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Results Based on Generalised Aluthge Transformation of Composition Operators

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Abstract

In this paper , we will be concerned with various results based on generalised Aluthge transformation $C_r = |C|^r U |C|^{1-r}$ for $0 < r \leq 1$ and $C_{s,t} = |C|^s U |C|^t$ for $s > 0$ and $t > 0$ of the composition operator C.

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1 Introduction

Throughout this paper let (X, Σ, λ) be a sigma finite measure space and let T be a measurable transformation from X into itself . The equation Cf = $f \circ T$, $f \epsilon\, L^2(\lambda)$ defines a composition transformation on $L^2(\lambda)$. Where the measure $\lambda \circ T^{-1}$ is absolutely continuous with respect to λ and the Radon Nikodym derivative of $\lambda \circ T^{-1}$ with respect to λ is denoted by h and that of $\lambda \circ T^{-k}$ with respect to λ is denotes by h_k where $k \,\epsilon\, N$,the set of naturals.

More generally in [1]Lambert has defined family of operators $A_r : 0 < r \leq 1$ where $A_r = |A|^r U |A|^{1-r}$ and for composition operators as $C_r = |C|^r U |C|^{1-r}$ for $0 < r \leq 1$ and $C_r f = (\frac{h}{h \circ T})^{r/2} f \circ T$. [3] introduced a very interesting class of bounded linear Hilbert space operators class A. Recently Jeon and Kim [4] introduced quasiclass A operator. [5] consider an extension of quasi class A operators. [2]introduced posinormal operators in Hilbert space.[7] studied on generalised Aluthge transformation of composition operators.

In this paper we are interested with various results based on generalised Aluthge transformation of composition operator .

2 Preliminary Notes

Let H be a Hilbert space and let T be a bounded linear operator on H. A weighted composition operator W is a linear transformation defined by Wf = $w(f \circ T)$, ω is a complex valued Σ measurable function.From [6] w_k denotes $w(w \circ T)(w \circ T^2)...(w \circ T^{k-1})$ so that

$$\begin{aligned} W^k f &= w_k (f \circ T^k) \\ W^{*^k} f &= h_k E(w_k f) \circ T^{-k} \\ W^{*^k} W^k f &= h_k E(w_k^2) \circ T^{-k} f \\ W^k W^{*^k} f &= w_k (h_k \circ T^k) E(w_k f) \end{aligned}$$

For a composition operator C, the polar decomposition is given by $C = U |C|$ where $|C| f = \sqrt{hf}$ and U f = $\frac{1}{\sqrt{h \circ T}}$ $f \circ T$. This is valid for all composition operators.

Generally Aluthge transformation for composition operator C_r is a weighted composition operator with weight $\pi = (\frac{h}{h \circ T})^{r/2}$ where $0 < r \leq 1$. Since C_r is a weighted composition operator it is easy to show that

$$\begin{aligned} |C_r| f &= \sqrt{hE(\pi^2) \circ T^{-1} f} \\ \Rightarrow C_r^* C_r f &= hE(\pi^2) \circ T^{-1} f \\ and\, |C_r^*| f &= P_v f \\ &= vE(vf), where \\ v &= \frac{\pi \sqrt{h \circ T}}{[E(\pi \sqrt{h \circ T})^2]^{1/4}} \\ \Rightarrow C_r C_r^* f &= P_v^2 f \\ \Rightarrow (C_r C_r^*)^k f &= P_v^{2k} f \end{aligned}$$

Also we have

$$\begin{aligned}
C_r^k f &= \pi_k(f \circ T^k) \\
C_r^{*^k} f &= h_k E(\pi_k f) \circ T^{-k} \\
(C_r^* C_r)^k f &= h_k (E(\pi^2) \circ T^{-1})^k f \\
C_r^{*^k} C_r^k f &= h_k E(\pi_k^2) \circ T^{-k} f \\
C_r^k C_r^{*^k} f &= \pi_k (h_k \circ T^k) E(\pi_k f)
\end{aligned}$$

For any s and t such as s> 0 and t> 0 , define $C_{s,t} = |C|^s U |C|^t$ where $|C| = \sqrt{h}$ and $U = \frac{C}{\sqrt{h \circ T}}$.

