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Almost (g, m)-Continuous Functions

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Abstract

The purpose of this paper is to introduce a new notion of almost (g, m)-continuous functions as functions from a generalized topological space into a set satisfying some minimal conditions. We obtain some characterizations and several properties of functions.

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1 Introduction

The concept of minimal structure (briefly m-structure) was introduced by V. Popa and T. Noiri [6] in 2000. Also they introduced the notion of m_X-open set and m_X-closed set and characterize those sets using m_X-closure and m_X-interior operators respectively. Further they introduced M-continuous functions and studied some of its basic properties. Á. Császár [1] introduced the concepts of generalized neighborhood systems and generalized topological spaces. He also introduced the concepts of continuous functions and associated interior and closure operators on generalized neighborhood systems and generalized topological spaces. In particular, he investigated characterizations for the generalized continuous function by using a closure operator defined on generalized neighborhood systems. In [2], he introduced and studied the notions of g-α-open sets, g-semi-open sets, g-preopen sets and g-β-open sets in generalized topological spaces. W.K. Min [4] introduced the notion of almost (g, g')-continuity and investigated properties of such functions and relationships among (g, g')-continuity, almost (g, g')-continuity, and weak

(g, g')-continuity. In this paper, we introduce a new notion of almost (g, m)-continuous functions as functions from a generalized topological space (X, g_X) into a set satisfying some minimal conditions. We obtain several characterizations and properties of such functions.

2 Preliminaries

We recall some notions and notations defined in [1]. Let X be a nonempty set and g_X be a collection of subsets of X. Then g is called a *generalized topology* (briefly GT) on X iff $\emptyset \in g_X$ and $G_i \in g_X$ for $i \in I \neq \emptyset$ implies $G = \cup_{i \in I} G_i \in g_X$. We call the pair (X, g_X) a *generalized topological space* (briefly GTS) on X. The elements of g_X are called g_X*-open* sets and the complements are called g_X*-closed* sets. Set $gO(X) = \{U \subseteq X : U \in g_X\}$ and $gO(x) = \{U \in g_X : x \in U\}$. The closure of a subset A of X, denoted by $c_X(A)$, is the intersection of generalized closed sets including A. And the interior of A, denoted by $i_X(A)$, is the union of generalized open sets contained in A.

Theorem 2.1. [1] *Let (X, g_X) be a generalized topological space. Then*

(1) $c_X(A) = X - i_X(X - A)$;

(2) $i_X(A) = X - c_X(X - A)$.

Proposition 2.2. [4] *Let (X, g_X) be a generalized topological space and $A \subseteq X$. Then*

(1) $x \in i_X(A)$ *if and only if there exists* $V \in gO(x)$ *such that* $V \subseteq A$.

(2) $x \in c_X(A)$ *if and only if* $V \cap A \neq \emptyset$ *for every* $V \in gO(x)$.

Proposition 2.3. *Let (X, g_X) be a generalized topological space. For subsets A and B of X, the following properties holds:*

(1) $c_X(X - A) = X - i_X(A)$ *and* $i_X(X - A) = X - c_X(A)$;

(2) If $(X - A) \in g_X$, *then* $c_X(A) = A$ *and if* $A \in g_X$, *then* $i_X(A) = A$;

(3) If $A \subseteq B$, *then* $c_X(A) \subseteq c_X(B)$ *and* $i_X(A) \subseteq i_X(B)$;

(4) $A \subseteq c_X(A)$ *and* $i_X(A) \subseteq A$;

(5) $c_X(c_X(A)) = c_X(A)$ *and* $i_X(i_X(A)) = i_X(A)$.

Definition 2.4. [5] Let X be a nonempty set and $P(X)$ the power set of X. A subfamily m_X of $P(X)$ is called a *minimal structure* (briefly *m-structure*) on X if $\emptyset \in m_X$ and $X \in m_X$

By (X, m_X), we denote a nonempty set X with an m-structure m_X on X and it is called an m-space. Each member of m_X is said to be m_X*-open* and the complement of an m_X-open set is said to be m_X*-closed*.

