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Compactness and Norm of the Sum of Weighted Composition Operators on $A(\mathbb{D})$

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Abstract. Let $A(\mathbb{D})$ denote the disk algebra and uC_φ be the weighted composition operator on $A(\mathbb{D})$. In this paper, we characterize the compactness of sum of weighted composition operators on $A(\mathbb{D})$. Furthermore, we also find the norm of sum of weighted composition operators on $A(\mathbb{D})$.

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1. Introduction

Let $\mathbb{D}$ denote the standard unit disk in the complex plane $\mathbb{C}$ and $\bar{\mathbb{D}}$ be its closure in $\mathbb{C}$. Let $H^\infty = H^\infty(\mathbb{D})$ be the space of all bounded analytic functions on the open unit disk $\mathbb{D}$. Then H^∞ is a Banach algebra with the supremum norm

$$\|f\|_\infty = \sup\{|f(z)| : z \in \mathbb{D}\}.$$

The disk algebra $A(\mathbb{D})$ is the Banach algebra of all continuous functions on $\bar{\mathbb{D}}$ which are analytic in $\mathbb{D}$, under the supremum norm

$$\|f\| = \sup\left\{|f(z)| : z \in \overline{\mathbb{D}}\right\}.$$

By the maximum modulus principle, we can see that this norm is equal to $\|f\|_\infty$. For functions u and $\varphi \in A(\mathbb{D})$ with $\|\varphi\| \leq 1$, we define a weighted composition operator uC_φ on $A(\mathbb{D})$ by

$$(uC_\varphi f)(z) = u(z) f(\varphi(z)) \quad \forall\ z \in \bar{\mathbb{D}}.$$

It is easy to see that uC_φ is a bounded linear operator and $\|uC_\varphi\| = \|u\|$. When $u(z) \equiv 1$, we just have the composition operator C_φ on $A(\mathbb{D})$. For z and $w \in \bar{\mathbb{D}}$, the pseudo-hyperbolic distance between z and w is given by

$$\rho(z,w) = \begin{cases} \left|\dfrac{z-w}{1-\overline{z}w}\right|, & \text{if } z \neq w \\ 0, & \text{if } z = w. \end{cases}$$

Weighted composition operators appear in the literature in different ways. For example, the endomorphisms of semi-simple commutative Banach algebras can be represented as weighted composition operators. Hence these operators, and also their sums, are interesting to study from different point of views such as compactness, norm and so on. The disk algebra is an illuminating example of semi-simple commutative Banach algebra. Therefore it is instructive to study weighted composition operators and their sums on it. The aim of this paper is to clarify the compactness and norm of sum of weighted composition operators on the disk algebra.

The various properties of weighted composition operators have been extensively studied on Hardy spaces H^p into itself in [2], [12]. In the H^∞ setting, some results were obtained in [3], [13]. These operators have been studied in other spaces of analytic functions as well (See for [11], [1], [8], [9] and references therein). Operator theoretic properties of a weighted composition operator depends canonically on function theoretic properties of u and φ. Consequently, the study of weighted composition operator uC_φ on the spaces of analytic functions is a meeting point between functional analysis and analytic function theory. This interplay has resulted in deep and profound theorems which have illuminated the whole of operator theory. This apart, these operators are also source of rich variety of examples of linear operators.

In 1979, Herbert Kamowitz characterized the compactness of weighted composition operators on $A(\mathbb{D})$ and studied their spectra [10]. We state his result as follows.

Theorem 1.1. *Let u, φ in $A(\mathbb{D})$ with $\|\varphi\| \leq 1$ and suppose φ is a non-constant function. Then uC_φ is a compact operator on $A(\mathbb{D})$ if and only if $|\varphi(z)| < 1$ whenever $u(z) \neq 0$.*

In [6] Gorkin and Mortini studied norms and essential norms of finite linear combinations of composition operators on uniform algebras. Further, Izuchi and S.Ohno have characterized compactness of finite linear combinations of compostion operators on H^∞ in [9]. In this article we study properties of the sum of weighted composition operators on $A(\mathbb{D})$. In section 2, we give a characterization of compact operators which are the sum of weighted composition operators on $A(\mathbb{D})$. Furthermore, we also give a necessary and sufficient condition under which the norm of the infinite sum of weighted composition operators on $A(\mathbb{D})$ is equal to the sum of norm of each component in section 3.

In what follows, we denote by $\mathbb{T}$ the boundary of the unit disk $\mathbb{D}$ and $\mathcal{S}(\mathbb{D})$ the set of all analytic self map of $\mathbb{D}$. We refer to the reader to [4], [5], [7] and [14] for the results about the Banach spaces of analytic functions. For function $u \in H^\infty$ and $\varphi \in \mathcal{S}(\mathbb{D})$, we define a weighted composition operator uC_φ on

H^∞ by $uC_\varphi f = u(f\circ\varphi)$ for $f \in H^\infty$. It is clear that uC_φ is linear and bounded on H^∞ and its properties were investigated in [3] and [13].

