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A Subclass of Close-to-Convex Functions

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Abstract

Let $J(A,B)$ denote the class of functions $f(z)=z+\sum_{k=2}^{\infty}a_k z^k$, regular in the unit disc $E=\{z:|z|<1\}$ such that $\frac{zf'(z)}{h(z)}\prec\frac{1+Az}{1+Bz}$, $-1\le B<A\le 1$, $z\in E$, where $h(z)=z+\sum_{k=2}^{\infty}b_k z^k$ is convex univalent in E . In this paper , we determine the coefficient estimates , distortion theorems , argument theorem and radius of convexity for the class $J(A,B)$.

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1. Introduction

Let U be the class of bounded functions

$$w(z) = \sum_{k=1}^{\infty} c_k z^k \qquad (1.1)$$

which are regular in the unit disc $E = \{z : |z| < 1\}$ and satisfying the conditions

$$w(0) = 0 \text{ and } |w(z)| < 1, \ z \in E .$$

Let K denote the class of functions

$$h(z) = z + \sum_{k=2}^{\infty} b_k z^k \qquad (1.2)$$

regular and convex univalent in *E*.

Let $J(A,B)$ denote the class of functions

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (1.3)$$

regular in E and satisfying the conditions

$$\frac{zf'(z)}{h(z)} \prec \frac{1+Az}{1+Bz} \ , \ -1 \le B < A \le 1, \ z \in E \ , \ h(z) \in K .$$

Obviously $J(A,B)$ is a subclass of the class $J \equiv J(1,-1)$ of close-to-convex functions discussed by Gawad and Thomas [1].

By definition of subordination it follows that $f(z) \in J(A,B)$ if ,and only if $f(z)$ can be represented in the form

$$\frac{zf'(z)}{h(z)} = \frac{1+Aw(z)}{1+Bw(z)} \ , \ w(z) \in U \ , \ -1 \le B < A \le 1 \ , \ z \in E \ . \qquad (1.4)$$

To avoid repetition , we lay down once for all that $-1 \le B < A \le 1, \ z \in E$.

We study the class $J(A,B)$ and obtain coefficient estimates , distortion theorems, argument theorem and radius of convexity .

2. Some Preliminary Lemmas

We need the following lemmas:

Lemma 2.1. Let

$$\frac{zf'(z)}{h(z)} = P(z) = 1+\sum_{k=1}^{\infty} p_k z^k \quad , \tag{2.1}$$

then $$|p_n| \le (A-B)\ , \quad n \ge 1\ . \tag{2.2}$$

The bounds are sharp , being attained for the functions

$$P_n(z) = \frac{1+A\delta z^n}{1+B\delta z^n}\ , \quad |\delta| = 1.$$

This lemma is due to Goel and Mehrok [2] .

Lemma 2.2. If $w(z) \in U$, then for $|z| = r < 1$,

$$|zw'(z) - w(z)| \le \frac{r^2 - |w(z)|^2}{1-r^2}\ .$$

Singh and Goel proved this result in [4] .

Lemma 2.3. Let $p(z) = \dfrac{1+Bw(z)}{1+Aw(z)}$, $w(z) \in U$, then for $|z| = r < 1$,

$$\operatorname{Re}\left[Ap(z) + \frac{B}{p(z)}\right] + \frac{r^2|Ap(z)-B|^2 - |1-p(z)|^2}{(1-r^2)|p(z)|}$$

$$\le \begin{cases} \dfrac{AB(A+B)r^2 - 4ABr + (A+B)}{(1-Ar)(1-Br)}, R_1 \le R_0, \\ \dfrac{2}{(1-r^2)}\left[(1-ABr^2) - \left((1-A)(1-B)(1+Ar^2)(1+Br^2)\right)^{1/2}\right], R_1 \ge R_0, A \ne 1, \end{cases}$$

where $R_1 = \dfrac{1-Br}{1-Ar}$ and $R_0^2 = \dfrac{(1-B)(1+Br^2)}{(1-A)(1+Ar^2)}$.

The bounds are sharp .

Goel and Mehrok [3] established this result.

3. Coefficient Inequalities

Theorem. 3.1. If $f(z) \in J(A,B)$, then

$$|a_n| \le \frac{1}{n} + \frac{(n-1)(A-B)}{n} \ , \ n \ge 2 . \tag{3.1}$$

The bounds are sharp .

