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Weighted Composition Operators between weighted Bergman-Nevanlinna and Growth Spaces

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Abstract. In this paper, we characterize the boundedness and compactness of weighted composition operators $\psi C_\varphi f = \psi(f \circ \varphi)$ acting between Bergman-Nevanlinna and Growth spaces of holomorphic functions on the open unit disk $\mathbb{D}$. In fact, we prove that ψC_φ maps compactly from weighted Bergman-Nevanlinna spaces to Growth spaces if and only if it maps boundedly between these spaces.

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1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. Denote by $H(\mathbb{D})$, the space of holomorphic functions on $\mathbb{D}$. Let $dA(z)$ be the area measure on $\mathbb{D}$ normalized so that area of $\mathbb{D}$ is 1. For each $\lambda \in (-1, \infty)$, we set $d\nu_\lambda(z) = (\lambda+1)(1-|z|^2)^\lambda dA(z),\ z \in \mathbb{D}$. Then $d\nu_\lambda$ is a probability measure on $\mathbb{D}$. For $0 < p < \infty$ the weighted Bergman space $\mathcal{A}^p_\lambda$ is defined as

$$\mathcal{A}^p_\lambda = \left\{ f \in H(\mathbb{D}) : ||f||_{\mathcal{A}^p_\lambda} = \Big(\int_{\mathbb{D}} |f(z)|^p d\nu_\lambda(z) \Big)^{1/p} < \infty \right\}.$$

The weighted Bergman-Nevanlinna class $\mathcal{A}^0_\lambda$ is defined by

$$\mathcal{A}^0_\lambda = \left\{ f \in H(\mathbb{D}) : ||f||_{\mathcal{A}^0_\lambda} = \int_D \log(1+|f(z)|) d\nu_\lambda(z) < \infty \right\}.$$

Of course, we are abusing the term norm since $||f||_{\mathcal{A}^0_\lambda}$ fails to satisfy the properties of norm, but in this case $(f, g) \to ||f-g||_{\mathcal{A}^0_\lambda}$ defines a translation invariant metric on $\mathcal{A}^0_\lambda$ and this turns $\mathcal{A}^0_\lambda$ into a complete metric space. Also, by the subharmonicity of $\log(1+|f(z)|)$, we have

$$\log(1+|f(z)|) \le C_\lambda \frac{||f||_{\mathcal{A}^0_\lambda}}{(1-|z|^2)^{\lambda+2}}, \qquad z \in \mathbb{D} \tag{1.1}$$

for all $f \in \mathcal{A}^0_\lambda$. For any $\alpha > 0$, the space $\mathcal{A}^{-\alpha}$ consists of analytic functions f in $\mathbb{D}$ such that

$$||f||_{\mathcal{A}^{-\alpha}} = \sup\{(1-|z|^2)^\alpha |f(z)| : z \in \mathbb{D}\} < \infty.$$

Each $\mathcal{A}^{-\alpha}$ is a non-separable Banach space with the norm defined above and contains all bounded analytic functions on $\mathbb{D}$. The closure in $\mathcal{A}^{-\alpha}$ of the set of polynomials will be denoted by $\mathcal{A}^{-\alpha}_0$, which is a separable Banach space and consists of exactly those functions f in $\mathcal{A}^{-\alpha}$ with

$$\lim_{|z|\to 1^-} (1-|z|^2)^\alpha |f(z)| = 0.$$

For general background on weighted Bergman spaces $\mathcal{A}^p_\lambda$, weighted Bergman-Nevanlinna spaces $\mathcal{A}^0_\lambda$, and Growth spaces $\mathcal{A}^{-\alpha}$ and $\mathcal{A}^{-\alpha}_0$, one may consult [**5**] and [**14**] and the references therein.