2. COMPACTNESS

In this section we study the compactness of those operators which can be expressed as a sum of weighted composition operators on $A(\mathbb{D})$. It is well known that a composition operator C_φ on, say a Banach space X of analytic functions in the disk , is compact if and only if whenever $\{f_m\}$ is a bounded sequence in X and $f_m \to 0$ uniformly on compact subsets of $\mathbb{D}$, then $\|C_\varphi(f_m)\|_X \to 0$. It can be readly seen that the above result also holds for those compact operators which are sum of weighted composition operators on H^∞. The particular instance of this result, which we will need, is the following proposition.

Proposition 2.1. *Let $\varphi_1, \varphi_2, \dots, \varphi_N$ be distinct functions in $\mathcal{S}(\mathbb{D})$ and $u_1, u_2, \dots, u_N$ in H^∞ with $u_i \neq 0$ for every i. Then $\sum_{i=1}^{N} u_i C_{\varphi_i}$ is compact on H^∞ if and only if whenever $\{f_n\}_n$ is a bounded sequence in H^∞ such that $\{f_n\}_n \to 0$ uniformly on every compact subsets of $\mathbb{D}$ then $\left\|\sum_{i=1}^{N} u_i C_{\varphi_i} f_n\right\|_\infty \to 0$ as $n \to \infty$.*

Proof. The proof is the special case of the proof of proposition 3.11 in [4]. $\square$

Let $\varphi_1, \varphi_2, \dots, \varphi_N$ be distinct functions in $\mathcal{S}(\mathbb{D})$ and $N \geq 2$. Let $\mathcal{Z} = \mathcal{Z}(\varphi_1, \varphi_2, \dots, \varphi_N)$ be the family of sequences $\{z_n\}_n$ in $\mathbb{D}$ satisfying the following three conditions:
(a) $|\varphi_i(z_n)| \to 1$ as $n \to \infty$ for some i,
(b) $\{\varphi_i(z_n)\}_n$ is convergent sequence for every i,
(c) $\left\{\dfrac{\varphi_j(z_n) - \varphi_i(z_n)}{1 - \overline{\varphi_j(z_n)}\varphi_i(z_n)}\right\}_n$ is a convergent sequence for every i, j.
Condition (c) implies that $\{\rho(\varphi_i(z_n), \varphi_j(z_n))\}_n$ is a convergent sequence for every i, j.
Note that if $|\varphi_i(z_n)| \to 1$ as $n \to \infty$ for some i, then it is easy to see that there exists a subsequence $\{z_{n_j}\}_j$ of $\{z_n\}_n$ satisfying $\{z_{n_j}\}_j \in \mathcal{Z}$.
For $\{z_n\}_n \in \mathcal{Z}$, we write $I(\{z_n\}) = \{i : 1 \leq i \leq N, |\varphi_i(z_n)| \to 1 \text{ as } n \to \infty\}$.
By condition (a), $I(\{z_n\}) \neq \varphi$. By (b), there exists δ with $0 < \delta < 1$ such that $|\varphi_j(z_k)| < \delta < 1$ for every $j \notin I(\{z_n\})$ and for every k. For each $t \in I(\{z_n\})$, we write

$$I_0(\{z_n\}, t) = \{j \in I(\{z_n\}) : \rho(\varphi_j(z_n), \varphi_t(z_n)) \to 0 \text{ as } n \to \infty\}. \tag{2.1}$$

For $s, t \in I(\{z_n\})$, we have either $I_0(\{z_n\}, s) = I_0(\{z_n\}, t)$ or $I_0(\{z_n\}, s) \cap I_0(\{z_n\}, t) = \varphi$. Hence there is a subset $\{t_1, t_2 \dots, t_l\} \subset I(\{z_n\})$ such that $I(\{z_n\}) = \bigcup_{p=1}^{l} I_0(\{z_n\}, t_p)$ and $I_0(\{z_n\}, t_p) \cap I_0(\{z_n\}, t_q) = \varphi$, for $p \neq q$.

We begin with the following theorem which generalises the result given by Ohno and Izuchi in [9].

Theorem 2.1. *Let $\varphi_1, \varphi_2, \ldots\ldots, \varphi_N$ be distinct functions in $\mathcal{S}(\mathbb{D})$ and $u_1, u_2, \ldots, u_N$ in H^∞ with $u_i \neq 0$ for every i. Then the following conditions are equivalent.*

(i) $\displaystyle\sum_{i=1}^{N} u_i C_{\varphi_i}$ *is compact on H^∞.*

(ii) $\displaystyle\sum_{i \in I_0(\{z_n\},t)} u_i(z_n) \to 0$ *as $n \to \infty$ for every $\{z_n\}_n \in \mathcal{Z}$ and $t \in I(\{z_n\})$.*

Proof. The idea of the proof of this theorem is same as that of Theorem 2.2 in [9]. $\square$

In the next theorem we show that compactness of finite sum of weighted compostition operator on $A(\mathbb{D})$ is equivalent to that on H^∞.