Proof. Using (1.2) and (1.3) in (2.1) , we get

$$z\left(1+\sum_{k=2}^{\infty} k a_k z^{k-1}\right) = \left(z+\sum_{k=2}^{\infty} b_k z^k\right)\left(1+\sum_{k=1}^{\infty} p_k z^k\right) . \tag{3.2}$$

Equating the coefficients of z^n in (3.2) , we have

$$n a_n = b_n + p_1 b_{n-1} + p_2 b_{n-2} + \dots\dots + p_{n-1} . \tag{3.3}$$

Therefore using (2.2) ,

$$n|a_n| \le |b_n| + (A-B)\left[|b_{n-1}| + |b_{n-2}| + \dots\dots + |b_2| + 1\right] .$$

Also it is well known that $|b_n| \le 1$, $n \ge 2$. Hence

$$|a_n| \le \frac{1}{n} + \frac{(n-1)(A-B)}{n} .$$

For $n = 2$, equality signs in (3.1) hold for the functions $f_n(z)$ defined by

$$f_n'(z) = \frac{1}{(1-\delta_1 z)} \frac{1+A\delta_2 z^{n-1}}{1+B\delta_2 z^{n-1}} \ , \ |\delta_1| = 1 \ , \ |\delta_2| = 1 . \tag{3.4}$$

On putting $A = 1, B = -1$ in the above theorem , we get the following result due to Gawad and Thomas [1] .

Corollary. Let $f(z) \in J$, then

$$|a_n| \le 2 - \frac{1}{n} .$$

4. Distortion Theorems

Theorem. 4.1. If $f(z) \in J(A,B)$, then for $|z| = r$, $0 < r < 1$, we have

$$\frac{1-Ar}{(1-Br)(1+r)} \le |f'(z)| \le \frac{1+Ar}{(1+Br)(1-r)} \tag{4.1}$$

And

$$\int_0^r \frac{(1-At)}{(1-Bt)(1+t)}dt \le \left| f(z) \right| \le \int_0^r \frac{(1+At)}{(1+Bt)(1-t)}dt \quad . \tag{4.2}$$

Estimates are sharp.

Proof. From (1.4) , we have

$$\left| zf'(z) \right| = \left| h(z) \right| \left| \frac{1+Aw(z)}{1+Bw(z)} \right| \ , \quad w(z) \in U \quad . \tag{4.3}$$

It is easy to show that the transformation

$$\frac{zf'(z)}{h(z)} = \frac{1+Aw(z)}{1+Bw(z)}$$

maps $\left| w(z) \right| \le r$ onto the circle

$$\left| \frac{zf'(z)}{h(z)} - \frac{1-ABr^2}{1-B^2r^2} \right| \le \frac{(A-B)r}{(1-B^2r^2)} \ , \ |z| = r \ .$$

This implies that

$$\frac{1-Ar}{1-Br} \le \left| \frac{1+Aw(z)}{1+Bw(z)} \right| \le \frac{1+Ar}{1+Br} \quad . \tag{4.4}$$

Since $h(z)$ is convex , so

$$\frac{r}{1+r} \le \left| h(z) \right| \le \frac{r}{1-r} \ . \tag{4.5}$$

(4.3) together with (4.4) and (4.5) yields (4.1) . On integrating (4.1) , (4.2) follows. For $n = 2$, the function $f_n(z)$ defined by (3.4), gives sharp estimates.

For $A = 1, B = -1$, we have the following :

Corollary. Let $f \in J$, then for $z = re^{i\theta} \in E$, we have

$$\frac{1-r}{(1+r)^2} \le \left| f'(z) \right| \le \frac{1+r}{(1-r)^2} \qquad \text{and}$$

$$-\log(1+r) + \frac{2r}{1+r} \le \left| f(z) \right| \le \log(1-r) + \frac{2r}{1-r} \ .$$

These results were proved by Gawad and Thomas [1].

5. Argument of $f'(z)$

Theorem. 5.1. If $f(z) \in J(A,B)$, then

$$\left|\arg f'(z)\right| \le \sin^{-1} r + \sin^{-1}\frac{(A-B)r}{1-ABr^2} \ . \tag{5.1}$$

The result is sharp.

Proof. As discussed in theorem 4.1 , the transformation

$$\frac{zf'(z)}{h(z)} = \frac{1+Aw(z)}{1+Bw(z)}$$

maps $|w(z)| \le r$ onto the circle

$$\left|\frac{zf'(z)}{h(z)} - \frac{1-ABr^2}{1-B^2r^2}\right| \le \frac{(A-B)r}{(1-B^2r^2)} \ , \ |z| = r \ .$$

So

$$\left|\arg\frac{zf'(z)}{h(z)}\right| \le \sin^{-1}\frac{(A-B)r}{1-ABr^2} \ . \tag{5.2}$$

Also

$$\left|\arg\frac{h(z)}{z}\right| \le \sin^{-1} r \ . \tag{5.3}$$

Since

$$\left|\arg f'(z)\right| \le \left|\arg\frac{h(z)}{z}\right| + \left|\arg\frac{zf'(z)}{h(z)}\right| \ . \tag{5.4}$$

Using (5.2) and (5.3) in (5.4) , (5.1) follows . The result (5.1) is sharp for the function $f_n(z)$ ($n=2$) defined by (3.4), where

$$\delta_2 = \frac{z}{r}\left[\frac{-(A+B)r + i\left((1-A^2r^2)(1-B^2r^2)\right)^{1/2}}{1+ABr^2}\right] \ .$$