Let $\psi \in H(\mathbb{D})$ and φ be a holomorphic self-map on $\mathbb{D}$. Define a linear operator

$$\psi C_{\varphi} f = \psi (f \circ \varphi), \qquad f \in H(\mathbb{D})$$

The operator ψC_{φ} is the weighted composition operator. The weighted composition operator is a generalization of the composition operator C_{φ} defined as $C_{\varphi} f = (f \circ \varphi)$ and the multiplication operator defined as $M_{\psi} f = \psi f$. Weighted composition operators have gained increasing recognitation during the last three decades, mainly due to the fact that they provide, just as, for example, Hankel and Toeplitz operators, ways and means to link classical function theory to functional analysis and operator theory. For general back ground on composition operators, we refer [**4**], [**11**] and references therein. Recently, several authors have studied weighted composition operators on different spaces of analytic functions. For more information on weighted composition operators, one can refer to [**1**], [**2**], [**3**], [**6**], [**7**], [**8**], [**9**], [**10**], [**12**] and [**13**].

2. Boundedness and compactness

In this section, we characterize the boundedness and compactness of weighted composition operators between weighted Bergman-Nevanlinna and Growth spaces. The boundedness and compactness of weighted composition operators from weighted Bergman spaces $\mathcal{A}^p_\lambda$ to Growth spaces $\mathcal{A}^{-\alpha}$ and $\mathcal{A}^{-\alpha}_0$ were studied in [**12**].

The following criterion for compactness is a useful tool to us and it follows from standard arguments, for example, to those outlined in Proposition 3.11 of [**4**].

Lemma 2.1. *Let $\lambda \in (-1, \infty)$. Then $\psi C_{\varphi} : \mathcal{A}^0_\lambda \to \mathcal{A}^{-\alpha}$ is compact if and only if for any sequence $\{f_n\}$ in $\mathcal{A}^0_\lambda$ with $\sup_n \|f_n\|_{\mathcal{A}^0_\lambda} = M < \infty$ and which converges to zero locally uniformly on $\mathbb{D}$, we have $\lim_{n\to\infty} \|\psi C_{\varphi} f_n\|_{\mathcal{A}^{-\alpha}} \to 0$ in $\mathcal{A}^{-\alpha}$.*

Theorem 2.2. *Let $\alpha > 0, \lambda > -1, \psi \in H(\mathbb{D})$ and φ be a holomorphic self-map of $\mathbb{D}$. Then the following are equivalent:*

(i) *ψC_{φ} maps $\mathcal{A}^0_\lambda$ boundedly into $\mathcal{A}^{-\alpha}$.*
(ii) *ψC_{φ} maps $\mathcal{A}^0_\lambda$ compactly into $\mathcal{A}^{-\alpha}$.*

(iii) *For all* $c > 0$,

$$\lim_{|\varphi(z)|\to 1} \left(1-|z|^2\right)^{\alpha} |\psi(z)| \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\lambda+2}}\right] = 0$$

and $\psi \in \mathcal{A}^{-\alpha}$.

Proof. It suffices to check only two implications: $(i) \Rightarrow (iii)$ and $(iii) \Rightarrow (ii)$.

$(i) \Rightarrow (iii)$. Suppose (i) holds. By taking $f(z) = 1$, the constant function one in $\mathcal{A}^0_\lambda$, we get $\psi \in \mathcal{A}^{-\alpha}$. Fix $z_0 \in \mathbb{D}$ and $c > 0$ and let $w = \varphi(z_0)$. Consider the function

$$f_w(z) = \left(\frac{1-|w|^2}{(1-\overline{w}z)^2}\right)^{(\lambda+2)} \exp\left\{\frac{c(1-|w|^2)^{\lambda+2}}{(1-\overline{w}z)^{2(\lambda+2)}}\right\}.$$

Using the inequalities, $\log(1+x) \le 1 + \log^+ x$, $\log(1+x) \le x$ and

$$\log(1+xy) \le \log(1+x) + \log(1+y) \quad \text{for} \quad x, y \ge 0,$$

we have

$$\begin{aligned} \log(1+|f_w(z)|) &\le \log\left[1+\left(\frac{1-|w|^2}{|1-\overline{w}z|^2}\right)^{\lambda+2}\right] + 1 + \left\{\frac{c(1-|w|^2)^{\lambda+2}}{|1-\overline{w}z|^{2(\lambda+2)}}\right\} \\ &\le 1 + (1+c)\left(\frac{1-|w|^2}{|1-\overline{w}z|^2}\right)^{\lambda+2} \end{aligned}$$

and so

$$\|f_w\|_{\mathcal{A}^0_\lambda} \le 1 + (1+c)\int_{\mathbb{D}} \left(\frac{1-|w|^2}{|1-\overline{w}z|^2}\right)^{\lambda+2} d\nu_\lambda(z) \le 2 + c.$$

