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Emergence in Random Noisy Environments

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Abstract

We investigate the emergent behavior of four types of generic dynamical systems under random environmental perturbations. Sufficient conditions for nearly-emergence in various scenarios are presented. Recent fundamental works of F. Cucker and S. Smale on the construction and analysis of flocking models directly inspired our present work.

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1 Introduction

The emergent behaviors of a large number of autonomous interacting agents such as flocking of birds [12, 17], multi-agent cooperative coordination in mobile networks [1, 3] and emergence of a common language in primitive societies [8, 11] have been attracting great research attentions since the last two decades from biologists, physicists, sociologists, engineers and mathematicians.

Recently, Cucker and Smale [6] have proposed a remarkable model aiming to exploring the flocking phenomenon and mathematical analysis is performed to show the convergence results only depend on some initial states of the population. This notable feature is in contrast with the previous models (e.g. the so called Vicsek model [17]) where convergence relies on the global behavior of the agents' trajectories (or on the neighborhood graphs of the underlying dynamical systems), which are quite hard to verify in general. The same authors [5] extend the model later to a more general setting beyond flocking. [14] further develops a hierarchical leadership architecture in the Cucker-Smale flocking model. The work in [4] focuses on a situation where uniform or Gaussian noises are involved in the environments. A hydrodynamic description and the mean-field limit of this very model are also provided in [7].

The starting point of our present work is directly motivated by the aforementioned series work. Primarily, we want to refine the rudimental results (in the noisy environment) in [4] and extend them to more general scenarios such as those discussed by [5]; and try to shed some light on the understanding of various emergence behaviors observed in diverse natural, social and man-made complex systems [15]. To do so, we first introduce four types of non-autonomous, nonlinear dynamical systems; two (I(D) and II(D)) for discrete time and two (I(C) and II(C)) for continuous time. In each case, we provide a convergence analysis. Systems I(D) and I(C) are adapted from [5, 6] and the underlying idea stems from the birds flocking in a noisy environment. Whereas the original idea behind systems II(D) and II(C) is the linguistic evolution with some possible fluctuations in a primitive society. The random noises considered here may reflect the change of the environment which is usually unclear to the objects. Moreover, information interaction among agents may be contaminated or corrupted by errors. Hence, it becomes significant to analyze systems in the presence of random noises. We mention that the systems tackled in this paper are quintessential in the sense of reflecting some typical mechanisms behind emergence (see Remark 1 in Section 2.1), but by no means limited to flocking or language evolution since we will treat them in a quite general manner with emphasis on the methodology. Some other related work about emergent behaviors under random environmental perturbation can be found in e.g. [9, 10, 16] and references therein.

The rest of this paper is organized as follows. In Section 2, we will study the discrete time models I(D), II(D). Section 3 is devoted to the continuous counterparts I(C), II(C). We then draw our conclusion and discuss future direction in Section 4.

2 Discrete-time Emergence

Let $k \in \mathbb{N}$. We assume the population under consideration consists of k agents throughout the paper.

2.1 Models Setup (I(D), II(D))

We shall first introduce the dynamical system I(D), which is developed similarly with that considered in [5].

Suppose X and Y are two given inner product spaces whose elements are denoted as x and y, respectively. Let $x(t) = (x_1(t), \cdots, x_k(t)) \in X$ and $(y_1(t), \cdots, y_k(t)) \in Y^k$ represent two kinds of characteristics of the agents at time instant t. Convergence of $x \in X$ (or $y \in Y^k$) is naturally understood as entrywise convergence as t approaches infinity. Let $\triangle$ signify the diagonal of Y^k, that is, $\triangle := \{(y, \cdots, y)|\ y \in Y\}$. Denote $\widetilde{Y} := Y^k/\triangle$ and fix an inner product $\langle \cdot, \cdot \rangle$ in $\widetilde{Y}$, which induces a norm $||\cdot||$. (Here in the discrete case, we do not really need an inner product; what we want is $\widetilde{Y}$ should be a normed space. The same remark applies to X and Y.) Since $\widetilde{Y}$ is a finite dimensional space, $\hat{y} := (y_1(t), \cdots, y_k(t)) \to (y_0, \cdots, y_0)$ for some $y_0 \in Y$ if and only if $||\bar{\hat{y}} - \bar{0}|| \to 0$, where $\bar{a} := a + \triangle \in \widetilde{Y}$ for $a \in Y^k$. In what follows, we denote norms in all different spaces as $||\cdot||$ with some ambiguity, but the proper meaning will be clear in the context.