On putting $A=1, B=-1$, we obtain the following

Corollary. Let $f \in J$, then

$$\left|\arg f'(z)\right| \le \sin^{-1} r + \sin^{-1}\frac{2r}{1+r^2} \ , \text{ as shown in [1] .}$$

6. Radius Of Convexity

Theorem. 7.1. Let $f(z) \in J(A,B)$, then

(i) For $\frac{1}{3} \le A \le 1$, $f(z)$ is convex in $|z| < r_0$, where r_0 is the smallest positive root of

$$1 - 2Ar + (AB - A + B)r^2 = 0 \ ; \tag{6.1}$$

(ii) For $-1 < A \le \frac{1}{3}$, $f(z)$ is convex in $|z| < r_1$, where r_1 is the smallest positive root of

$$4(A-1) - 4(A-1)r + (A - 5B + 4AB)r^2 + 2(A + B - 2AB)r^3 + (A-B)r^4 = 0. \tag{6.2}$$

Results are sharp .

Proof. After differentiating (1.4) logarithmically, we get

$$\left(1 + \frac{zf''(z)}{f'(z)}\right) = \frac{zh'(z)}{h(z)} + (A - B)\frac{zw'(z)}{(1 + Aw(z))(1 + Bw(z))} \ . \tag{6.3}$$

Using lemma (2.2) , (6.3) becomes

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \ge \operatorname{Re}\frac{zh'(z)}{h(z)}$$

$$+(A-B)\left[\operatorname{Re}\frac{w(z)}{(1+Aw(z))(1+Bw(z))} - \frac{r^2 - |w(z)|^2}{(1-r^2)|(1+Aw(z))(1+Bw(z))|}\right]. \tag{6.4}$$

Put $p(z) = \dfrac{1 + Bw(z)}{1 + Aw(z)}$, $w(z) \in U$.

Thus from (6.4) ,it follows that

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \ge \operatorname{Re}\frac{zh'(z)}{h(z)} + \frac{(A+B)}{(A-B)}$$

$$-\frac{1}{(A-B)}\left[\operatorname{Re}\left(Ap(z) + \frac{B}{p(z)}\right) + \frac{r^2|Ap(z) - B|^2 - |1 - p(z)|^2}{(1-r^2)|p(z)|}\right]. \tag{6.5}$$

Since $h(z)$ is convex , so

$$\operatorname{Re}\frac{zh'(z)}{h(z)} \ge \frac{1}{1+r} \ . \tag{6.6}$$

Using (6.6) and lemma 2.3 , (6.5) gives

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right)$$

$$\geq \begin{cases} \dfrac{1-2Ar+(AB-A+B)r^2}{(1+r)(1-Ar)(1-Br)}, R_1 \leq R_0, \\ \dfrac{\left[2(A-1)-(A-B)r-(A+B-2AB)r^2\right] + 2\left[(1-A)(1-B)(1+Ar^2)(1+Br^2)\right]^{1/2}}{(A-B)(1-r^2)}, R_1 \geq R_0, A \neq 1. \end{cases} \quad (6.7)$$

On equating the right hand sides of (6.7) to zero, we obtain (6.1) and (6.2).

The equation $R_1 = R_0$ yields

$$1-2r+(2A+2B-AB-1)r^2-2ABr^3+ABr^4=0 . \quad (6.8)$$

After eliminating r between (6.1) and (6.8) , we get

$$(1+B)^2\left[(4B-8)A^3+7A^2-(3B+1)A+B\right]=0 . \quad (6.9)$$

If $1+B\neq 0$, we have

$$B=\frac{8A^3-7A^2+A}{4A^3-3A+1} \ , \ A\neq 1 .$$

Then $B<A$ implies that $0<4A^2(A-1)^2$ which holds.

Also $B=\dfrac{8A^3-7A^2+A}{4A^3-3A+1}>-1$ implies $A<\dfrac{1}{3}<1$.

For $B=-1$, elimination of r between (6.1) and (6.8) gives

$$12A^3-7A^2-2A+1=0 .$$

Therefore $A=\dfrac{1}{3}=A_0$, say .

The results in the theorem are sharp , being attained respectively for the functions $f_1(z)$ and $f_\theta(z)$ defined by

$$f_1'(z)=\frac{1+Az}{(1-z)(1+Bz)} ,$$

$$f_\theta'(z)=\frac{1-(1+A)z\cos\theta+Az^2}{(1-z)\left(1-(1+B)z\cos\theta+Bz^2\right)} ,$$

where $R_0=\dfrac{1-(1+B)r\cos\theta+Br^2}{1-(1+A)r\cos\theta+Ar^2}$, $R_0^2=\dfrac{(1-B)(1+Br^2)}{(1-A)(1+Ar^2)}$.

Corollary. Let $f \in J$, then $r = \frac{1}{3}$ which is the radius of convexity of J due to Gawad and Thomas [1] .

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