Since ψC_φ maps $\mathcal{A}^0_\lambda$ boundedly into $\mathcal{A}^{-\alpha}$, there is a constant $M > 0$ such that

$$\begin{aligned} M &\ge (1-|z_0|^2)^{\alpha} |\psi(z_0)| \, |f(\varphi(z_0))| \\ &= \frac{(1-|z_0|^2)^{\alpha}|\psi(z_0)|}{(1-|\varphi(z_0)|^2)^{\lambda+2}} \exp\left\{\frac{c}{(1-|\varphi(z_0)|^2)^{\lambda+2}}\right\} \end{aligned}$$

That is,

$$(1-|z_0|^2)^{\alpha}|\psi(z_0)| \exp\left\{\frac{c}{(1-|\varphi(z_0)|^2)^{\lambda+2}}\right\} \le M(1-|\varphi(z_0)|^2)^{\lambda+2}.$$

Taking $\lim_{|\varphi(z_0)|\to 1}$ on both sides of above inequality, we get (iii).

$(iii) \Rightarrow (ii)$. Assume that (iii) is valid for all $c > 0$. Note that if $f \in \mathcal{A}^0_\lambda$, then

$$|f(z)| \le \exp\left\{\frac{M_0 \|f\|_{\mathcal{A}^0_\lambda}}{(1-|z|^2)^{\lambda+2}}\right\}.$$

Choose any sequence f_n in $\mathcal{A}_\lambda^0$ such that $\|f_n\|_{\mathcal{A}_\lambda^0} \leq M$ and $f_n \longrightarrow 0$ locally uniformly on $\mathbb{D}$. By Lemma 2.1, it is sufficient to show that $\psi C_\varphi f_n \longrightarrow 0$ as $n \longrightarrow \infty$. For $r \in (0,1)$

$$\begin{aligned}\sup_{|\varphi(z)|\leq r}(1-|z|^2)^\alpha |\psi C_\varphi f_n(z)| &= \sup_{|\varphi(z)|\leq r}(1-|z|^2)^\alpha |\psi(z)| |f_n(\varphi(z))| \\ &\leq A \sup_{|\varphi(z)|\leq r}|f_n(\varphi(z))| \longrightarrow 0\end{aligned}$$

as $n \longrightarrow \infty$, where $A = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |\psi(z)| < \infty$. On the other hand, whenever $r \longrightarrow 1$, we have

$$\sup_{|\varphi(z)|>r}(1-|z|^2)^\alpha |\psi C_\varphi f_n(z)|$$

$$\leq \sup_{|\varphi(z)|>r}(1-|z|^2)^\alpha |\psi(z)| \exp\left\{\frac{M_0 \|f\|_{\mathcal{A}_\lambda^0}}{(1-|z|^2)^{\lambda+2}}\right\} \longrightarrow 0.$$

Combining the above estimates, we see that $\|\psi C_\varphi f_n\|_{\mathcal{A}^{-\alpha}} \longrightarrow 0$ as $n \longrightarrow \infty$. This completes the proof.

Corollary 2.3. *Let $\alpha > 0, \lambda > -1$ and φ be a holomorphic self map of $\mathbb{D}$. Then the following are equivalent:*

(i) *C_φ maps $\mathcal{A}_\lambda^0$ boundedly into $\mathcal{A}^{-\alpha}$.*

(ii) *C_φ maps $\mathcal{A}_\lambda^0$ compactly into $\mathcal{A}^{-\alpha}$.*

(iii) $\lim_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\lambda+2}}\right] = 0$

Lemma 2.4. *Let $\alpha > 0, \lambda > -1, \psi \in H(\mathbb{D})$ and φ be a holomorphic self map of $\mathbb{D}$. Then the following are equivalent:*

(i) *For all $c > 0$,*

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha |\psi(z)| \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\lambda+2}}\right] = 0. \tag{2.1}$$

(ii) *For all $c > 0$,*

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha |\psi(z)| \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\lambda+2}}\right] = 0 \tag{2.2}$$

and $\psi \in \mathcal{A}_0^{-\alpha}$.