For $x \in X$, $y \in \widetilde{Y}$, consider the following dynamical system:

$$I(D): \qquad \begin{cases} x(t+h) = x(t) + hJ(x(t), y(t)) \\ y(t+h) = S(x(t))y(t) + hH(t) \end{cases}$$

Here h is the time step and we shall denote in the sequel $x[t] := x(th)$, $y[t] := y(th)$ and $H[t] := H(th)$ for brevity. Take $t \in \mathbb{N}$ herein. We now explain the notations in system I(D). Let $J : X \times \widetilde{Y} \to X$ be a Lipschitz or C^1 operator satisfying, for some $C, \delta > 0$, $0 \leq \gamma < 1$, that

$$||J(x, y)|| \leq C(1 + ||x||)^{\gamma} ||y||^{\delta} \tag{1}$$

for all $x \in X$, $y \in \widetilde{Y}$. Let $S : X \to \mathrm{End}(\widetilde{Y})$ be an operator satisfying, for some $G > 0$, $\beta \geq 0$, that

$$||S(x)|| \leq 1 - \frac{hG}{(1 + ||x||)^{\beta}} \tag{2}$$

for all $x \in X$. The operator norm in (2) is defined as $||S(x)|| = \sup_{\substack{y \neq 0 \\ y \in \widetilde{Y}}} \frac{||S(x)y||}{||y||}$. Let $H : (\Omega, \mathcal{F}, P) \to (\widetilde{Y}, \mathcal{B}(\widetilde{Y}))$ be a random element. $(\Omega, \mathcal{F}, P)$ is some probability space and $\mathcal{B}(\widetilde{Y})$ is the Borel σ-algebra on $\widetilde{Y}$. We assume $H[t]$ is independent and identically distributed for different $t \in \mathbb{N}$. Notice that $||H|| : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a random variable and let $F(x) := P(||H|| \leq x)$ for $x \in \mathbb{R}$ be the distribution function of $||H||$.

Next, we present our dynamical system II(D) as follows. The spaces $X, Y, \widetilde{Y}$ are defined as before. For $x \in X$, $y \in \widetilde{Y}$, consider the following dynamical system:

$$II(D): \qquad \begin{cases} x(t_1 + h_1) = S_1(y(t_2))x(t_1) + h_1 H_1(t_1) \\ y(t_2 + h_2) = S_2(x(t_1))y(t_2) + h_2 H_2(t_2) \end{cases}$$

Here h_1, h_2 are the time steps w.r.t. x and y. Take $t_1, t_2 \in \mathbb{N}$ and we denote $x[t_1] = x(t_1 h_1)$, $y[t_2] = y(t_2 h_2)$, $H_1[t_1] = H_1(t_1 h_1)$ and $H_2[t_2] = H_2(t_2 h_2)$. In light of these notations, the system II(D) can be rewritten as follows

$$\begin{cases} x[t+1] = S_1(y[t])x[t] + h_1 H_1[t] \\ y[t+1] = S_2(x[t])y[t] + h_2 H_2[t] \end{cases}$$

In analogy with the system I(D), suppose $S_1 : \widetilde{Y} \to \mathrm{End}(X)$ and $S_2 : X \to \mathrm{End}(\widetilde{Y})$ are two operators satisfying, for some $G_1, G_2 > 0$, $\beta_1, \beta_2 \geq 0$, that

$$||S_1(y)|| \leq 1 - \frac{h_1 G_1}{(1+||y||)^{\beta_1}}, \qquad ||S_2(x)|| \leq 1 - \frac{h_2 G_2}{(1+||x||)^{\beta_2}} \tag{3}$$

for all $y \in \widetilde{Y}$, $x \in X$. Let $H_1 : (\Omega_1, \mathcal{F}_1, P_1) \to (X, \mathcal{B}(X))$ and $H_2 : (\Omega_2, \mathcal{F}_2, P_2) \to (\widetilde{Y}, \mathcal{B}(\widetilde{Y}))$ be two random elements as before. We assume $H_1[t]$ is independent and identically distributed for different $t \in \mathbb{N}$ and so is $H_2[t]$. Furthermore, H_1 is assumed to be independent with H_2. The distribution functions of random variables $||H_1||$ and $||H_2||$ are defined as $F_1(x) := P_1(||H_1|| \leq x)$ and $F_2(y) := P_2(||H_2|| \leq y)$ for $x, y \in \mathbb{R}$, respectively. It is worth noting that we do not ask the time scales h_1, h_2 to be the same; and the coupled system may thus work in a kind of asynchronous way.