$$\begin{aligned}
C_{s,t} f &= \frac{h^{s/2} C h^{t/2}}{\sqrt{h \circ T}} f \\
&= \frac{h^{s/2} (h^{t/2} \circ T)}{h^{1/2} \circ T} C f \\
&= (h^{s/2} (h^{(t-1)/2} \circ T)) f \circ T
\end{aligned}$$

let

$$\begin{aligned}
\pi_{(s,t)} &= h^{s/2} (h^{(t-1)/2} \circ T) \\
\Rightarrow C_{s,t} f &= \pi_{(s,t)} f \circ T
\end{aligned}$$

And also,

$$\begin{aligned}
C_{s,t}^* f &= h E(\pi_{(s,t)} f) \circ T^{-1} \\
C_{s,t}^k f &= \pi_{(s,t)}^k (f \circ T^k) \\
C_{s,t}^{*^k} f &= h_k E(\pi_{(s,t)}^k f) \circ T^{-k} \\
C_{s,t}^* C_{s,t} f &= h (E(\pi_{(s,t)}^2) \circ T^{-1}) f \\
C_{s,t} C_{s,t}^* f &= P_{v(s,t)}^2 f \\
(C_{s,t} C_{s,t}^*)^k f &= h_k (E(\pi_{(s,t)}^2) \circ T^{-1})^k f \\
(C_{s,t} C_{s,t}^*)^k f &= P_{v(s,t)}^{2k} f, where \\
v_{(s,t)} &= \frac{\pi_{(s,t)} \sqrt{h \circ T}}{(E(\pi_{(s,t)} \sqrt{h \circ T})^2)^{1/4}}
\end{aligned}$$

Definition 2.1 : [3] *An operator T belongs to class A iff* $\left(T^* |T|^2 T\right)^{1/2} \geq T^* T$.

Definition 2.2 : [4] *An operator T belongs to quasiclass A iff* $T^* \left(|T^2| - |T|^2\right) T \geq 0$.

Definition 2.3 : [5] *An operator T belongs to k- quasiclass A operator for a positive integer k if $T^{*^k}\left(|T^2|-|T|^2\right)T^k \geq 0$.*

Definition 2.4 : [2] *An operator T in a Hilbert space H is called posinormal if $TT^* \leq c^2T^*T$ for some $c > 0$.*

Definition 2.5 : [2] *An operator T in a Hilbert space H is quasiposinormal if $(TT^*)^2 \leq c^2T^{*^2}T^2$.*

3 Class A operators

Theorem 3.1 :
C_r is of quasiclass A iff $h_4E(\pi_4^2)\circ T^{-4} \geq (h_2E(\pi_2^2)\circ T^{-2})^2$.
Proof:
C_r is of quasiclass A if $C_r^*\left|C_r^2\right|C_r - C_r^*\left|C_r\right|^2C_r \geq 0$

$$C_r^*(C_r^{*^2}C_r^2)^{1/2}C_r - C_r^*C_r^*C_rC_r \geq 0$$

$$\begin{aligned}
&\Leftrightarrow (C_r^{*^4}C_r^4)^{1/2} - C_r^{*^2}C_r^2 && \geq 0\\
&\Leftrightarrow \left\langle((C_r^{*^4}C_r^4)^{1/2} - C_r^{*^2}C_r^2)f, f\right\rangle && \geq 0\\
&\Leftrightarrow \int_E [(h_4E(\pi_4^2)\circ T^{-4})^{1/2} - (h_2E(\pi_2^2)\circ T^{-2})]\left|f\right|^2 d\lambda && \geq 0
\end{aligned}$$

for every $E \ \epsilon\ \Sigma$

$$\Leftrightarrow h_4E(\pi_4^2)\circ T^{-4} \qquad \geq (h_2E(\pi_2^2)\circ T^{-2})^2.$$

Theorem 3.2 :
Suppose $T^{-1}\Sigma = \Sigma$. Then C_r is quasiclass A iff
$h_4\pi_4^2\circ T^{-4} \geq (h_2\pi_2^2\circ T^{-2})^2$.