Definition 2.5. [5] Let X be a nonempty set and m_X an m-structure on X. For a subset A of X, the m_X*-closure* of A and the m_X*-interior* of A are defined as follows:

(1) $m_X\text{-}Cl(A) = \cap\{F : A \subseteq F, X - F \in m_X\}$;

(2) $m_X\text{-}Int(A) = \cup\{U : U \subseteq A, U \in m_X\}$.

Lemma 2.6. [3] *Let X be a nonempty set and m_X a minimal structure on X. For subset A and B of X, the following properties hold:*

(1) $m_X\text{-}Cl(X - A) = X - (m_X\text{-}Int(A))$ and $m_X\text{-}Int(X - A) = X - (m_X\text{-}Cl(A))$;

(2) If $(X - A) \in m_X$, then $m_X\text{-}Cl(A) = A$ and if $A \in m_X$, then $m_X\text{-}Int(A) = A$;

(3) $m_X\text{-}Cl(\emptyset) = \emptyset$, $m_X\text{-}Cl(X) = X$, $m_X\text{-}Int(\emptyset) = \emptyset$ and $m_X\text{-}Int(X) = X$;

(4) If $A \subseteq B$, then $m_X\text{-}Cl(A) \subseteq m_X\text{-}Cl(B)$ and $m_X\text{-}Int(A) \subseteq m_X\text{-}Int(B)$;

(5) $A \subseteq m_X\text{-}Cl(A)$ and $m_X\text{-}Int(A) \subseteq A$;

(6) $m_X\text{-}Cl(m_X\text{-}Cl(A)) = m_X\text{-}Cl(A)$ and $m_X\text{-}Int(m_X\text{-}Int(A)) = m_X\text{-}Int(A)$.

Lemma 2.7. [3] *Let X be a nonempty set with a minimal structure m_X and A a subset of X. Then $x \in m_X\text{-}Cl(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m_X$ containing x.*

Definition 2.8. [3] An m-structure m_X on a nonempty set X is said to have *property* $\mathcal{B}$ if the union of any family of subsets belong to m_X belong to m_X.

Lemma 2.9. [5] *Let X be a nonempty set and m_X an m-structure on X sastisfying property $\mathcal{B}$. For a subset A of X, the following properties hold:*

(1) $A \in m_X$ if and only if $m_X\text{-}IntA = A$;

(2) If A is m_X-closed if and only if $m_X\text{-}Cl(A) = A$;

(3) $m_X\text{-}Int(A) \in m_X$ and $m_X\text{-}Cl(A)$ is m_X-closed.

3 Almost (g, m)-continuous functions

In this section, we introduce and study almost (g, m)-continuous functions.

Definition 3.1. A function $f : (X, g_X) \to (Y, m_Y)$ is said to be (g, m)-*continuous* at a point $x \in X$ if for each m_Y-open set V containing $f(x)$, there exists a g_X-open set U containing x such that $f(U) \subseteq V$. A function $f : (X, g_X) \to (Y, m_Y)$ is said to be (g, m)-*continuous* if it has this property at each point $x \in X$.

Theorem 3.2. *For a function $f : (X, g_X) \to (Y, m_Y)$, the following properties are equivalent:*

(1) f is (g, m)-continuous at $x \in X$;

(2) $x \in i_X(f^{-1}(V))$ for every $V \in m_Y$ containing $f(x)$;

(3) $x \in f^{-1}(m_Y\text{-}Cl(f(A)))$ for every subset A of X with $x \in c_X(A)$;

(4) $x \in f^{-1}(m_Y\text{-}Cl(B))$ for every subset B of Y with $x \in c_X(f^{-1}(B))$;

(5) $x \in i_X(f^{-1}(B))$ for every subset B of Y with $x \in f^{-1}(m_Y\text{-}Int(B))$;

(6) $x \in f^{-1}(K)$ for every m_Y-closed set K of Y such that $x \in c_X(f^{-1}(K))$.