Before going further, we give a definition for *nearly-emergence* that we adopt in this paper.

Definition 1. *Let $\mu, \nu > 0$, $x \in X$, $y \in \widetilde{Y}$, ν(or μ)-nearly-emergence occurs for the population $\{1, \cdots, k\}$ if $||y|| \leq \nu$(or $||x|| \leq \mu$).*

Clearly, the exact emergence is no longer possible due to the random perturbation.

Remark 1. *The two features x and y of agents in the system I(D) are asymmetric and y is the object whose emergence behavior is of interest. In the system II(D), the status of x and y is symmetric and both emergence behaviors may be of interest. The same can be said for the continuous case in Section 3 below.*

2.2 Main Results

We define several constants that are only related with the initial state $(x(0), y(0))$ of the population.

For system I(D):

$$Q(\delta) = 1 \vee \frac{1}{\delta}, \quad a = \frac{2C}{G} Q(\delta)||y(0)||^{\delta}, \quad b = 1 + ||x(0)||$$

$$B_0 = U_0 - 1, \quad \mathcal{H}_0 = \frac{2^{-\beta-1}G}{U_0^{\beta}}, \quad U_0 = \begin{cases} \max\{(2a)^{\frac{1}{1-\gamma-\beta}}, 2b\}, & \text{if } \beta+\gamma<1 \\ \frac{b}{1-a}, & \text{if } \beta+\gamma=1 \\ \frac{(\beta+\gamma)b}{\beta+\gamma-1}, & \text{if } \beta+\gamma>1 \end{cases}$$

For system II(D):

$$\mathcal{H}_1 = \frac{G_1}{2(1+||y(0)||)^{\beta_1}}, \qquad \mathcal{H}_2 = \frac{G_2}{2(1+||x(0)||)^{\beta_2}}$$

The main results in this section are stated as follows.

Theorem 1. *For dynamical system I(D), we assume*

$$h < \min\Big\{\frac{1}{G},\ \frac{1}{2^{1-\gamma}C||y(0)||^{\delta}}\Big(\frac{G}{2\mathcal{H}_0}\Big)^{\frac{1-\gamma}{\beta}}\Big\},$$

and one of the following hypotheses holds:
(i) $\beta+\gamma<1$,
(ii)$\beta+\gamma=1$, *and* $||y(0)|| < \big(\frac{G}{2CQ(\delta)}\big)^{\frac{1}{\delta}}$,
(iii)$\beta+\gamma>1$, *and* $\big(\frac{1}{a(\beta+\gamma)}\big)^{\frac{1}{\beta+\gamma-1}}\frac{\beta+\gamma-1}{\beta+\gamma} > b + h\big((\frac{\beta+\gamma}{\beta+\gamma-1})b\big)^{\gamma}\frac{aG}{2Q(\delta)}$.
Then, for $\nu < ||y(0)||$, ν*-nearly-emergence occurs in a number of iterations bounded by* $T_0 := \frac{2U_0^{\beta}}{hG}\ln\big(\frac{||y(0)||}{\nu}\big)$ *with probability at least* $F(\mathcal{H}_0\nu)^{T_0}$. *In addition, if* $\mu < aU_0^{\beta+\gamma}$, *let* $T_1 := \frac{2U_0^{\beta}}{\delta hG}\ln\big(\frac{aU_0^{\beta+\gamma}}{\mu}\big)$, *then the events* $\{||x[t]-x[\tau]|| \le \mu,\ for\ \tau > t \ge T_0 \vee T_1\}$ *and* $\{\nu$*-nearly-emergence occurs in a number of iterations bounded by* $T_0 \vee T_1\}$ *hold simultaneously with probability at least* $F(\mathcal{H}_0\nu)^{T_0\vee T_1}$.