Theorem 2.2. *Let $\varphi_1, \varphi_2, \ldots\ldots, \varphi_N$ be distinct functions in $A(\mathbb{D})$ with $\|\varphi_i\| = 1$ for each i and $u_1, u_2, \ldots, u_N$ in $A(\mathbb{D})$ with $u_i \neq 0$ for every i. Then the following is equivalent.*

(i) $\sum_{i=1}^{N} u_i C_{\varphi_i}$ *is compact on H^∞.*

(ii) $\sum_{i=1}^{N} u_i C_{\varphi_i}$ *is compact on $A(\mathbb{D})$.*

Proof. $(i) \Rightarrow (ii)$. Suppose $\sum_{i=1}^{N} u_i C_{\varphi_i}$ is compact on H^∞. We claim that $\sum_{i=1}^{N} u_i C_{\varphi_i}$ is compact on $A(\mathbb{D})$. Let $\{f_n\}$ be a bounded sequence in $A(\mathbb{D})$. Since $A(\mathbb{D}) \subset H^\infty$, hence we get a subsequence $\{f_{n_k}\}$ and a function $g \in H^\infty$ such that $\lim_{k\to\infty} \left(\sum_{i=1}^{N} u_i C_{\varphi_i}\right) f_{n_k} = g$. Clearly $g \in A(\mathbb{D})$. Thus $\sum_{i=1}^{N} u_i C_{\varphi_i}$ is compact on $A(\mathbb{D})$.

$(ii) \Rightarrow (i)$ Suppose that $\sum_{i=1}^{N} u_i C_{\varphi_i}$ is not compact on H^∞. Then for some $\epsilon > 0$ there exists $\{f_n\}_n \subset$ ball H^∞ such that $f_n \to 0$ uniformly on compact subset of $\mathbb{D}$ and $\left\|\left(\sum_{i=1}^{N} u_i C_{\varphi_i}\right) f_n\right\|_\infty > \epsilon$ for all n. Hence there is $\{z_n\}_n \subset \mathbb{D}$ such that $\left|\sum_{i=1}^{N} u_i(z_n) f_n(\varphi_i(z_n))\right| > \epsilon$. For $0 < r < 1$, define $f_{n,r}(z) = f_n(rz)$. Then $f_{n,r} \in$ ball $A(\mathbb{D})$. There exists $\{r_n\}_n, 0 < r_n < 1$, such that $r_n \to 1$ and $\left|\sum_{i=1}^{N} u_i(z_n) f_{n,r_n}(\varphi_i(z_n))\right| > \epsilon$ for all n. Since $f_{n,r_n} \to 0$ uniformly on compact subset of $\mathbb{D}$, it follows that $\sum_{i=1}^{N} u_i C_{\varphi_i}$ is not compact on $A(\mathbb{D})$. $\square$

The following theorem gives a necessary condition for compactness of infinite sum of weighted composition operators on disk algebra.

Theorem 2.3. *Let* $\{\varphi_n\}_{n=1}^{\infty}$ *be distinct functions in* $A(\mathbb{D})$ *with* $\|\varphi_n\| = 1$ *for each* $n \geq 1$ *and* $\{u_n\}_{n=1}^{\infty}$ *a sequence in* $A(\mathbb{D})$ *with* $\sum\limits_{n=1}^{\infty} \|u_n\| < \infty$. *If* $\sum\limits_{n=1}^{\infty} u_n C_{\varphi_n}$ *is compact on* $A(\mathbb{D})$ *then* $\sum\limits_{n \in \mathcal{N}(\{z_o\},k)} u_n(z_o) = 0$ *for every* $z_o \in \mathrm{Z}$ *and for every* $k \in \mathcal{N}(\{z_o\})$, *where* $\mathrm{Z} = \{z \in \mathbb{T} : |\varphi_n(z)| = 1$ *for some* $n\}$, $\mathcal{N}(\{z_o\}) = \{n \geq 1 : |\varphi_n(z_o)| = 1\}$ *and* $\mathcal{N}(\{z_o\}, k) = \{m \in \mathcal{N}(\{z_o\}) : \varphi_k(z_o) = \varphi_m(z_o)\}$.