Proof. (1) $\Rightarrow$ (2): Let V be any m_Y-open subset of Y containing $f(x)$. Then, there exists a m_X-open subset U of X containing x such that $f(U) \subseteq V$. Since $U \in g_X$, we have $x \in i_X(f^{-1}(V))$.

(2) $\Rightarrow$ (3): Let A be any subset of X. Let $x \in c_X(A)$ and $V \in m_Y$ containing $f(x)$. Then $x \in i_X(f^{-1}(V))$. There exists $U \in g_X$ such that $x \in U \subseteq f^{-1}(V)$. Since $x \in c_X(A)$, by Proposition 2.2, $U \cap A \neq \emptyset$ and $\emptyset \neq f(U \cap A) \subseteq f(U) \cap f(A) \subseteq V \cap f(A)$. Since $V \in m_Y$ containing $f(x)$, $f(x) \in m_Y\text{-}Cl(f(A))$ and hence $x \in f^{-1}(m_Y\text{-}Cl(f(A)))$.

(3) $\Rightarrow$ (4): Let B be any subset of Y and $x \in c_X(f^{-1}(B))$. By (3), $x \in f^{-1}(m_Y\text{-}Cl(f(f^{-1}(B)))) \subseteq f^{-1}(m_Y\text{-}Cl(B))$. Hence, we have $x \in f^{-1}(m_Y\text{-}Cl(B))$.

(4) $\Rightarrow$ (5): Let B be any subset of Y such that $x \notin i_X(f^{-1}(B))$. Then $x \in X - i_X(f^{-1}(B)) = c_X(X - f^{-1}(B)) = c_X(f^{-1}(Y - B))$. By (4), we have $x \in f^{-1}(m_Y\text{-}Cl(Y - B)) = f^{-1}(Y - (m_Y\text{-}Int(B))) = X - f^{-1}(m_Y\text{-}Int(B))$. Hence, $x \notin f^{-1}(m_Y\text{-}Int(B))$.

(5) $\Rightarrow$ (6): Let K be any m_Y-closed set of Y such that $x \notin f^{-1}(K)$. Then $x \in X - f^{-1}(K) = f^{-1}(Y - K) = f^{-1}(m_Y\text{-}Int(Y - K))$ because $Y - K$ is m_Y-open. By (5), $x \in i_X(f^{-1}(Y - K)) = i_X(X - f^{-1}(K)) = X - c_X(f^{-1}(K))$. Hence, $x \notin c_X(f^{-1}(K))$.

(6) $\Rightarrow$ (2): Let $x \in X$ and $V \in m_Y$ containing $f(x)$. Suppose that $x \notin i_X(f^{-1}(V))$. Then $x \in X - i_X(f^{-1}(V)) = c_X(X - f^{-1}(V)) = c_X(f^{-1}(Y - V))$.

By (6), $x \in f^{-1}(Y - V) = X - f^{-1}(V)$. Hence $x \notin f^{-1}(V)$. This contraries to the hypothesis.

(2) $\Rightarrow$ (1): Let $V \in m_Y$ containing $f(x)$. By (2), $x \in i_X(f^{-1}(V)$ and hence there exists $U \in g_X$ containing x such that $x \in U \subseteq f^{-1}(V)$. Therefore, $f(U) \subseteq V$ and f is (g, m)-continuous at x. □

Definition 3.3. A function $f : (X, g_X) \to (Y, m_Y)$ is said to be *almost* (g, m)-*continuous* at a point $x \in X$ if for each m_Y-open set V containing $f(x)$, there exists a g_X-open set U containing x such that $f(U) \subseteq m_Y\text{-}Int(m_Y\text{-}Cl(V))$. A function $f : (X, m_X) \to (Y, m_Y)$ is said to be *almost* (g, m)-*continuous* if it has this property at each point $x \in X$.