Theorem 2. *For dynamical system II(D), we assume* $h_1 < \frac{1}{G_1}$ *and* $h_2 < \frac{1}{G_2}$. *Then, for* $\mu < ||x(0)||$, $\nu < ||y(0)||$ *with* $T_2 := \frac{1}{h_1\mathcal{H}_1}\ln\big(\frac{||x(0)||}{\mu}\big)$ *and* $T_3 := \frac{1}{h_2\mathcal{H}_2}\ln\big(\frac{||y(0)||}{\nu}\big)$, μ*-nearly-emergence and* ν*-nearly-emergence both occur in a number of iterations bounded by* $T_2 \vee T_3$ *with probability at least* $(F_1(\mathcal{H}_1\mu)F_2(\mathcal{H}_2\nu))^{T_2\vee T_3}$.

We now give some concrete substances to illustrate emergence behaviors of the general models I(D) and II(D).

For the system I(D), take $Y=\mathbb{R}^3$ with standard inner product $\langle\cdot,\cdot\rangle$, and $X=\widetilde{Y}=(\mathbb{R}^3)^k/\triangle \cong \triangle^{\perp}$. For $u=(u_1,\cdots,u_k), v=(v_1,\cdots,v_k)\in\widetilde{Y}$, define the inner product on $\widetilde{Y}$ as $\langle u,v\rangle_{\widetilde{Y}} = \frac{1}{2}\sum_{i,j=1}^{k}\langle u_i-u_j, v_i-v_j\rangle$. Here $x\in X$ represents the spatial positions of agents (e.g. birds, fishes, robots,...) and $y\in\widetilde{Y}$ their velocities both projected to the subspace $\triangle^{\perp}$ [6]. Given $x\in X$, let the $k\times k$ matrix A_x has entries $a_{ij}\ge\frac{K}{(1+||x_i-x_j||)^{\beta}}$. Let D_x be the $k\times k$ diagonal matrix whose ith diagonal element is $d_i=\sum_{j\le k}a_{ij}$ and $L_x := D_x - A_x$. Then L_x is the Laplacian of A_x. For $t\in\mathbb{N}$, take $J(x[t],y[t])=y[t]$, $S(x[t])=I_k-hL_x$, here I_k is the identity matrix of order k, and let the noise term H has the uniform distribution $U_{3k}(0,r)$ for some $r>0$ or the Gaussian distribution $N(0,\sigma^2 I_{3k})$ in the model I(D), and then we recover the situations encountered in [4]. Thm.1 in [4] is clearly a special case of Theorem 1 (and note that we really said more). Other kinds of flocking scenarios such as flocking with unrelated pairs and flocking with leader-follower schemes can also be dealt with under our present framework (c.f. [5] Sect. 3). We mention here that the asymmetric conclusions of x and y in Theorem 1 indeed give what we desire in a flocking phenomenon; see [6] (Rem. 2).

For the system II(D), let $\triangle_X$ be the diagonal of $(\mathbb{R}^3)^k$ and take $X=(\mathbb{R}^3)^k/\triangle_X$ with inner product defined as $\langle\cdot,\cdot\rangle_{\widetilde{Y}}$ above. Let Y be the space of languages with some appropriate distance defined on it (c.f. [6]); and the metric of $\widetilde{Y}$ is inherited from that of Y. Given $x \in X, y \in \widetilde{Y}$, let $A_x = (a_{ij})$, $B_y = (b_{ij})$ be the $k \times k$ matrices with entries $a_{ij} = f(||x_i - x_j||)$ and $b_{ij} = g(||y_i - y_j||)$. $f, g : \mathbb{R}^+ \to \mathbb{R}^+$ are some bounded non-increasing functions. For $t \in \mathbb{N}$, take $S_1(y[t]) = I_k - h_1 L_1 y$ and $S_2(x[t]) = I_k - h_2 L_2 x$ with $L_1 y := D_y - B_y$, $L_2 x := D_x - A_x$ in the model II(D). Here D_x and D_y are $k \times k$ diagonal matrices defined similarly as above. Computation of the distributions of $||H_1||$ and $||H_2||$ from some proper random noises H_1, H_2 is a routine [2]. Here $x \in X$ is interpreted as the geographical positions of agents projected to $\triangle_X^{\perp}$ and $y \in \widetilde{Y}$ as the space of languages projected to $\triangle^{\perp}$. This specification of system II(D) can be used to model emergence behavior in linguistic evolution, since each agent tends to move to others using similar languages and meanwhile the influence from other agents' languages decreases according to distances [6] (Sect. 6).