Proof. Suppose $\sum\limits_{n=1}^{\infty} u_n C_{\varphi_n}$ is compact on $A(\mathbb{D})$. We claim that $\sum\limits_{n \in \mathcal{N}(\{z_o\},k)} u_n(z_o) = 0$ for every $z_o \in \mathrm{Z}$ and for every $k \in \mathcal{N}(\{z_o\})$. Let $z_o \in \mathrm{Z}$ and $k \in \mathcal{N}(\{z_o\})$. For each positive integer m define f_m as $f_m(z) = \left(\dfrac{z + \varphi_k(z_o)}{2\varphi_k(z_o)}\right)^m$. Then $f_m \in A(\mathbb{D})$, $\|f_m\| = 1$ and $f_m \to 0$ uniformly on every compact subset of $\mathbb{D}$. Since $\sum\limits_{n=1}^{\infty} u_n C_{\varphi_n}$ is compact on $A(\mathbb{D})$, there exists a subsequence $\{f_{m_i}\}$ and a function F in $A(\mathbb{D})$ such that $\left(\sum\limits_{n=1}^{\infty} u_n C_{\varphi_n}\right) f_{m_i} \to F$ in $A(\mathbb{D})$. That is $\sum\limits_{n=1}^{\infty} u_n f_{m_i}(\varphi_n(z)) \to F(z)$ uniformly for $z \in \bar{\mathbb{D}}$. But $f_{m_i}(\varphi_n(z)) \to 0$ for $|z| < 1$ and $\sum\limits_{n=1}^{\infty} \|u_n\|$ is uniformly bounded. So $F(z) = 0$ on $\mathbb{D}$. However, as F is continuous on $\bar{\mathbb{D}}$, therefore $F(z) = 0$ on $\bar{\mathbb{D}}$. Hence $\left(\sum\limits_{n=1}^{\infty} u_n C_{\varphi_n}\right) f_{m_i} \to 0$

uniformly on $\bar{\mathbb{D}}$. In particular, $\lim\limits_{i\to\infty}\sum\limits_{n=1}^{\infty} u_n(z_o) f_{m_i}(\varphi_n(z_o)) = 0$. Now interchanging limits, which is permissible under the given conditions, we get

$$\begin{aligned}
0 = \lim_{i\to\infty}\sum_{n=1}^{\infty} u_n(z_o) C_{\varphi_n} f_{m_i}(z_o) &= \lim_{i\to\infty}\sum_{n=1}^{\infty} u_n(z_o) f_{m_i}(\varphi_n(z_o)) \\
&= \lim_{n\to\infty}\lim_{i\to\infty}\sum_{j=1}^{n} u_j(z_o) f_{m_i}(\varphi_j(z_o)) \\
&= \lim_{n\to\infty}\lim_{i\to\infty}\sum_{j=1}^{n} u_j(z_o)\left(\frac{\varphi_j(z_o)+\varphi_k(z_o)}{2\varphi_k(z_o)}\right)^{m_i} \\
&= \lim_{n\to\infty}\sum_{\substack{j=1\\ j\in\mathcal{N}(\{z_o\},k)}}^{n} u_j(z_o) \\
&= \sum_{\substack{n=1\\ j\in\mathcal{N}(\{z_o\},k)}}^{\infty} u_n(z_o).
\end{aligned}$$

Thus $\sum\limits_{n=1}^{\infty} u_n C_{\varphi_n}$ is compact on $A(\mathbb{D})$ implies that $\sum\limits_{n\in\mathcal{N}(\{z_o\},k)} u_n(z_o) = 0$ for every $z_o \in \mathrm{Z}$ and for every $k \in \mathcal{N}(\{z_o\})$. $\square$

The converse of the above theorem is not true and we give the following example to show this.

Example 2.1. Let $\varphi_1(z) = \frac{z+1}{2}, \varphi_2(z) = \frac{z+2}{3}, u_1(z) = 1$ and $u_2(z) = -1$. Then $\mathrm{Z} = \{1\}$, $\mathcal{N}(\{1\}) = \{1,2\}$, $\mathcal{N}(\{1\},1) = \{1,2\}$ and $\sum\limits_{n\in\mathcal{N}(\{1\},1)} u_n(1) = 0$. But $u_1C_{\varphi_1} + u_2C_{\varphi_2}$ is not compact on H^∞. Therefore by Theorem 2.2, $u_1C_{\varphi_1} + u_2C_{\varphi_2}$ is not compact on $A(\mathbb{D})$.

The following corollaries are simple consequences of Theorem 2.3.

Corollary 2.1. Let $\varphi_1, \varphi_2, \varphi_3, \dots, \varphi_K$ $(K \geq 2)$ be distinct functions in $A(\mathbb{D})$ with $\|\varphi_n\| = 1$ and $u_1, u_2, u_3, \dots, u_K$ in $A(\mathbb{D})$ with $u_n \neq 0$ for every n, $1 \leq n \leq K$. If there exists a point $z_o \in \mathrm{Z}$ such that $\sum\limits_{n\in J} u_n(z_o) \neq 0$ for every subset J of $\{1,2,\dots,K\}$, then $\sum\limits_{n=1}^{K} u_n C_{\varphi_n}$ is not compact on $A(\mathbb{D})$.