Remark 1. *From the above definitions of* (g, m)*-continuity and almost* (g, m)*-continuity, we have the following implication but the reverse relation may not be true in general:*

$$(g, m)\text{-}continuous \Rightarrow almost\ (g, m)\text{-}continuous.$$

Example 3.4. *Let* $X = \{a, b\} = Y$*,* $g_X = \{\emptyset, X\}$ *and* $m_Y = \{\emptyset, \{a\}, Y\}$*. Let* $f : (X, g_X) \to (Y, m_Y)$ *be the identity function. Then* f *is almost* (g, m)*-continuous but it is not* (g, m)*-continuous.*

Definition 3.5. A subset A of a m-space (X, m_X) is said to be

(1) m_X-regular open if $A = m_X\text{-}Int(m_X\text{-}Cl(A))$;

(2) m_X-semi-open if $A \subseteq m_X\text{-}Cl(m_X\text{-}Int(A))$;

(3) m_X-preopen if $A \subseteq m_X\text{-}Int(m_X\text{-}Cl(A))$;

(4) m_X-α-open if $A \subseteq m_X\text{-}Int(m_X\text{-}Cl(m_X\text{-}Int(A)))$;

(5) m_X-β-open if $A \subseteq m_X\text{-}Cl(m_X\text{-}Int(m_X\text{-}Cl(A)))$.

The complement of a m_X-regular open (resp. m_X-semi-open, m_X-preopen, m_X-α-open, m_X-β-open) set is called *m_X-regular closed* (resp. *m_X-semi-closed*, *m_X-preclosed*, *m_X-α-closed*, *m_X-β-closed*).

Lemma 3.6. *Let* (X, m_X) *be a* m*-space and* A *a subset of* X*.*

(1) A *is* m_X*-regular closed if and only if* $A = m_X\text{-}Cl(m_X\text{-}Int(A))$*;*

(2) A *is* m_X*-semi-closed if and only if* $m_X\text{-}Int(m_X\text{-}Cl(A)) \subseteq A$*;*

(3) A *is* m_X*-preclosed if and only if* $m_X\text{-}Cl(m_X\text{-}Int(A)) \subseteq A$*;*

(4) A *is* m_X*-*β*-closed if and only if* $m_X\text{-}Int(m_X\text{-}Cl(m_X\text{-}Int(A))) \subseteq A$*.*

Theorem 3.7. *For a function $f : (X, g_X) \to (Y, m_Y)$, the following properties are equivalent:*

(1) f *is almost* (g, m)*-continuous at* $x \in X$*;*

(2) $x \in i_X(f^{-1}(m_Y\text{-}Int(m_Y\text{-}Cl(V))))$ *for every* m_Y*-open set* V *containing* $f(x)$*;*

(3) $x \in i_X(f^{-1}(V))$ *for every* m_Y*-regular open set* V *containing* $f(x)$*;*

(4) For every m_Y*-regular open set* V *containing* $f(x)$*, there exists a* g_X*-open set* U *containing* x *such that* $f(U) \subseteq V$*.*

Proof. (1) ⇒ (2): Let V be any m_Y-open subset of Y containing $f(x)$. Then, there exists a g_X-open subset U of X containing x such that $f(U) \subseteq m_Y\text{-}Int(m_Y\text{-}Cl(V))$. Thus $x \in U \subseteq f^{-1}(m_Y\text{-}Int(m_Y\text{-}Cl(V)))$. Since $U \in g_X$, we have $x \in i_X(f^{-1}(m_Y\text{-}Int(m_Y\text{-}Cl(V))))$.

(2) ⇒ (3): Let V be any m_Y-regular open subset of Y containing $f(x)$. Then since $V = m_Y\text{-}Int(m_Y\text{-}Cl(V))$ and by (2), we have $x \in i_X(f^{-1}(V))$.

(3) ⇒ (4): Let V be any m_Y-regular open subset of Y containing $f(x)$. By (3), there exists a g_X-open set U containing x such that $U \subseteq f^{-1}(V)$. Hence, we have (4).