2.3 Proof of Theorem 1 and 2

The proof closely follows that of [4], and we prove Theorem 1 and 2 through some intermediate steps. Due to the limitation of space, we refer the reader to [13] for more details.

Proposition 1. *Let $T \in \mathbb{N}$ or $T = \infty$. Suppose $||H|| \leq \mathcal{H}_0 ||y[t]||$ for $0 \leq t < T$ and $h < \min\left\{\frac{1}{G}, \frac{1}{2^{1-\gamma} C ||y(0)||^{\delta}} \left(\frac{G}{2\mathcal{H}_0}\right)^{\frac{1-\gamma}{\beta}}\right\}$, and one of the following hypotheses holds:*
(i) $\beta + \gamma < 1$,
(ii)$\beta + \gamma = 1$, and $||y(0)|| < \left(\frac{G}{2CQ(\delta)}\right)^{\frac{1}{\delta}}$,
(iii)$\beta + \gamma > 1$, and $\left(\frac{1}{a(\beta+\gamma)}\right)^{\frac{1}{\beta+\gamma-1}} \frac{\beta+\gamma-1}{\beta+\gamma} > b + h\left(\left(\frac{\beta+\gamma}{\beta+\gamma-1}\right)b\right)^{\gamma} \frac{aG}{2Q(\delta)}$.
Then $1 - \frac{hG}{2U_0^{\beta}} \in (0, 1)$, for $0 \leq t < T$, $||x[t]|| \leq B_0$ and $||y[t]|| \leq ||y(0)||\left(1 - \frac{hG}{2U_0^{\beta}}\right)^t$. If $T = \infty$, then $||y[t]|| \to 0$ as $t \to \infty$, and moreover, there exists $\hat{x} \in X$ such that $x[t] \to \hat{x}$ as $t \to \infty$, and $||x[t] - \hat{x}|| \leq aU_0^{\beta+\gamma}\left(1 - \frac{hG}{2U_0^{\beta}}\right)^{\delta t}$ for $t \geq 0$.

Proof of Theorem 1. Suppose the conditions of Proposition 1 hold for some $T > 0$. Then we have $||y[t]|| \leq ||y(0)||\left(1 - \frac{hG}{2U_0^{\beta}}\right)^t$ for $t < T$ and $||x[\tau] - x[t]|| \leq aU_0^{\beta+\gamma}\left(1 - \frac{hG}{2U_0^{\beta}}\right)^{\delta t}$ for $t < \tau \leq T$. If $T = \infty$, $||x[t] - \hat{x}|| \leq aU_0^{\beta+\gamma}\left(1 - \frac{hG}{2U_0^{\beta}}\right)^{\delta t}$ for $t \geq 0$. By the proof of Proposition 1 and straightforward calculations, we have $||y[T]|| \leq \nu$ when $T \geq T_0$; $||x[\tau] - x[t]|| \leq \mu$ when $\tau > t \geq T_1$; and $||x[t] - \hat{x}|| \leq \mu$ when $t \geq T_1$. If ν-nearly-emergence has not occurred, then $||y[t]|| \geq \nu$, wherefore by the definition of function F,

$$P(||H[t]|| \leq \mathcal{H}_0 ||y[t]||) \geq P(||H[t]|| \leq \mathcal{H}_0 \nu) = F(\mathcal{H}_0 \nu).$$

Since $\{H[t]\}$ are i.i.d. for varying t, we get

$$P(||H[t]|| \leq \mathcal{H}_0 ||y[t]|| \text{ for } t = 0, \cdots, T_0 - 1) \geq F(\mathcal{H}_0 \nu)^{T_0},$$

which yields the first part of the conclusions. Likewise, we have

$$P(||H[t]|| \le \mathcal{H}_0||y[t]|| \text{ for } t = 0, \cdots, T_0 \vee T_1 - 1) \ge F(\mathcal{H}_0\nu)^{T_0 \vee T_1}.$$

