delta$,

$$\begin{aligned}
\alpha^*_{0,n}(\xi) &= \overline{\alpha^*_{0,-n-1}(\xi)} \\
&= \overline{\frac{b(|\xi|)}{\sqrt{2\pi}} e^{i(-n-1)\xi} e^{\frac{i\xi}{2}} e^{i\frac{\delta}{\pi}\xi} e^{i(sgn\xi)(n+1)\pi} e^{\frac{-i\pi(sgn\xi)}{2}} e^{-i(sgn\xi)\delta}} \\
&= \frac{b(|\xi|)}{\sqrt{2\pi}} e^{i(n+1)\xi} e^{\frac{-i\xi}{2}} e^{-i\frac{\delta}{\pi}\xi} (-1)^{n+1} i(sgn\xi) e^{i(sgn\xi)\delta} \\
&= (-1)^n (-i) e^{in\xi} \gamma^*(\xi).
\end{aligned}$$

Thus (4) holds for all integers n. $\square$

To get our main results, we define w^* by

$$w^*(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} \gamma^*(\xi) d\xi,$$

then we have $\widehat{w^*} = \sqrt{2\pi}\gamma^*$ and for $k, n \,\epsilon\, \mathbb{Z}$,

$$w^*_{k,n} = 2^{-\frac{k}{2}} w^*(2^{-k}x - n) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} \gamma^*_{k,n}(\xi) d\xi.$$

Theorem 3.4 *$\{w^*_{k,n}\}$, $k, n \epsilon \mathbb{Z}$, is a Riesz basis for $L^2(\mathbb{R})$, and generates a Lemarié - Meyer wavelet basis.*

Proof : By the Plancherel theorem and Theorem 3.3,

$$\frac{A}{2\pi}\|f\|^2 \leq \frac{1}{2\pi}\sum \left|< \widehat{f}, \gamma^*_{k,n} >\right|^2 = \sum \left|< f, w^*_{k,n} >\right|^2 \leq \frac{B}{2\pi}\|f\|^2,$$

where A and B are frame bounds for a Riesz basis $\gamma^*_{k,n}(\xi)$. Thus $\{w^*_{k,n}\}$ is a frame. To show that $\{w^*_{k,n}\}$ is a Riesz basis in $L^2(\mathbb{R})$, it remains to verify that it is complete. For any f in $L^2(\mathbb{R})$, suppose that $< f, w^*_{k,n} > = 0$ for all $k, n \epsilon \mathbb{Z}$, then

$$\begin{aligned} 0 &=< f, w^*_{k,n} > \\ &= \frac{1}{2\pi} < \widehat{f}, \widehat{w^*_{k,n}} > \\ &= \frac{1}{\sqrt{2\pi}} < \widehat{f}, \gamma^*_{k,n} > . \end{aligned}$$

Thus $< \widehat{f}, \gamma^*_{k,n} >= 0$ for all $k, n \epsilon \mathbb{Z}$. Since $\{\gamma^*_{k,n}\}$ is a Riesz basis for $L^2(\mathbb{R})$, $\widehat{f}$ $= 0$, therefore $f = 0$.

Since Supp $\widehat{w^*} =$ Supp $b(|\xi|) = [-\frac{8\pi}{3}, -\frac{2\pi}{3}] \cup [\frac{8\pi}{3}, \frac{2\pi}{3}]$, w^* is a mother function of the type that we mentioned at the beginning of the section. Thus w^* generates a Lemarié - Meyer wavelets. □

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