(4) ⇒ (1): Let V be any m_Y-open subset of Y containing $f(x)$. Then $f(x) \in V \subseteq m_Y\text{-}Int(m_Y\text{-}Cl(V))$. Since $m_Y\text{-}Int(m_Y\text{-}Cl(V))$ is m_Y-regular open, by (4) there exists a g_X-open set U containing x such that $f(U) \subseteq m_Y\text{-}Int(m_Y\text{-}Cl(V))$. Hence, f is almost (g, m)-continuous. □

Theorem 3.8. *For a function $f : (X, g_X) \to (Y, m_Y)$, where m_Y have property $\mathcal{B}$, the following properties are equivalent:*

(1) f *is almost* (g, m)*-continuous,*

(2) $f^{-1}(V) \subseteq i_X(f^{-1}(m_Y\text{-}Int(m_Y\text{-}Cl(V))))$ *for every* m_Y*-open subset* V *of* Y*;*

(3) $c_X(f^{-1}(m_Y\text{-}Cl(m_Y\text{-}Int(F)))) \subseteq f^{-1}(F)$ *for every* m_Y*-closed subset* F *of* Y*;*

(4) $c_X(f^{-1}(m_Y\text{-}Cl(m_Y\text{-}Int(m_Y\text{-}Cl(B))))) \subseteq f^{-1}(m_Y\text{-}Cl(B))$ *for every subset* B *of* Y*;*

(5) $f^{-1}(m_Y\text{-}Int(B)) \subseteq i_X(f^{-1}(m_Y\text{-}Int(m_Y\text{-}Cl(m_Y\text{-}Int(B)))))$ *for every subset* B *of* Y*;*

(6) $f^{-1}(V)$ *is* g_X*-open in* X *for every* m_Y*-regular open subset* V *of* Y*;*

(7) $f^{-1}(F)$ *is* g_X*-closed in* X *for every* m_Y*-regular closed subset* V *of* Y*.*

Proof. (1) $\Rightarrow$ (2): Let V be a m_Y-open subset of Y and $x \in f^{-1}(V)$. There exists a g-open subset U of X containing x such that $f(U) \subseteq m_Y\text{-}Int(m_Y\text{-}Cl(V))$. This implies $x \in i_X(f^{-1}(m_Y\text{-}Int(m_Y\text{-}Cl(V))))$. Hence $f^{-1}(V) \subseteq i_X(f^{-1}(m_Y\text{-}Int(m_Y\text{-}Cl(V))))$.

(2) $\Rightarrow$ (3): Let F be a m_Y-closed subset of Y. By Lemma 3.6(1), it follows $f^{-1}(Y-F) \subseteq i_X(f^{-1}(m_Y\text{-}Int(m_Y\text{-}Cl(Y-F)))) = m_X\text{-}Int(f^{-1}(Y-(m_Y\text{-}Cl(m_Y\text{-}Int(F))))) = X-(m_X\text{-}Cl(f^{-1}(m_Y\text{-}Cl(m_Y\text{-}Int(F)))))$. Hence, $c_X(f^{-1}(m_Y\text{-}Cl(m_Y\text{-}Int(F)))) \subseteq f^{-1}(F)$.

(3) $\Rightarrow$ (4): Let B be a subset of Y. Since $m_Y\text{-}Cl(B)$ is m_Y-closed and by (3), we have $c_X(f^{-1}(m_Y\text{-}Cl(m_Y\text{-}Int(m_Y\text{-}Cl(B))))) \subseteq f^{-1}(m_Y\text{-}Cl(B))$.

(4) $\Rightarrow$ (5): Let B be any subset of Y. By (4), $f^{-1}(m_Y\text{-}Int(B)) = X - f^{-1}(m_Y\text{-}Cl(Y-B)) \subseteq X-(m_X\text{-}Cl(f^{-1}(m_Y\text{-}Cl(m_Y\text{-}Int(m_Y\text{-}Cl(Y-B)))))) = i_X(f^{-1}(m_Y\text{-}Int(m_Y\text{-}Cl(m_Y\text{-}Int(B)))))$.