We then conclude the proof. □

Proposition 2. *Let $T^1, T^2 \in \mathbb{N} \cup \{\infty\}$. Suppose $||H_1|| \le \mathcal{H}_1||x[t_1]||$ for $0 \le t_1 < T^1$; $||H_2|| \le \mathcal{H}_2||y[t_2]||$ for $0 \le t_2 < T^2$ and $h_1 < \frac{1}{G_1}$, $h_2 < \frac{1}{G_2}$. Then, for $0 \le t < T^1 \wedge T^2$, $||x[t]|| \le ||x(0)||(1 - h_1\mathcal{H}_1)^t$ and $||y[t]|| \le ||y(0)||(1 - h_2\mathcal{H}_2)^t$.*

Proof of Theorem 2. Suppose the conditions of Proposition 2 hold for some $T^1, T^2 > 0$. Then we have $||x[t]|| \le ||x(0)||(1 - h_1\mathcal{H}_1)^t$ and $||y[t]|| \le ||y(0)||(1 - h_2\mathcal{H}_2)^t$ for $t < T := T^1 \wedge T^2$. By direct calculations, we get $||x[T]|| \le \mu$ when $T \ge T_2$ and $||y[T]|| \le \nu$ when $T \ge T_3$. If μ-nearly-emergence and ν-nearly-emergence have not occurred at time t_1h_1 and t_2h_2 resp., then $||x[t_1]|| \ge \mu$ and $||y[t_2]|| \ge \nu$. Whence,

$$P_1(||H_1[t_1]|| \le \mathcal{H}_1||x[t_1]||) \ge P_1(||H_1[t_1]|| \le \mathcal{H}_1\mu) = F_1(\mathcal{H}_1\mu),$$

and

$$P_2(||H_2[t_2]|| \le \mathcal{H}_2||y[t_2]||) \ge P_2(||H_2[t_2]|| \le \mathcal{H}_2\nu) = F_2(\mathcal{H}_2\nu).$$

Since $\{H_i[t_i]\}$ are i.i.d. for varying t_i, $i = 1, 2$, and H_1 is independent with H_2, by letting $P = P_1 \times P_2$ be the independent product of P_1 and P_2 (c.f. [2]), we get

$$\begin{aligned} P(||H_1[t]|| \le \mathcal{H}_1||x[t]|| \text{ and } ||H_2[t]|| \le \mathcal{H}_2||y[t]||, \text{ for } t = 0, \cdots, T_2 \vee T_3 - 1) \\ \ge (F_1(\mathcal{H}_1\mu)F_2(\mathcal{H}_2\nu))^{T_2 \vee T_3}, \end{aligned}$$

which concludes the proof. □

3 Continuous-time Emergence

3.1 Models Setup (I(C), II(C))

In principle, by letting the time steps h, h_i approach zero, we may derive the continuous counterparts of systems I(D) and II(D). We shall, however, make some modifications for technical reason and it is at this time the inner product structures of spaces $X, Y, \widetilde{Y}$ take effect.

Let $t \in \mathbb{R}^+$, for $x \in X$, $y \in \widetilde{Y}$, consider the following dynamical system:

$$I(C): \qquad \begin{cases} x'(t) = J(x(t), y(t)) \\ y'(t) = -L_x y(t) + hH(t) \end{cases}$$

Here, as in the system I(D), $J : X \times \widetilde{Y} \to X$ is a Lipschitz or C^1 operator. We now require, for some $C, \delta > 0$, $0 \le \gamma < 1$, that

$$||J(x,y)|| \le C(1 + ||x||^2)^{\frac{\gamma}{2}}||y||^{\delta} \tag{4}$$

for all $x \in X$, $y \in \widetilde{Y}$. Denote $\mathbb{R}^{k\times k}$ as the space of $k \times k$ real matrices. Let $L : X \to \mathbb{R}^{k\times k}$ be a Lipschitz or C^1 operator with $L : x \mapsto L_x$. L_x can be seen as a linear transformation on Y^k by mapping $(y_1, \cdots, y_k) \in Y^k$ to $(L_x(i,1)y_1 + \cdots + L_x(i,k)y_k)_{i\le k}$. Here, $L_x(i,j)$ is the (i,j) entry of matrix L_x. For $x \in X$, define $\phi_x := \min_{\substack{y\neq 0\\ y\in\widetilde{Y}}} \frac{\langle L_x y, y\rangle}{||y||^2}$. We impose the following two hypotheses on L: (i) For $x \in X$, $y \in Y$, $L_x(y, \cdots, y) = 0$. (ii) there exists $K > 0, \beta \ge 0$ such that