Corollary 2.2. Let $\varphi_1, \varphi_2, \dots, \varphi_K$ $(K \geq 2)$ be distinct functions in $A(\mathbb{D})$ with $\|\varphi_n\| = 1$ and $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_K \in \mathbb{C}$ with $\lambda_n \neq 0$ for every n, $1 \leq n \leq K$.

If $\sum\limits_{n\in J} \lambda_n \neq 0$ for every subset J of $\{1,2,\dots,K\}$, then $\sum\limits_{n=1}^{K} \lambda_n C_{\varphi_n}$ is not compact on $A(\mathbb{D})$.

Thus the sum $\sum_{n=1}^{K} C_{\varphi_n}$ is never compact on $A(\mathbb{D})$ provided $\|\varphi_n\| = 1$ for each n, $1 \le n \le K$.

An example of the non-compact operator of the form $\sum_{n=1}^{\infty} u_n C_{\varphi_n}$ on $A(\mathbb{D})$ is the following.

Example 2.2. Let $\varphi_n(z) = \frac{z+(-1)^{n-1}n}{n+1}$.Then Z $= \{1, -1\}$, $\mathcal{N}(\{1\}) = \{1, 3, 5, \ldots\ldots\}$, $\mathcal{N}(\{-1\}) = \{2, 4, 6, \ldots\ldots\}$,$\mathcal{N}(\{1\}, k) = \{1, 3, 5, \ldots\ldots\}$ for every $k \in \mathcal{N}(\{1\})$ and $\mathcal{N}(\{-1\}, l) = \{2, 4, 6, \ldots\ldots\}$ for every $l \in \mathcal{N}(\{-1\})$.
For $n \ge 1$, define u_n as follows:
$u_{2n-1}(z) = \frac{z}{2^n}$ and $u_{2n}(z) = -\frac{z}{2^n}$.
Since $\sum_{n \in \mathcal{N}(\{-1\}, l)} u_n(-1) \neq 0$, hence by Theorem 2.3 the operator $\sum_{n=1}^{\infty} u_n C_{\varphi_n}$ is non-compact on $A(\mathbb{D})$.

3. Norm

In this section we estimate the norm of sum of weighted composition operators on $A(\mathbb{D})$. Let $\varphi_1, \varphi_2, \ldots, \varphi_K$ $(K \ge 2)$ be distinct functions in $A(\mathbb{D})$ with $\|\varphi_n\| \le 1$ and $u_1, u_2, u_3, \ldots, u_K$ in $A(\mathbb{D})$ with $u_n \neq 0$ for every n, $1 \le n \le K$. Let B $= \{z \in \mathbb{T} : \|u_n\| = |u_n(z)|$, for all n , $1 \le n \le K\}$ and Z $= \{z \in \mathbb{T} : |\varphi_n(z)| = 1, \text{for some } n\}$.

Case 1. If B $= \phi$, then $\left\| \sum_{n=1}^{K} u_n C_{\varphi_n} \right\| < \sum_{n=1}^{K} \|u_n\|$.

Proof. Let $f \in A(\mathbb{D})$ with $\|f\| \le 1$. Then

$$\begin{aligned}
\left\| \sum_{n=1}^{K} u_n C_{\varphi_n} f \right\| &= \sup_{|z| \le 1} \left| \sum_{n=1}^{K} u_n(z) f(\varphi_n(z)) \right| \\
&\le \sup_{|z| \le 1} \sum_{n=1}^{K} |u_n(z)| \\
&= \sup_{|z| \le 1} \{ |u_1(z)| + |u_2(z)| + \cdots + |u_K(z)| \}.
\end{aligned}$$

Since $|u_1(z)| + |u_2(z)| + \cdots + |u_K(z)|$ is continuous, this implies there exists a point $z_o \in \mathbb{T}$ such that $\sup_{|z| \le 1} \{|u_1(z)| + |u_2(z)| + \cdots + |u_K(z)|\} = |u_1(z_o)| + |u_2(z_o)| + \cdots + |u_K(z_o)|$.
Now B $= \phi$ implies that exists a u_n such that $|u_n(z_o)| < \|u_n\|$ and also $|u_m(z_o)| \le \|u_m\|$ for every $m \neq n$.

Let $\epsilon = \frac{\|u_n\| - |u_n(z_o)|}{2}$. This implies $|u_n(z_o)| < \|u_n\| - \epsilon$. Hence

$$\sup_{|z|\le 1} \{|u_1(z)| + |u_2(z)| + \cdots + |u_K(z)|\} = |u_1(z_o)| + |u_2(z_o)| + \cdots + |u_K(z_o)|$$
$$< \|u_1\| + \|u_2\| + \cdots + \|u_K\| - \epsilon.$$

Therefore $\left\|\sum_{n=1}^{K} u_n C_{\varphi_n} f\right\| < \sum_{n=1}^{K} \|u_n\| - \epsilon$ for all $f \in A(\mathbb{D})$ with $\|f\| \le 1$. Thus we have $\sup_{\|f\|\le 1} \left\|\sum_{n=1}^{K} u_n C_{\varphi_n} f\right\| \le \sum_{n=1}^{K} \|u_n\| - \epsilon$.