(5) $\Rightarrow$ (6): Let V be any m_Y-regular open subset of Y. Since $m_Y\text{-}Int(m_Y\text{-}Cl(m_Y\text{-}Int(V))) = V$, from (5), it follows $f^{-1}(V) \subseteq i_X(f^{-1}(V))$ and so $f^{-1}(V) = c_X(f^{-1}(V))$.

(6) $\Rightarrow$ (7): Let F be any m_Y-regular closed subset of Y. Then by (6), we have $X - f^{-1}(F) = f^{-1}(Y-F) = i_X(f^{-1}(Y-F)) = X - c_X(f^{-1}(F))$.

(7) $\Rightarrow$ (1): Let V be any m_Y-regular open subset of Y containing $f(x)$. By (7), $X - f^{-1}(V) = f^{-1}(Y-V) = c_X(f^{-1}(Y-V)) = X - i_X(f^{-1}(V))$. Since $x \in f^{-1}(V) = i_X(f^{-1}(V))$, there exists a g-open set U containing x such that $U \subseteq f^{-1}(V)$. Hence by Theorem 3.7(4), f is almost (g,m)-continuous. □

Theorem 3.9. *For a function $f:(X,g_X)\to(Y,m_Y)$, the following properties are equivalent:*

(1) f is almost (g,m)-continuous,

(2) $c_X(f^{-1}(U)) \subseteq f^{-1}(m_Y\text{-}Cl(U))$ for every m_Y-β-open subset U of Y;

(3) $c_X(f^{-1}(U)) \subseteq f^{-1}(m_Y\text{-}Cl(U))$ for every m_Y-semi-open subset U of Y;

(4) $f^{-1}(U) \subseteq i_X(f^{-1}(m_Y\text{-}Int(m_Y\text{-}Cl(U))))$ for every m_Y-preopen subset U of Y.

Proof. (1) $\Rightarrow$ (2): Let U be any m_Y-β-open subset of Y. Since $m_Y\text{-}Cl(U)$ is m_Y-regular closed, by Theorem 3.8(7), $c_X(f^{-1}(m_Y\text{-}Cl(U))) = f^{-1}(m_Y\text{-}Cl(U))$. Thus $c_X(f^{-1}(U)) \subseteq c_X(f^{-1}(m_Y\text{-}Cl(U))) = f^{-1}(m_Y\text{-}Cl(U))$.

(2) $\Rightarrow$ (3): It is obvious since every m_Y-semi-open set is m_Y-β-open.

(3) $\Rightarrow$ (1): Let F be any m_Y-regular closed subset of Y. Then since F is m_Y-semi-open, we have $c_X(f^{-1}(F)) \subseteq f^{-1}(m_Y\text{-}Cl(F)) = f^{-1}(F)$. Thus from Theorem 3.8(7), f is almost (g,m)-continuous.

(1) $\Rightarrow$ (4): Let U be any m_Y-preopen subset of Y. Then $U \subseteq m_Y\text{-}Int(m_Y\text{-}Cl(U)))$ and $m_Y\text{-}Int(m_Y\text{-}Cl(U))$ is m_Y-regular open. By Theorem

3.8(6), $f^{-1}(m_Y\text{-}Int(m_Y\text{-}Cl(U))) = i_X(f^{-1}(m_Y\text{-}Int(m_Y\text{-}Cl(U))))$. Thus, we have $f^{-1}(U) \subseteq f^{-1}(m_Y\text{-}Int(m_Y\text{-}Cl(U))) = i_X(f^{-1}(m_Y\text{-}Int(m_Y\text{-}Cl(U))))$.

(4) $\Rightarrow$ (1): Let U be any m_Y-regular open subset of Y. Then U is m_Y-preopen and $f^{-1}(U) \subseteq i_X(f^{-1}(m_Y\text{-}Int(m_Y\text{-}Cl(U)))) = i_X(f^{-1}(U))$. Hence, by Theorem 3.8(6), f is almost (g, m)-continuous. □

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