$$\phi_x \ge \frac{K}{(1 + ||x||^2)^\beta} \tag{5}$$

for all $x \in X$. It is easy to see from (i) that L_x induces a linear transformation on $\widetilde{Y}$, which will also be denoted as L_x for notational simplicity. Let $H(t)$ be a continuous time stochastic process defined on some probability space $(\Omega, \mathcal{F}, P)$ taking value in $(\widetilde{Y}, \mathcal{B}(\widetilde{Y}))$. Let $F : \mathbb{R}^+ \times \mathbb{R}^+ \to [0,1]$ be a real function (not necessarily a distribution function) such that $P(\max_{0\le t\le T} ||H(t)|| \le x) \ge F(x,T)$, for $x, T \in \mathbb{R}^+$. We may observe that $F(x,T)$ is non-decreasing w.r.t. x while non-increasing w.r.t. T.

Next, we introduce a continuous version of II(D). For $x \in X$, $y \in \widetilde{Y}$, consider the following dynamical system:

$$II(C): \qquad \begin{cases} x'(t) = -L_{1y}x(t) + H_1(t) \\ y'(t) = -L_{2x}y(t) + H_2(t) \end{cases}$$

Here operators L_1, L_2 are given similarly as L above with L_1 defined on $\widetilde{Y}$ and L_2 on X. For $x \in X$, $y \in \widetilde{Y}$, define $\xi_x := \min_{\substack{y\neq 0\\ y\in\widetilde{Y}}} \frac{\langle L_{2x} y, y\rangle}{||y||^2}$ and $\eta_y := \min_{\substack{x\neq 0\\ x\in X}} \frac{\langle L_{1y} x, x\rangle}{||x||^2}$. The corresponding hypothesis (ii) above becomes (ii′): there exist $K_1, K_2 > 0$, $\beta_1, \beta_2 \ge 0$ such that

$$\xi_x \ge \frac{K_1}{(1 + ||x||^2)^{\beta_1}}, \qquad \eta_y \ge \frac{K_2}{(1 + ||y||^2)^{\beta_2}} \tag{6}$$

for all $x \in X$, $y \in \widetilde{Y}$. Let $H_1(t) : (\Omega_1, \mathcal{F}_1, P_1) \to (X, \mathcal{B}(X))$ and $H_2(t) : (\Omega_2, \mathcal{F}_2, P_2) \to (\widetilde{Y}, \mathcal{B}(\widetilde{Y}))$ be two continuous time stochastic processes, which are independent with each other. Let $i = 1, 2$, $F_i : \mathbb{R}^+ \times \mathbb{R}^+ \to [0,1]$ be real functions such that $P_i(\max_{0\le t\le T} ||H_i(t)|| \le x) \ge F_i(x,T)$, for $x, T \in \mathbb{R}^+$.

For notational convenience we sometimes write $L_t := L_{x(t)}$, $\phi_t := \phi_{x(t)}$, $\xi_t := \xi_{x(t)}$ and $\eta_t := \eta_{y(t)}$.

3.2 Main Results

As in the discrete case, we define several constants which are only dependent on the initial state $(x(0), y(0))$ of the population.

For system I(C):

$$a = 2^{\frac{1+\gamma+2\beta}{1-\gamma}} \frac{((1-\gamma)C)^{\frac{2}{1-\gamma}}||y(0)||^{\frac{2\delta}{1-\gamma}}}{(\delta K)^{\frac{2}{1-\gamma}}}, \quad b = 2^{\frac{1+\gamma}{1-\gamma}}(1+||x(0)||^2), \quad \alpha = \frac{2\beta}{1-\gamma}$$

$$B_0 = U_0 - 1, \quad B_1 = \frac{2C||y(0)||^{\delta} B_0^{\frac{\gamma}{2}+\beta}}{\delta K}, \quad \mathcal{H}_0 = \frac{2^{-\beta-1}G}{U_0^{\beta}}$$

$$U_0 = \begin{cases} \max\{(2a)^{\frac{1-\gamma}{1-\gamma-2\beta}}, 2b\}, & \text{if } 2\beta+\gamma<1 \\ \frac{b}{1-a}, & \text{if } 2\beta+\gamma=1 \\ (\frac{1}{a\alpha})^{\frac{1}{\alpha-1}}, & \text{if } 2\beta+\gamma>1 \end{cases}$$

For system II(C):

$$\mathcal{H}_1 = \frac{K_2}{2(1+||y(0)||^2)^{\beta_2}}, \qquad \mathcal{H}_2 = \frac{K_1}{2(1+||x(0)||^2)^{\beta_1}}$$

The main results in this section are stated as follows.