Since $\epsilon > 0$, this implies $\left\|\sum_{n=1}^{K} u_n C_{\varphi_n}\right\| < \sum_{n=1}^{K} \|u_n\|$. $\square$

Thus we get the following lemma.

Lemma 3.1. *Let $\varphi_1, \varphi_2, \ldots, \varphi_K \, (K \ge 2)$ be distinct functions in $A(\mathbb{D})$ with $\|\varphi_n\| \le 1$ and $u_1, u_2, u_3, \ldots, u_K \in A(\mathbb{D})$ with $u_n \ne 0$ for every n, $1 \le n \le K$. If $\left\|\sum_{n=1}^{K} u_n C_{\varphi_n}\right\| = \sum_{n=1}^{K} \|u_n\|$, then there exists a point $z_o \in \mathbb{T}$ such that $\|u_n\| = |u_n(z_o)|$ for all $n, 1 \le n \le K$.*

Case 2. If $\mathrm{B} \ne \phi$ and there exists a $z' \in \mathrm{B}$ such that $\frac{u_n(z')}{\|u_n\|} = \frac{u_m(z')}{\|u_m\|}$ for all n and m . Then one can easily seen that

$$\left\|\sum_{n=1}^{K} u_n C_{\varphi_n}\right\| = \sum_{n=1}^{K} \|u_n\|$$

Case 3. If $\mathrm{B} \ne \phi$ and for each $z' \in \mathrm{B}$, $\frac{u_n(z')}{\|u_n\|} \ne \frac{u_m(z')}{\|u_m\|}$ for some n and m. Then we have the following theorem :

Theorem 3.1. *let $\varphi_1, \varphi_2, \ldots, \varphi_K \, (K \ge 2)$ be distinct functions in $A(\mathbb{D})$ with $\|\varphi_n\| \le 1$ and $u_1, u_2, u_3, \ldots, u_K \in A(\mathbb{D})$ with $u_n \ne 0$ for every n, $1 \le n \le K$. Suppose that for each $z' \in \mathrm{B}$, $\frac{u_n(z')}{\|u_n\|} \ne \frac{u_m(z')}{\|u_m\|}$ for some n and m. Then $\left\|\sum_{n=1}^{K} u_n C_{\varphi_n}\right\| = \sum_{n=1}^{K} \|u_n\|$ if and only if there exists a $z_o \in \mathrm{B} \cap \mathrm{Z}$ satisfying $\rho(\varphi_n(z_o), \varphi_m(z_o)) = 1$ for each n and m with $\frac{u_n(z_o)}{\|u_n\|} \ne \frac{u_m(z_o)}{\|u_m\|}$.*

To prove this, we need a lemma due to Gorkin and Mortini, whose easy modification is as follows. see ([6], Proof of theorem 5).

Lemma 3.2. Let $\varphi_1, \varphi_2, \ldots, \varphi_K$ be distinct functions in $A(\mathbb{D})$ with $\|\varphi_n\| \le 1$. Let $z_o \in \bar{\mathbb{D}}$ with $|z_o| = 1$. If $\min_{n \ne m} \rho(\varphi_n(z_o), \varphi_m(z_o)) = 1$, then for $e^{i\theta_j} \in \mathbb{T}, 1 \le j \le K$, there exists a sequence f_k in ball $A(\mathbb{D})$ such that $f_k(\varphi_n(z_o)) \to e^{i\theta_j}$ as $k \to \infty$ for every n.

Proof of Theorem 3.1. Suppose $\left\|\sum\limits_{n=1}^{K} u_n C_{\varphi_n}\right\| = \sum\limits_{n=1}^{K} \|u_n\|$. Then by Lemma 3.1 it follows that B $\neq \phi$. Further $\left\|\sum\limits_{n=1}^{K} u_n C_{\varphi_n}\right\| = \sum\limits_{n=1}^{K} \|u_n\|$ implies there exists a sequence of function $\{f_k\} \in$ ball $A(\mathbb{D})$ satisfying

$$\left\|\sum_{n=1}^{K} u_n C_{\varphi_n} f_k\right\| \to \sum_{n=1}^{K} \|u_n\| \ \ as\ k \to \infty.$$

Consequently, there exists a sequence in $\{z_p\} \subset \mathbb{D}$ such that

$$\left|\sum_{n=1}^{K} u_n\left(z_p\right) f_k\left(\varphi_n\left(z_p\right)\right)\right| \to \sum_{n=1}^{K} \|u_n\| \ \ as\ k \to \infty \text{ and } \ p \to \infty.$$