Theorem 3.3 :
C_r is of k quasiclass A iff
$h_{2(k+1)}E(\pi_{2(k+1)}^2)\circ T^{-2(k+1)} \geq (h_{k+1}E(\pi_{k+1}^2)\circ T^{-(k+1)})^2$
Proof:
C_r is a k quasiclass A if $C_r^{*^k}(|C_r^2| - |C_r|^2)C_r^k \geq 0$

$$C_r^{*^k}(C_r^{*^2}C_r^2)^{1/2}C_r^k - C_r^{*^k}C_r^*C_rC_r^k \geq 0$$

$$\begin{aligned}
&\Leftrightarrow (C_r^{*^{2k}}C_r^{*2}C_r^2C_r^{2k})^{1/2} - C_r^{*^{k+1}}C_r^{k+1} && \geq 0\\
&\Leftrightarrow \left\langle((C_r^{*^{2(k+1)}}C_r^{2(k+1)})^{1/2} - C_r^{*^{k+1}}C_r^{k+1})f, f\right\rangle && \geq 0\\
&\Leftrightarrow \int_E [(h_{2(k+1)}E(\pi_{2(k+1)}^2)\circ T^{-2(k+1)})^{1/2} - h_{k+1}E(\pi_{k+1}^2)\circ T^{-(k+1)}]\left|f\right| d\lambda && \geq 0
\end{aligned}$$

for every $E \ \epsilon \ \Sigma$

$$\Leftrightarrow h_{2(k+1)}E(\pi^2_{2(k+1)}) \circ T^{-2(k+1)} \qquad \geq (h_{k+1}E(\pi^2_{k+1}) \circ T^{-(k+1)})^2.$$

Theorem 3.4 :
Suppose $T^{-1}\Sigma = \Sigma$. *Then* C_r *is of k quasiclass A iff* $h_{2(k+1)}\pi^2_{2(k+1)} \circ T^{-2(k+1)} \geq (h_{k+1}\pi^2_{k+1} \circ T^{-(k+1)})^2$.

4 Posinormal operators

Theorem 4.1 :
C_r *is posinormal iff* $P_v^2 \leq c^2 h_1(E(\pi^2) \circ T^{-1})$.
Proof:
C_r *is posinormal if* $C_rC_r^* - c^2C_r^*C_r \leq 0$.

$$\langle (C_rC_r^* - c^2C_r^*C_r)f, f\rangle \leq 0$$

$$\Leftrightarrow \int_E (P_v^2 - c^2h_1(E(\pi^2) \circ T^{-1}))\,|f|^2\,d\lambda \leq 0, E\ \epsilon\ \Sigma$$

$$\Leftrightarrow P_v^2 \leq c^2h_1(E(\pi^2) \circ T^{-1}).$$

Theorem 4.2 :
$C_{s,t}$ *is posinormal iff* $P^2_{v(s,t)} \leq c^2h_1(E(\pi^2_{(s,t)}) \circ T^{-1})$.
Proof:
$C_{s,t}$ *is posinormal if* $C_{s,t}C^*_{s,t} - c^2C^*_{s,t}C_{s,t} \leq 0$

$$\left\langle (C_{s,t}C^*_{s,t} - c^2C^*_{s,t}C_{s,t})f, f\right\rangle \leq 0$$

$$\Leftrightarrow \int_E [P^2_{v(s,t)} - c^2h_1(E(\pi^2_{(s,t)}) \circ T^{-1})]\,|f|^2\,d\lambda \leq 0, E\ \epsilon\ \Sigma$$

$$\Leftrightarrow P^2_{v(s,t)} \leq c^2h_1(E(\pi^2_{(s,t)}) \circ T^{-1}).$$

Theorem 4.3 :
C_r *is quasiposinormal iff* $P_v^4 \leq c^2h_2E(\pi_2^2) \circ T^{-2}$.
Proof:
C_r *is quasiposinormal if* $(C_rC_r^*)^2 \leq c^2C_r^{*^2}C_r^2$

$$\left\langle ((C_rC_r^*)^2 \leq c^2C_r^{*^2}C_r^2)f, f\right\rangle \leq 0$$

$$\Leftrightarrow \int_E [P_v^4 - c^2h_2(E(\pi_2^2) \circ T^{-2})]\,|f|^2\,d\lambda \leq 0, E\ \epsilon\ \Sigma$$

$$\Leftrightarrow P_v^4 - c^2h_2(E(\pi_2^2) \circ T^{-2}) \leq 0$$

$$\Leftrightarrow P_v^4 \leq c^2h_2(E(\pi_2^2) \circ T^{-2}).$$

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