Theorem 3. *Let $x(0) \in X$ and $y(0) \in \widetilde{Y}$, then there exists a unique solution $(x(t), y(t))$ of the dynamical system I(C) for all $t \in \mathbb{R}$. Moreover, assume one of the following hypotheses holds:*
(i) $2\beta+\gamma<1$,
(ii)$2\beta+\gamma=1$, and $||y(0)|| < \big(\frac{(\delta K)^2}{2^{1+\gamma+2\beta}((1-\gamma)C)^2}\big)^{\frac{1}{2\delta}}$,
(iii)$2\beta+\gamma>1$, and $\big(\frac{1}{a\alpha}\big)^{\frac{1}{\alpha-1}}\frac{\alpha-1}{\alpha} > b$.
Then, for $\nu < ||y(0)||$, ν-nearly-emergence occurs before time $T_0 := \frac{2B_0^{\beta}}{K}\ln\big(\frac{||y(0)||}{\nu}\big)$ with probability at least $F(\mathcal{H}_0\nu, T_0)$. In addition, if $\mu < B_1$, let $T_1 := \frac{2B_0^{\beta}}{K\delta}\ln\big(\frac{B_1}{\mu}\big)$, then the events $\{||x[t]-x[\tau]|| \le \mu,\ for\ \tau > t \ge T_0 \vee T_1\}$ and $\{\nu$-nearly-emergence occurs before time $T_0 \vee T_1\}$ hold simultaneously with probability at least $F(\mathcal{H}_0\nu, T_0 \vee T_1)$.

As in Section 2.2, we may readily recover the continuous-time result in [4] by letting L_x be the Laplacian of A_x, for $x \in X$; and taking $J(x,y)=y$ and the coordinate processes of $H(t)$ as independent smoothed Wiener processes.

Theorem 4. *Let $x(0) \in X$ and $y(0) \in \widetilde{Y}$, then there exists a unique solution $(x(t), y(t))$ of the dynamical system II(C) for all $t \in \mathbb{R}$. Furthermore, for $\mu < ||x(0)||$, $\nu < ||y(0)||$ with $T_2 := \frac{2(1+||y(0)||^2)^{\beta_2}}{K_2}\ln\big(\frac{||x(0)||}{\mu}\big)$ and $T_3 := \frac{2(1+||x(0)||^2)^{\beta_1}}{K_1}\ln\big(\frac{||y(0)||}{\nu}\big)$, either μ-nearly-emergence or ν-nearly-emergence occurs before time $T_2 \vee T_3$ with probability at least*
$F_1(\mathcal{H}_1\mu, T_2 \vee T_3)F_2(\mathcal{H}_2\nu, T_2 \vee T_3)$.

Compared with Theorem 2, the last result is weaker due to the fact that in the continuous case the stochastic processes $H_i(t)$ do not possess "independence property" among different "time steps". However, if the noise does not impose on both equations of system II(C), we have the following corollary.

Corollary 1. *Suppose $H_1(t) \equiv 0$. Under the assumptions of Theorem 4, the events $\{\mu$-nearly-emergence occurs before time $T_2\}$ and $\{\nu$-nearly-emergence occurs before time $T_3\}$ hold simultaneously with probability at least $F_2(\mathcal{H}_2\nu, T_3)$. An analogous result holds for the case $H_2(t) \equiv 0$.*

We mention that it is possible to have results similar with Corollary 1 when $H_1(t)$ is small enough or possesses independent increments. The main procedure of the proofs follow that of [4]. Those interested readers may consult [13] for details.

4 Conclusion

In this paper, we have studied the emergent behavior of four dynamical systems (I(D), I(C), II(D), II(C)) in the presence of random fluctuation contained in the environments. In all these cases, "nearly-emergence" phenomena of interested objectives are shown under certain conditions on the systems and the noises. Our results are presented in a quite general setting and reveal some intrinsic mechanisms of emergence which come up in a variety of disciplines [15]. We will extend the results herein onto other dynamical systems and different kinds of random environment will be treated in future work.

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