Proof. $(i) \implies (ii)$ Suppose that (i) holds. Then for some $C > 0$,

$$(1-|z|^2)^\alpha |\psi(z)| \leq C\,(1-|z|^2)^\alpha |\psi(z)| \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\lambda+2}}\right] \longrightarrow 0$$

as $|z| \longrightarrow 1$. Again, if $|\varphi(z)| \longrightarrow 1$, then $|z| \longrightarrow 1$, from which it follows that

$$\lim_{|\varphi(z)|\to 1} (1-|z|^2)^\alpha |\psi(z)| \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\lambda+2}}\right] = 0.$$

$(ii) \implies (i)$ Suppose that (ii) holds, but (i) is not true for some $c>0$, then there are c_0 and ϵ_0 and a sequence $\{z_n\}$ tending to $\partial\mathbb{D}$ such that

$$(1-|z_n|^2)^\alpha |\psi(z_n)| \exp\left[\frac{c_0}{(1-|\varphi(z_n)|^2)^{\lambda+2}}\right] \geq \epsilon_0. \tag{2.3}$$

Since $\psi \in \mathcal{A}_0^{-\alpha}$, (2.3) indicates that $\{z_n\}$ has a subsequence $\{z_{n_k}\}$ with $|\varphi(z_{n_k})| \longrightarrow 1$. Thus (2.2) produces the following limit:

$$(1-|z_{n_k}|^2)^\alpha |\psi(z_{n_k})| \exp\left[\frac{c}{(1-|\varphi(z_{n_k})|^2)^{\lambda+2}}\right] \longrightarrow 0,$$

which contradicts (2.3). Hence we are done.

Theorem 2.5. *Let $\alpha>0, \lambda>-1, \psi \in H(\mathbb{D})$ and φ be a holomorphic self map of $\mathbb{D}$. Then the following are equivalent:*

(i) *ψC_φ maps $\mathcal{A}_\lambda^0$ boundedly into $\mathcal{A}_0^{-\alpha}$.*
(ii) *ψC_φ maps $\mathcal{A}_\lambda^0$ compactly into $\mathcal{A}_0^{-\alpha}$.*
(iii) *For all $c>0$,*
$$\lim_{|z|\to 1}(1-|z|^2)^\alpha |\psi(z)| \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\lambda+2}}\right] = 0$$

Proof. $(i) \implies (iii)$ Using the same test function as in Theorem 2.2, we have for all $c>0$,

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha |\psi(z)| \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\lambda+2}}\right] = 0$$

and $\psi \in \mathcal{A}_0^{-\alpha}$ and so by Lemma 2.4, we have

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha |\psi(z)| \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\lambda+2}}\right] = 0.$$

The proof of the implications $(iii) \Rightarrow (ii)$ follows on the same lines as the proof of the corresponding case of Theorem 2.2. We omit the details.

Corollary 2.6. *Let $\alpha>0, \lambda>-1$ and φ be a holomorphic self map of $\mathbb{D}$. Then the following are equivalent:*

(i) *C_φ maps $\mathcal{A}_\lambda^0$ boundedly into $\mathcal{A}_0^{-\alpha}$.*
(ii) *C_φ maps $\mathcal{A}_\lambda^0$ compactly into $\mathcal{A}_0^{-\alpha}$.*
(iii) *For all $c>0$,*
$$\lim_{|z|\to 1}(1-|z|^2)^\alpha \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\lambda+2}}\right] = 0$$

Corollary 2.7. *Let $\alpha > 0, \lambda > -1$ and $\psi \in H(\mathbb{D})$. Then the following are equivalent:*

(i) M_ψ *maps* $\mathcal{A}_\lambda^0$ *boundedly into* $\mathcal{A}^{-\alpha}$.

(ii) M_ψ *maps* $\mathcal{A}_\lambda^0$ *compactly into* $\mathcal{A}^{-\alpha}$.

(iii) M_ψ *maps* $\mathcal{A}_\lambda^0$ *boundedly into* $\mathcal{A}_0^{-\alpha}$.

(iv) M_ψ *maps* $\mathcal{A}_\lambda^0$ *compactly into* $\mathcal{A}_0^{-\alpha}$.

(v) $\psi \equiv 0$.

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