Without any loss of generality, assume that $z_p \to z_o$. This gives

$$\left|\sum_{n=1}^{K} u_n\left(z_o\right) f_k\left(\varphi_n\left(z_o\right)\right)\right| \to \sum_{n=1}^{K} \|u_n\| \ \ as\ k \to \infty. \tag{3.1}$$

Since $\left|u_n\left(z_o\right) f_k\left(\varphi_n\left(z_o\right)\right)\right| \leq \left|u_n\left(z_o\right)\right|$ for every n and k, therefore we get

$$\begin{aligned} \sum_{n=1}^{K} \|u_n\| = \lim_{k\to\infty}\left|\sum_{n=1}^{K} u_n\left(z_o\right) f_k\left(\varphi_n\left(z_o\right)\right)\right| &\leq \sum_{n=1}^{K} \left|u_n\left(z_o\right)\right| \\ &\leq \sum_{n=1}^{K} \|u_n\|. \end{aligned}$$

Thus $\sum\limits_{n=1}^{K} \|u_n\| = \sum\limits_{n=1}^{K} \left|u_n\left(z_o\right)\right|$. This implies $z_o \in$ B.
Now by (3.1)

$$f_k\left(\varphi_n\left(z_o\right)\right) \to \frac{\overline{u_n\left(z_o\right)}}{\|u_n\|} \ as\ k \to \infty \ \text{ for every } n. \tag{3.2}$$

Let n and m with $\frac{u_n(z_o)}{\|u_n\|} \neq \frac{u_m(z_o)}{\|u_m\|}$. By Schwarz's lemma
$\rho\left(f_k\left(\varphi_n\left(z_p\right)\right), f_k\left(\varphi_m\left(z_p\right)\right)\right) \leq \rho\left(\varphi_n\left(z\right), \varphi_m\left(z_p\right)\right)$ for each $p \geq 1$ and for all $k \geq 1$. Hence $\lim\limits_{p\to\infty} \rho\left(f_k\left(\varphi_n\left(z_p\right)\right), f_k\left(\varphi_m\left(z_p\right)\right)\right) \leq \lim\limits_{p\to\infty} \rho\left(\varphi_n\left(z\right), \varphi_m\left(z_p\right)\right)$, which implies $\rho\left(f_k\left(\varphi_n\left(z_0\right)\right), f_k\left(\varphi_m\left(z_p\right)\right)\right) \leq \rho\left(\varphi_n\left(z\right), \varphi_m\left(z_0\right)\right)$ for all $k \geq 1$. Now by equation (3.2)

$$\begin{aligned} 1 &= \lim_{k\to\infty} \rho\left(f_k\left(\varphi_n\left(z_o\right)\right), f_k\left(\varphi_m\left(z_o\right)\right)\right) \\ &\leq \rho\left(\varphi_n\left(z_o\right), \varphi_m\left(z_o\right)\right) \end{aligned}$$

Hence $\rho\left(\varphi_n\left(z_o\right), \varphi_m\left(z_o\right)\right) = 1$. This shows that either $\left|\varphi_n\left(z_o\right)\right| = 1$ or $\left|\varphi_m\left(z_o\right)\right| = 1$. Therefore $z_o \in$ Z.
Conversely, suppose that there is a $z_o \in$ B$\cap$Z satisfying $\rho\left(\varphi_n\left(z_o\right), \varphi_m\left(z_o\right)\right) = 1$

for each n and m with $\frac{u_n(z_o)}{\|u_n\|} \neq \frac{u_m(z_o)}{\|u_m\|}$. Let $\rho\left(\varphi_n\left(z_o\right), \varphi_m\left(z_o\right)\right) = \gamma_{n,m}$. On the set $I = \{1, 2, \ldots, K\}$, we define an eqivalence relation '$\sim$' as follows. For n, $m \in I$, $n \sim m$ if and only if $\gamma_{n,m} < 1$. By given condition, it follows that, if $n \sim m$ then $\frac{u_n(z_o)}{\|u_n\|} = \frac{u_m(z_o)}{\|u_m\|}$. Now we choose one element from each equivalence class and let J denote the collection of all such elements of I. Note that for every $n, m \in J$ with $n \neq m$, we have $\rho\left(\varphi_n\left(z_o\right), \varphi_m\left(z_o\right)\right) = 1$.
By Lemma 3.2, there exists a sequence $\{f_k\}$ in ball $A(\mathbb{D})$ satisfying

$$f_k\left(\varphi_n\left(z_o\right)\right) \to \overline{\frac{u_n\left(z_o\right)}{\|u_n\|}} \text{ as } k \to \infty \text{ forevery } n \in J. \tag{3.3}$$

Now
$$\left\|\sum_{n=1}^{K} u_n C_{\varphi_n}\right\| \geq \lim_{k\to\infty}\left|\sum_{n=1}^{K} u_n\left(z_o\right) f_k\left(\varphi_n\left(z_o\right)\right)\right|$$
$$= \lim_{k\to\infty}\left|\sum_{m\in J} u_m\left(z_o\right) f_k\left(\varphi_m\left(z_o\right)\right) + \sum_{n\in I\backslash J} u_n\left(z_o\right) f_k\left(\varphi_n\left(z_o\right)\right)\right|.$$

Using equation (3.3), we get $\sum_{m\in J} u_m\left(z_o\right) f_k\left(\varphi_m\left(z_o\right)\right) \to \sum_{m\in J} \|u_m\|$ as $k \to \infty$. Also for each $n \in I\backslash J$ there is $m \in$ J with $n \sim m$ such that $\rho\left(\varphi_n\left(z_o\right), \varphi_m\left(z_o\right)\right) < 1$. Consequently

$$\frac{u_n(z_o)}{\|u_n\|} = \frac{u_m(z_o)}{\|u_m\|}. \tag{3.4}$$

Now using equations (3.2) and (3.4) it follows that

$$f_k\left(\varphi_n\left(z_o\right)\right) \to \overline{\frac{u_m\left(z_o\right)}{\|u_m\|}} \text{ as } k \to \infty. \tag{3.5}$$

Combining equations (3.4) and (3.5), we get

$$u_n\left(z_o\right) f_k\left(\varphi_n\left(z_o\right)\right) \to \frac{u_m\left(z_o\right)}{\|u_m\|}\left|u_n\left(z_o\right)\right| \overline{\frac{u_m\left(z_o\right)}{\|u_m\|}} = \left|u_n\left(z_o\right)\right| \text{ as } k \to \infty.$$

Thus we get $\left\|\sum_{n=1}^{K} u_n C_{\varphi_n}\right\| \geq \sum_{n=1}^{K} \|u_n\|$.
Therefore $\left\|\sum_{n=1}^{K} u_n C_{\varphi_n}\right\| = \sum_{n=1}^{K} \|u_n\|$. This completes the proof. $\square$

Theorem 3.2. *Let $\{\varphi_n\}_{n=1}^{\infty}$ be distinct functions in $A(\mathbb{D})$ with $\|\varphi_n\| \leq 1$ for each $n \geq 1$ and $\{u_n\}_{n=1}^{\infty}$ a sequence in $A(\mathbb{D})$ with $\sum_{n=1}^{\infty} \|u_n\| < \infty$.Suppose that for each $z' \in$ B, $\frac{u_n(z')}{\|u_n\|} \neq \frac{u_m(z')}{\|u_m\|}$ for some n and m. Then $\left\|\sum_{n=1}^{\infty} u_n C_{\varphi_n}\right\| =$*

$\sum_{n=1}^{\infty} \|u_n\|$ *if and only if there exists a* $z_o \in \mathrm{B} \cap \mathrm{Z}$ *satisfying* $\rho\left(\varphi_n\left(z_o\right), \varphi_m\left(z_o\right)\right) = 1$ *for each* n *and* m *with* $\frac{u_n(z_o)}{\|u_n\|} \neq \frac{u_m(z_o)}{\|u_m\|}$.

Proof. For $N \geq 2$, by Theorem 3.1, we get $\left\|\sum_{n=1}^{N} u_n C_{\varphi_n}\right\| = \sum_{n=1}^{N} \|u_n\|$ for each $N \geq 1$. Hence

$$\left\|\sum_{n=1}^{\infty} u_n C_{\varphi_n}\right\| = \lim_{N \to \infty} \left\|\sum_{n=1}^{N} u_n C_{\varphi_n}\right\|$$
$$= \lim_{N \to \infty} \sum_{n=1}^{N} \|u_n\| = \sum_{n=1}^{\infty} \|u_n\|.$$

□

Example 3.1. Let $\varphi_n(z) = \dfrac{z + (-1)^{n-1} n}{n+1}$. For n≥1, we define u_n as follows: $u_{2n-1}(z) = \frac{z}{2^n}$ and $u_{2n}(z) = -\frac{z}{2^n}$.
Then $\mathrm{B} = \{1, -1\}$ and $\mathrm{Z} = \{1, -1\}$. Let m be even and n be odd. Then we have $\rho\left(\varphi_n\left(z_o\right), \varphi_m\left(z_o\right)\right) = 1$ and $\dfrac{u_n(z_o)}{\|u_n\|} \neq \dfrac{u_m(z_o)}{\|u_m\|}$ for every $z_o \in \mathrm{B}$. Hence by Theorem 3.2.

$$\left\|\sum_{n=1}^{\infty} u_n C_{\varphi_n}\right\| = \sum_{n=1}^{\infty} \|u_n\|$$
$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^2} + \ldots\ldots$$
$$= 2$$

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